Title: Quantum complementarity: A novel resource for exclusion

Speakers: Chung-yun Hsieh

Collection/Series: Quantum Foundations

Subject: Quantum Foundations

Date: February 24, 2025 - 11:00 AM

URL: https://pirsa.org/25020041

Abstract:

Complementarity is a phenomenon explaining several core features of quantum theory, such as the well-known uncertainty principle. Roughly speaking, two objects are said to be complementary if being certain about one of them necessarily forbids useful knowledge about the other. Two quantum measurements that do not commute form an example of complementary measurements, and this phenomenon can also be defined for ensembles of states. Although a key quantum feature, it is unclear whether complementarity can be understood more operationally, as a necessary resource in some quantum information task. Here we show this is the case, and relates to a task which we term unambiguous exclusion. As well as giving complementarity a clear operational definition, this also uncovers the foundational underpinning of unambiguous exclusion tasks for the first time.

Quantum complementarity

A novel resource for exclusion tasks

University of Bristol

Chung-Yun Hsieh

Leverhulme Trust

Quantum complementarity

A novel resource for exclusion tasks

Chung-Yun Hsieh, Roope Uola, Paul Skrzypczyk, arXiv:2309.11968

University of BRISTOL

Chung-Yun Hsieh

LEVERHULME TRUST

What is complementarity


1

1

Here!

1

Details

Position

1

Details ⟷ Position

Complementary

(classical)

1

(momentum)
Details
of motion

Complementary
↔
(quantum)

Position

1

Spin X

Complementary

(quantum)

Spin Z

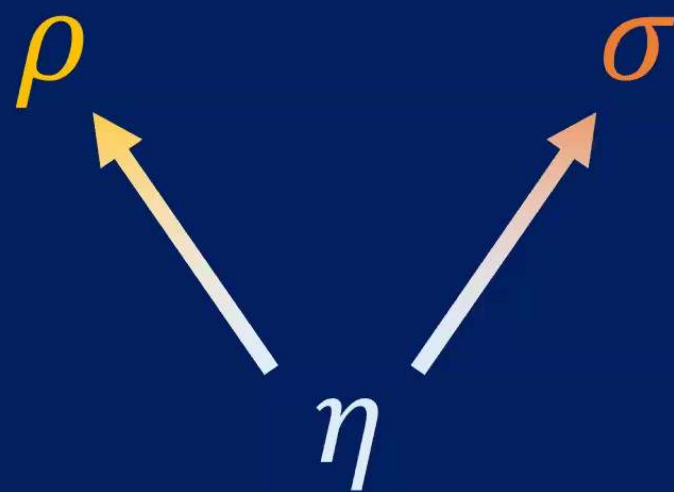
1

Complementarity

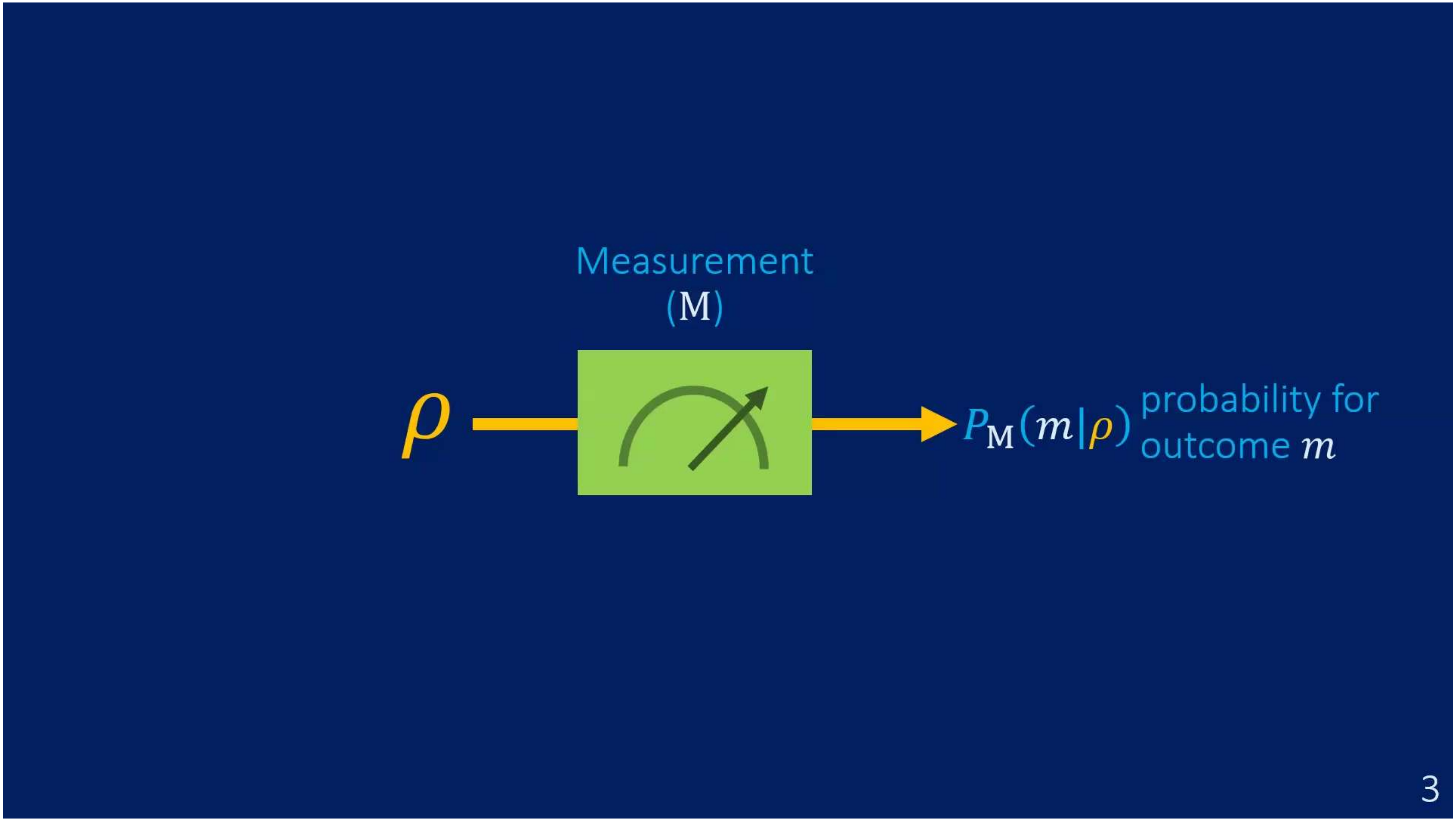
↕

Carrying no common Q info

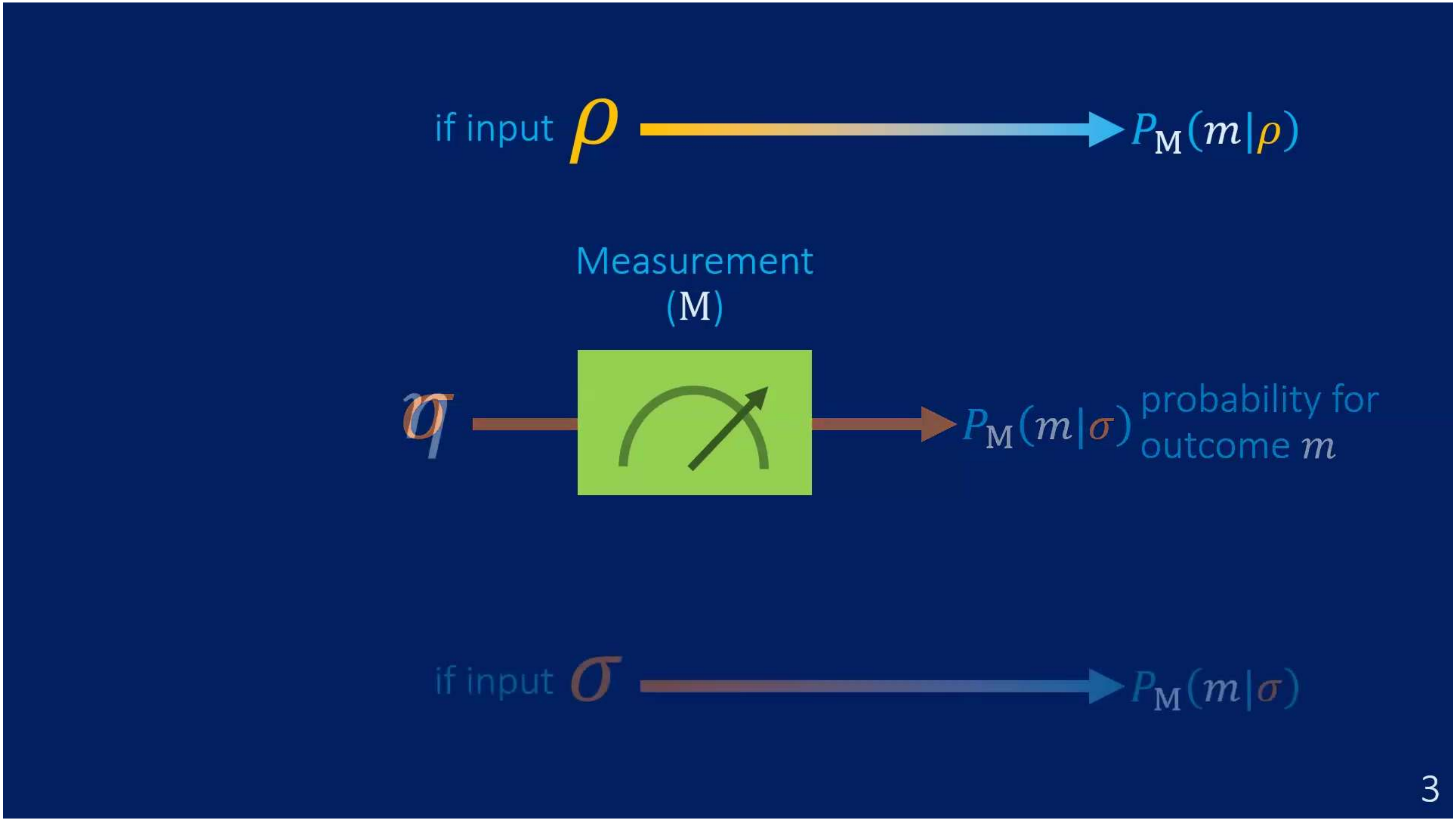
Can they carry common Q info?

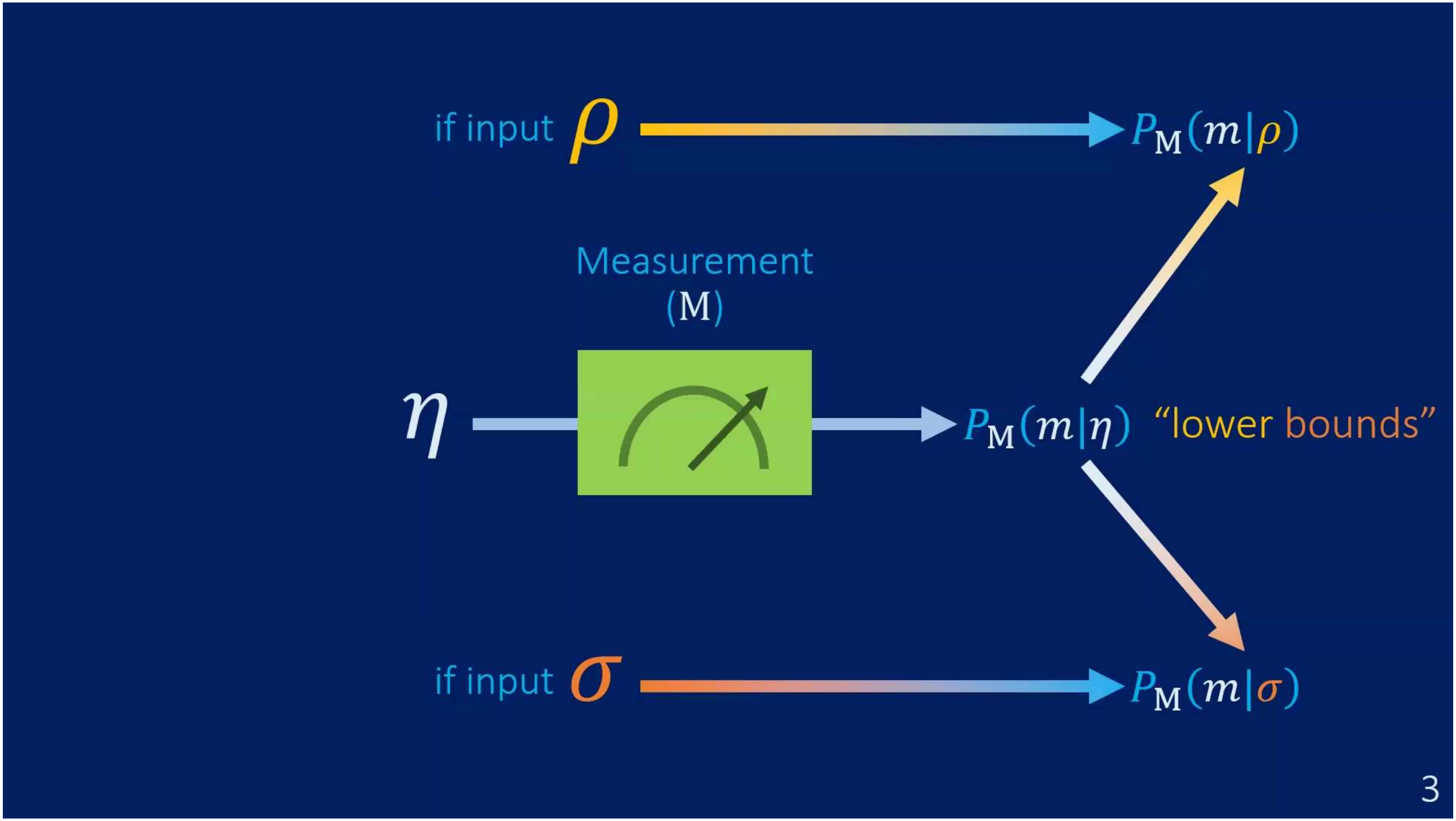


(partially) describes measurement statistics



Measurement
(M)
ρ
$P_{\mathrm{M}}(m|\rho)$
probability for outcome m
3


if input ρ
$P_{\mathrm{M}}(m|\rho)$
Measurement
(M)
σ
$P_{\mathrm{M}}(m|\sigma)$
probability for
outcome m
if input σ
$P_{\mathrm{M}}(m|\sigma)$
3

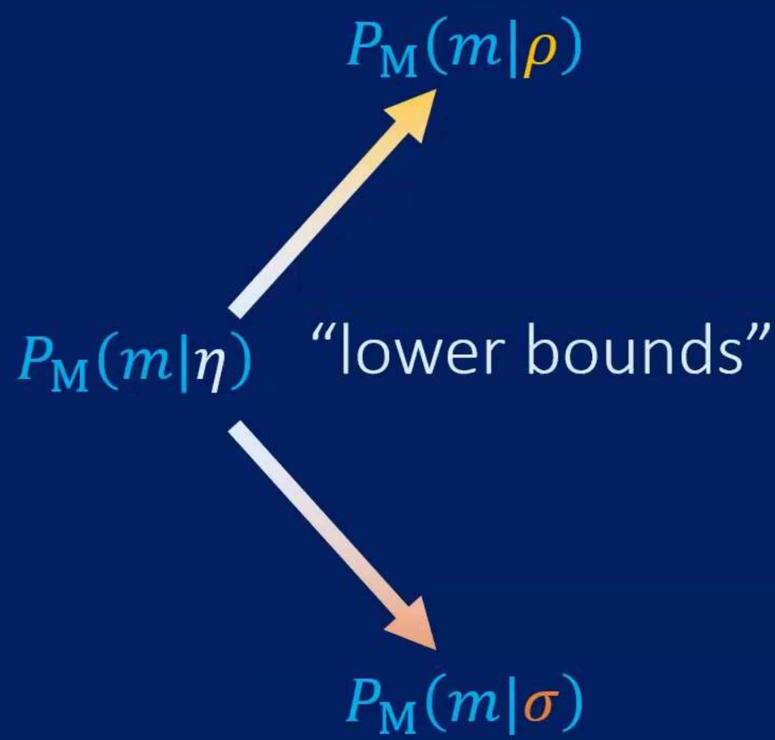

if input ρ
$P_{\mathrm{M}}(m|\rho)$
Measurement
(M)
η
$P_{\mathrm{M}}(m|\eta)$
"lower bounds"
if input σ
$P_{\mathrm{M}}(m|\sigma)$
3

We say ρ & σ carry common Q info η (quantified by $1 \geq \alpha > 0$) if

$$P_{\mathrm{M}}(m|\rho) \geq \alpha P_{\mathrm{M}}(m|\eta)$$

$$P_{\mathrm{M}}(m|\sigma) \geq \alpha P_{\mathrm{M}}(m|\eta)$$

for every measurement M and outcome m

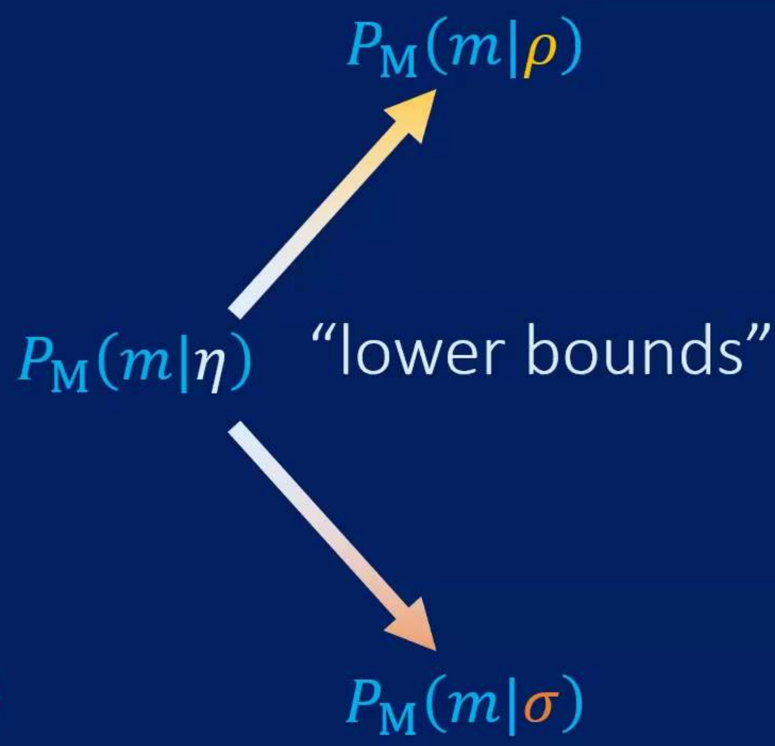


We say ρ & σ carry common Q info η (quantified by $1 \geq \alpha > 0$) if

$$P_{\mathrm{M}}(m|\rho) \geq \alpha P_{\mathrm{M}}(m|\eta)$$

$$P_{\mathrm{M}}(m|\sigma) \geq \alpha P_{\mathrm{M}}(m|\eta)$$

for every measurement M and outcome m

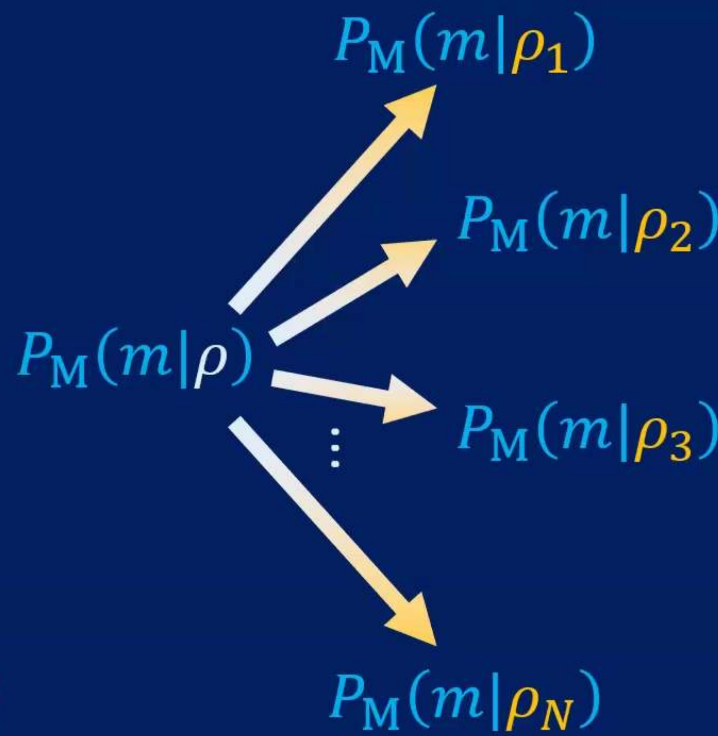


We say ρ & σ are complementary if they cannot carry common Q info (that is, $\alpha = 0$)

We say $\{\rho_x\}_{x=1}^{N}$ carry common Q info η (quantified by $1 \geq \alpha > 0$) if

$$P_{\mathrm{M}}(m|\rho_x) \geq \alpha P_{\mathrm{M}}(m|\eta) \quad \forall\, x$$

for every measurement M and outcome m



We say $\{\rho_x\}_{x=1}^{N}$ are complementary if they cannot carry common Q info (that is, $\alpha = 0$)

4

Complementary Q states are useful in tasks excluding classical info

Sending x, aiming to always output an index $\neq x$

5

Complementary Q states are useful in tasks excluding classical info

$|\phi_0\rangle$ $|\phi_1\rangle$ $|\phi_2\rangle$

Example: 3 qubit-states carry **no** common Q info

x —encode...→ $|\phi_x\rangle$ →

→ $P(\text{output} = x \,|\, |\phi_x\rangle) = 0$

Single-qubit three-outcome measurement

Example: Effective qubit in a qutrit

$$|\phi_x\rangle = \left(|0\rangle - e^{2\pi x i/3}|1\rangle\right)/\sqrt{2} \quad x = 0,1,2$$

$$|\omega_a\rangle = \left(|0\rangle + e^{2\pi a i/3}|1\rangle + e^{4\pi a i/3}|2\rangle\right)/\sqrt{3} \quad a = 0,1,2$$

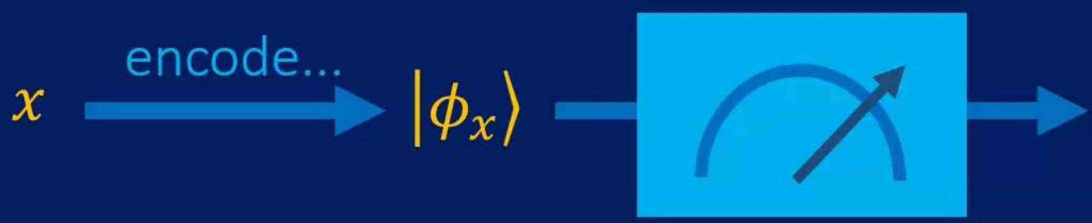
By considering the qutrit projective measurement $|\omega_a\rangle\langle\omega_a|$, we have...

Example: Effective qubit in a qutrit

$$|\phi_x\rangle = \left(|0\rangle - e^{2\pi x i/3}|1\rangle\right)/\sqrt{2} \quad x = 0,1,2$$

$$|\omega_a\rangle = \left(|0\rangle + e^{2\pi a i/3}|1\rangle + e^{4\pi a i/3}|2\rangle\right)/\sqrt{3} \quad a = 0,1,2$$

By considering the qutrit projective measurement $|\omega_a\rangle\langle\omega_a|$, we have...



a \ x	0	1	2
0	0	0.5	0.5
1	0.5	0	0.5
2	0.5	0.5	0

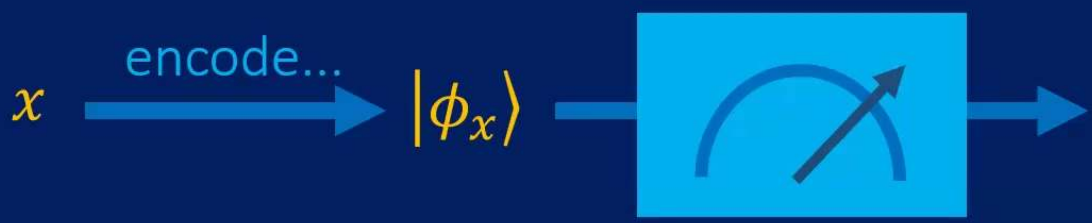
Is complementarity useful operationally?

Example: Effective qubit in a qutrit

$$|\phi_x\rangle = (|0\rangle - e^{2\pi x i/3}|1\rangle)/\sqrt{2} \quad x = 0,1,2$$

$$|\omega_a\rangle = (|0\rangle + e^{2\pi a i/3}|1\rangle + e^{4\pi a i/3}|2\rangle)/\sqrt{3} \quad a = 0,1,2$$

By considering the qutrit projective measurement $|\omega_a\rangle\langle\omega_a|$, we have...



a \ x	0	1	2
0	0	0.5	0.5
1	0.5	0	0.5
2	0.5	0.5	0

6

Complementary Q states are useful in tasks excluding classical info

$|\phi_0\rangle$ $|\phi_1\rangle$ $|\phi_2\rangle$

Example: 3 qubit-states carry **no** common Q info

x —encode...→ $|\phi_x\rangle$ —

→ $P(\text{output} = x \,|\, |\phi_x\rangle) = 0$

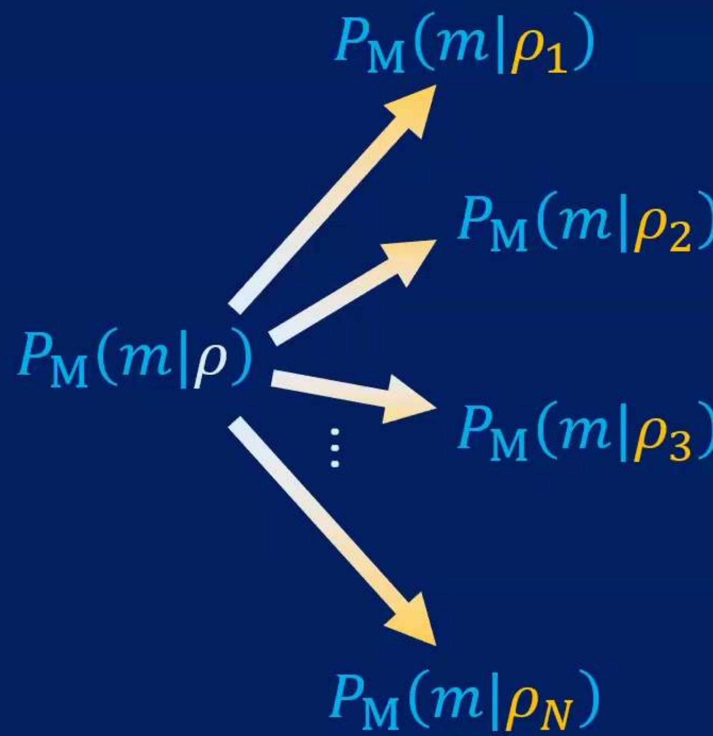
Q complementarity is a resource for exclusion tasks

5

Complementarity is useful in exclusion tasks

We say $\{\rho_x\}_{x=1}^{N}$ carry common Q info
if there is $P \gneqq 0$ such that

$$\rho_x \geq P \quad \forall\, x$$

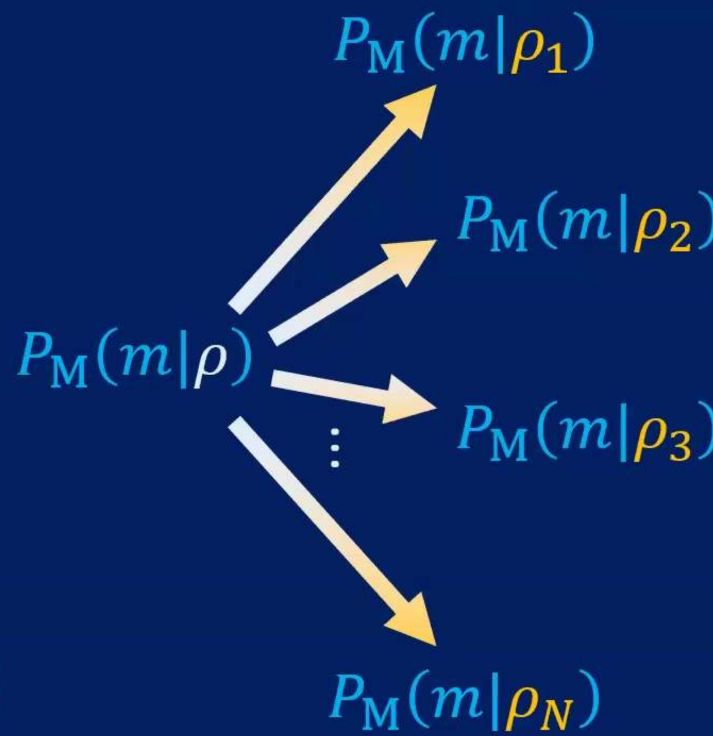


7

We say $\{\rho_x\}_{x=1}^{N}$ carry common Q info η (quantified by $1 \geq \alpha > 0$) if

$$P_{\mathrm{M}}(m|\rho_x) \geq \alpha P_{\mathrm{M}}(m|\eta) \quad \forall\, x$$

for every measurement M and outcome m



We say $\{\rho_x\}_{x=1}^{N}$ are complementary if they cannot carry common Q info (that is, $\alpha = 0$)

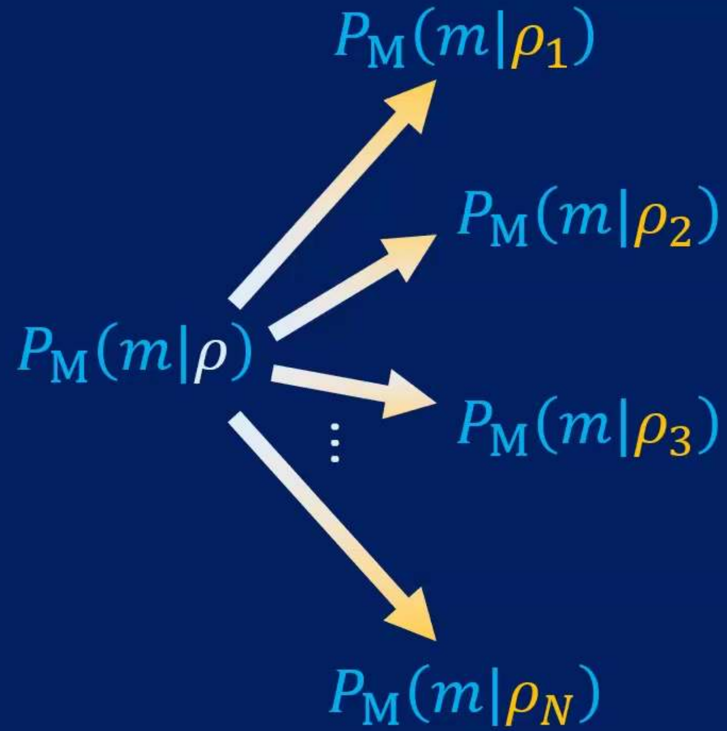
7

We say $\{\rho_x\}_{x=1}^{N}$ carry common Q info
if there is $P \gneqq 0$ such that

$$\rho_x \geq P \;\; \forall\, x$$

Quantifier:

$$q_{\mathrm{exc}}(\{\rho_x\}) \coloneqq \max_{\substack{\rho_x \geq P\ \forall\, x \\ P \geq 0}} \mathrm{tr}(P)$$



7

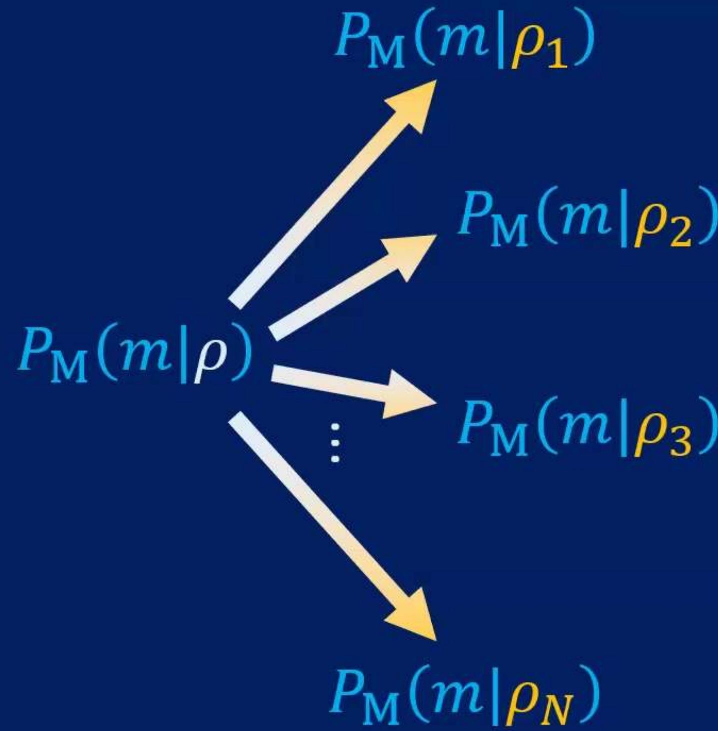
We say $\{\rho_x\}_{x=1}^{N}$ carry common Q info
if there is $P \gneqq 0$ such that

$$\rho_x \geq P \quad \forall\, x$$

Quantifier:

$$q_{\mathrm{exc}}(\{\rho_x\}) := \max_{\substack{\rho_x \geq P\ \forall\, x \\ P \geq 0}} \mathrm{tr}(P)$$

$q_{\mathrm{exc}}(\{\rho_x\}) = 0$ iff $\{\rho_x\}$ is complementary



7

Complementary Q states are useful in tasks excluding classical info

$|\phi_0\rangle$ $|\phi_1\rangle$ $|\phi_2\rangle$

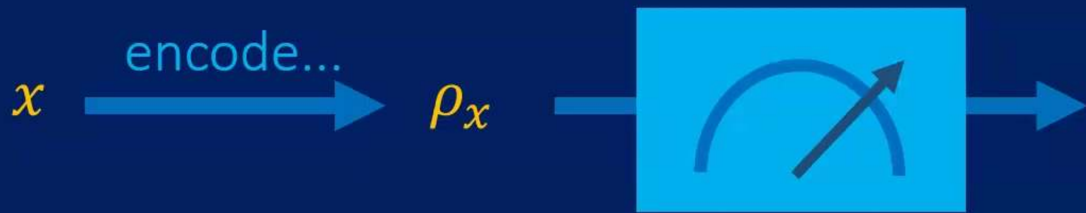
Example: 3 qubit-states carry **no** common Q info

x —encode...→ $|\phi_x\rangle$ →  → $P(\text{output} = x \,|\, |\phi_x\rangle) = 0$

Single-qubit three-outcome measurement

8

Complementary Q states are useful in tasks excluding classical info



Single-qudit (N+1)-outcome measurement

minimise $P(\text{output} = x|\rho_x)$

&

allow $P(\text{output} = \emptyset) < 1$

8

Complementary Q states are useful in tasks excluding classical info

ρ_1 ρ_2 $\cdots$ ρ_N

$$P_{\text{error}}(\{\rho_x\},\epsilon) \coloneqq \min\left\{ \sum_{x=1}^{N} \frac{1}{N}\operatorname{tr}(Q_x\rho_x) \,\middle|\, (1-\epsilon)\mathbb{I} \le \sum_{x=1}^{N} Q_x \le \mathbb{I}, Q_x \ge 0 \right\}$$

x — encode... → ρ_x →

Single-qudit (N+1)-outcome measurement

→ minimise $P(\text{output} = x|\rho_x)$

$= \operatorname{tr}(Q_x\rho_x)$

&

allow $P(\text{output} = \emptyset) \le \epsilon$

8

Result

For every $\{\rho_x\}$, there exists $\epsilon_* < 1$ such that

$$q_{\text{exc}}(\{\rho_x\}) = \frac{P_{\text{error}}(\{\rho_x\}, \epsilon)}{(1-\epsilon)/N} \quad \forall \epsilon_* \le \epsilon < 1$$

Complementarity = exclusion advantage!

$$P_{\text{error}}(\{\rho_x\}, \epsilon) \coloneqq \min \left\{ \sum_{x=1}^{N} \frac{1}{N} \operatorname{tr}(Q_x \rho_x) \,\middle|\, (1-\epsilon)\mathbb{I} \le \sum_{x=1}^{N} Q_x \le \mathbb{I}, Q_x \ge 0 \right\}$$

x encode... ρ_x

Single-qudit (N+1)-outcome measurement

minimise $P(\text{output} = x|\rho_x)$
$= \operatorname{tr}(Q_x \rho_x)$
&
allow $P(\text{output} = \emptyset) \le \epsilon$

9

Complementarity = carrying no common Q info

$$q_{\mathrm{exc}}(\{\rho_x\}) \coloneqq \max_{\substack{\rho_x \geq P\ \forall\, x \\ P \geq 0}} \mathrm{tr}(P)$$

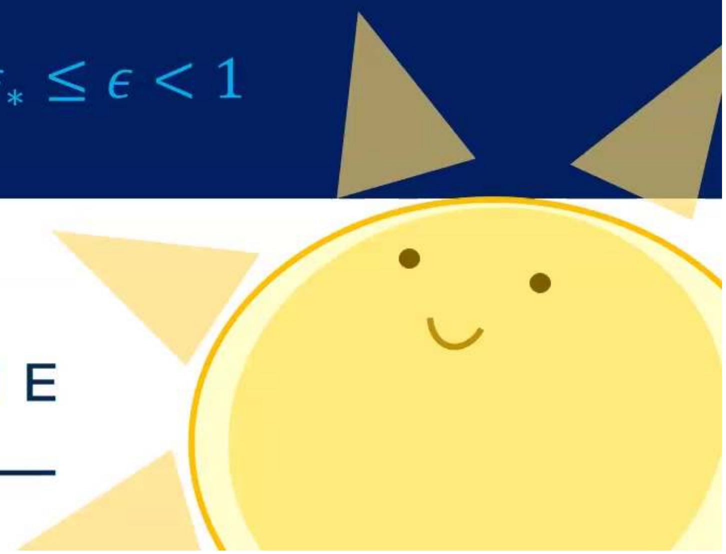
Complementarity = unambiguous exclusion advantage

$$q_{\mathrm{exc}}(\{\rho_x\}) = \frac{P_{\mathrm{error}}(\{\rho_x\}, \epsilon)}{(1-\epsilon)/N} \quad \forall \epsilon_* \leq \epsilon < 1$$

C.-Y. Hsieh, R. Uola, P. Skrzypczyk, arXiv:2309.11968

University of BRISTOL

LEVERHULME TRUST



Complementary Q states are useful in tasks excluding classical info

ρ_1 ρ_2 $\cdots$ ρ_N

$$P_{\text{error}}(\{\rho_x\},\epsilon) \coloneqq \min\left\{\sum_{x=1}^{N}\frac{1}{N}\operatorname{tr}(Q_x\rho_x)\;\middle|\;(1-\epsilon)\mathbb{I} \le \sum_{x=1}^{N} Q_x \le \mathbb{I}, Q_x \ge 0\right\}$$

x —encode...→ ρ_x →

→ minimise $P(\text{output} = x|\rho_x)$

$= \operatorname{tr}(Q_x\rho_x)$

&

allow $P(\text{output} = \emptyset) \le \epsilon$

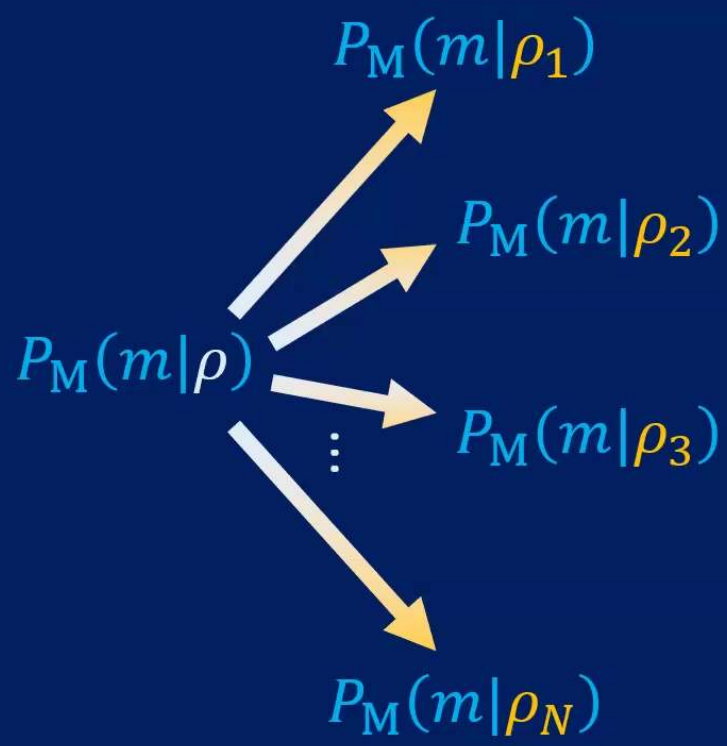
Single-qudit (N+1)-outcome measurement

We say $\{\rho_x\}_{x=1}^{N}$ carry common Q info
if there is $P \gneqq 0$ such that

$$\rho_x \geq P \;\; \forall\, x$$

Quantifier:

$$q_{\mathrm{exc}}(\{\rho_x\}) \coloneqq \max_{\substack{\rho_x \geq P\, \forall\, x \\ P \geq 0}} \mathrm{tr}(P)$$



Result

For every $\{\rho_x\}$, there exists $\epsilon_* < 1$ such that

$$q_{\text{exc}}(\{\rho_x\}) = \frac{P_{\text{error}}(\{\rho_x\}, \epsilon)}{(1-\epsilon)/N} \quad \forall \epsilon_* \leq \epsilon < 1$$

$$P_{\text{error}}(\{\rho_x\}, \epsilon) \coloneqq \min \left\{ \sum_{x=1}^{N} \frac{1}{N} \mathrm{tr}(Q_x \rho_x) \,\middle|\, (1-\epsilon)\mathbb{I} \leq \sum_{x=1}^{N} Q_x \leq \mathbb{I}, Q_x \geq 0 \right\}$$

x —encode...→ ρ_x →

→ minimise $P(\text{output} = x|\rho_x)$

$= \mathrm{tr}(Q_x \rho_x)$

&

allow $P(\text{output} = \emptyset) \leq \epsilon$

Single-qudit
(N+1)-outcome
measurement