Title: Statistical Fluctuations in the Causal Set-Continuum Correspondence

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Abstract:

Causal set theory is an approach to quantum gravity that proposes that spacetime is fundamentally discrete and the causal relations among the discrete elements play a prominent role in the physics. Progress has been made in recognizing and understanding how some continuumlike features can emerge from causal sets at macroscopic scales, i.e., when the number of elements is large. An important result in this context is that a causal set is well approximated by a continuum spacetime if there is a number-volume correspondence between the causal set and spacetime. This occurs when the number of elements within an arbitrary spacetime region is proportional to its volume. Such a correspondence is known to be best achieved when the number of causal set elements is randomly distributed according to the Poisson distribution. I will discuss the Poisson distribution and the statistical fluctuations it induces in the causal set-continuum correspondence, highlighting why it is important and interesting. I will also discuss new tools and techniques that facilitate such analyses.

Statistical Fluctuations in the Causal Set-Continuum Correspondence

Based on **arXiv:2407.03395**, work w/ H. Moradi & M. Zilhão

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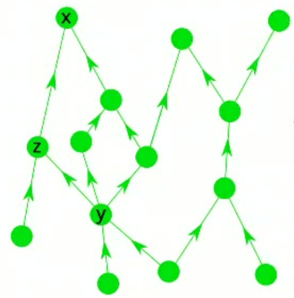
Questions in Quantum Gravity

What is the microscopic structure of spacetime that is more fundamental than the continuous Lorentzian manifold and metric of general relativity?

- **What is the resolution of the UV divergences in quantum field theory and spacetime singularities in GR?**
- **What happens in the early universe?**
- **What are the microstates of a black hole?**
- **How do matter fields backreact on spacetime?**
- **How do we think about superpositions of spacetimes or causal structures?**
- **How do we path integrate over Lorentzian geometries?**
- **...**

Causal Set Theory

Bombelli, Lee, Meyer, Sorkin, 1987, *Space-Time as a Causal Set*, PRL. 59, 521.



e.g.

A causal set is a fundamentally discrete spacetime with causal relations among some of its elements.

A **locally finite partially ordered set**. The set $\mathscr{C}$ (of spacetime elements) and ordering relation $\preceq$ (causal precedence) satisfy:

- **Reflexivity**: for all $x \in \mathscr{C}, \quad x \preceq x$
- **Antisymmetry**: for all $x, y \in \mathscr{C}, x \preceq y \preceq x$ implies $x = y$
- **Transitivity**: for all $x, y, z \in \mathscr{C}, \quad x \preceq y \preceq z$ implies $x \preceq z$
- **Local Finiteness**: for all $x, y \in \mathscr{C}, |I(x, y)| < \infty$, where $|\cdot|$ denotes cardinality and $I(x, y)$ is the causal interval defined by $I(x, y) := \{z \in \mathscr{C} \,|\, x \preceq z \preceq y\}$

We know that we can recover all essential aspects of a continuous spacetime from a causal set.

Zeeman, *Causality Implies the Lorentz Group*, J. Math. Phys. 5: 490-493 (1964). Hawking, King and McCarthy, *A New Topology for Curved Space-Time which Incorporates the Causal, Differential and Conformal Structures*, J. Math. Phys. 17: 174 (1976). Malament, *The Class of Continuous Timelike Curves Determines the Topology of Space-time*, J. Math. Phys 18: 1399 (1977)

Manifoldlike Causal Sets

What does it mean for a causal set $\mathscr{C}$ to be well approximated by a spacetime $(\mathscr{M}, g)$?

- **Causal Relations** $\mathscr{C}$ **should be embeddable in** $\mathscr{M}$ **such that the elements have the same causal relations as the corresponding points in** $\mathscr{M}$**: Sample points from** $\mathscr{M}$

- **Volumes The number of elements N within any arbitrary region with volume V should be statistically proportional to V (number-volume correspondence): Poisson distribution** (arXiv:1403.6429)

- **Scale True at scales larger than the discreteness scale.**

Manifoldlike Causal Sets

The best **number-volume correspondence** is achieved by the **Poisson distribution:**

$$P_N(V) = \frac{(\rho V)^N}{N!} e^{-\rho V} \qquad (1)$$

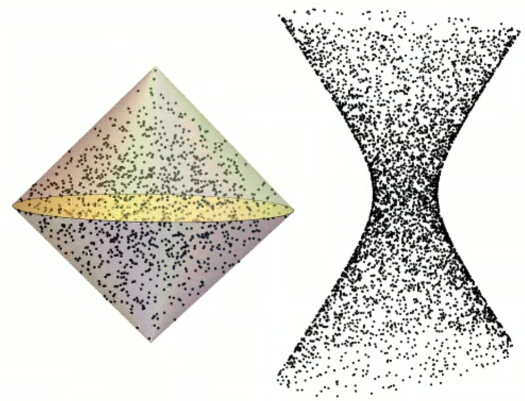
$$\langle N \rangle = \rho V$$
$$\Delta N = \sqrt{\rho V}$$

How? Place points at random in $\mathcal{M}$ using a **Poisson process**: divide spacetime into small subregions with volume dV and place at most one point in each subregion, with probability ρdV.

Then, $P_N(V) = \binom{V/dV}{N} (\rho dV)^N (1 - \rho dV)^{V/dV - N}$

which becomes (1) when $dV \to 0$

(Poisson limit theorem)



Poisson Sprinkling

No preferred frame (arXiv:gr-qc/0605006, arXiv:1909.06070)

Poisson Distribution

The best **number-volume correspondence** is achieved by the **Poisson distribution:**

$$P_N(V) = \frac{(\rho V)^N}{N!} e^{-\rho V}$$

$$\langle N \rangle = \rho V$$
$$\Delta N = \sqrt{\rho V}$$

Given a function $f : \mathrm{Sp}[\mathscr{M}] \longrightarrow \mathbb{C}$, we are interested in $\langle f \rangle$ and $\Delta f = \sqrt{\langle f^2 \rangle - \langle f \rangle^2}$

- How close is the correspondence between some continuum quantities (e.g. the d'Alembertian and gravitational action) and their discrete analogues in the causal set (arXiv:gr-qc/0703099, arXiv:1001.2725)
- Deviations can be a source of new physics: e.g. the Everpresent Λ cosmological model where $\Lambda \sim \frac{1}{\Delta N} = \frac{1}{\sqrt{V}} \sim H^2 \sim 10^{-122}$ (arXiv:astro-ph/0209274, arXiv:2304.03819)

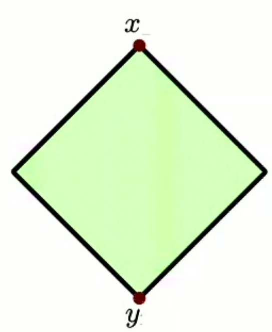
Notation

$$P_N(V) = \frac{(\rho V)^N}{N!} e^{-\rho V}$$

The **Cardinality Indicator function** $\zeta_i \; (i \geq 0)$ for a causal set $\mathscr{C}$ and region $\mathscr{R}$ is defined as $\zeta_i(\mathscr{R}) \equiv \delta_{\mathrm{Num}_{\mathscr{R}},i}$ which evaluates to 1 if there are i elements of $\mathscr{C}$ in $\mathscr{R}$, and evaluates to 0 otherwise. Note that $\langle \delta_{\mathrm{Num}_{\mathscr{R}},i} \rangle = P_i(V_{\mathscr{R}})$

It is useful to introduce a special variant when the region is a causal diamond $I(x,y) = \mathscr{J}^-(x) \cap \mathscr{J}^+(y)$:

$$\zeta_i(x,y) \equiv \Theta_{x,y} \, \zeta_i(I(x,y)) \quad \text{where} \quad \Theta_{x,y} \equiv \begin{cases} 1 & \text{if } I(x,y) \neq \emptyset, \\ 0 & \text{otherwise}. \end{cases}$$



arXiv:2407.03395

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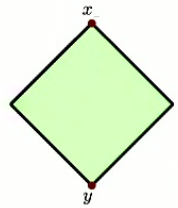


If a non-empty region $\mathscr{R}$ is split into n disjoint (possibly empty) subregions $\mathscr{R} = \bigsqcup_{a=1}^{n} \mathscr{R}_a$, then we have the ***Disjoint Decomposition Property***

$$\zeta_i\left(\bigsqcup_{a=1}^{n} \mathscr{R}_a\right) = \sum_{\substack{\alpha_a \geq 0 \\ \alpha_1 + \ldots + \alpha_n = i}} \prod_{a=1}^{n} \zeta_{\alpha_a}(\mathscr{R}_a)$$

arXiv:2407.03395

Notation

$$P_N(V) = \frac{(\rho V)^N}{N!} e^{-\rho V}$$

The **Cardinality Indicator function** ζ_i $(i \geq 0)$ for a causal set $\mathscr{C}$ and region $\mathscr{R}$ is defined as $\zeta_i(\mathscr{R}) \equiv \delta_{\text{Num}_{\mathscr{R}},i}$ which evaluates to 1 if there are i elements of $\mathscr{C}$ in $\mathscr{R}$, and evaluates to 0 otherwise. Note that $\langle \delta_{\text{Num}_{\mathscr{R}},i} \rangle = P_i(V_{\mathscr{R}})$

The **Occupation Indicator function** $\chi(\mathscr{R}) \equiv \sum_{i=1}^{\infty} \zeta_i(\mathscr{R}) = \begin{cases} 1 & \text{if } \text{Num}_{\mathscr{R}} > 0, \\ 0 & \text{otherwise.} \end{cases}$

$$\chi(\delta\mathscr{R}_x) \equiv \lim_{a\to 0} \chi(\Delta\mathscr{R}_x) = \lim_{a\to 0} \sum_{i=1}^{\infty} \frac{\delta_{\text{Num}_{\Delta\mathscr{R}_x},i}}{\Delta V_x} \Delta V_x$$

$V(\Delta\mathscr{R}_x) = a^d$

$$= \lim_{a\to 0} \frac{\delta_{\text{Num}_{\Delta\mathscr{R}_x},1}}{\Delta V_x} \Delta V_x = \sum_{z\in\mathscr{C}} \delta^{(d)}(z - x) dV_x$$

Note that $\langle \chi(\delta\mathscr{R}_x) \rangle = P_1(V_{\delta\mathscr{R}_x}) = \rho dV_x$. Can use $\chi(\delta\mathscr{R}_x)$ as an integration measure

$$\int_{\mathscr{R}} f(x)\chi(\delta\mathscr{R}_x) = \sum_{z\in\mathscr{C}} f(z)$$

arXiv:2407.03395

Correlations in $\langle f \rangle$ and $\Delta f = \sqrt{\langle f^2 \rangle - \langle f \rangle^2}$

$$P_N(V) = \frac{(\rho V)^N}{N!} e^{-\rho V}$$

We generically need to compute correlations

$$\langle \zeta_{i_1}(x_1, y_1) \cdots \zeta_{i_n}(x_n, y_n) \chi(\delta R_{x_1}) \chi(\delta R_{y_1}) \cdots \chi(\delta R_{x_n}) \chi(\delta R_{y_n}) \rangle$$

This is easy to compute if the functions are uncorrelated

$$\langle \zeta_{i_1}(x_1, y_1) \rangle \cdots \langle \zeta_{i_n}(x_n, y_n) \rangle \langle \chi(\delta R_{x_1}) \rangle \langle \chi(\delta R_{y_1}) \rangle \cdots \langle \chi(\delta R_{x_n}) \rangle \langle \chi(\delta R_{y_n}) \rangle$$

$$\langle \chi(\delta \mathscr{R}_x) \rangle = P_1(V_{\delta \mathscr{R}_x}) = \rho dV \qquad \langle \zeta_i(x_j, y_j) \rangle = P_i(V_{I(x_j, y_j)})$$

$\chi - \chi$ Correlations

For non-overlapping regions: $\langle \chi(\delta R_{x_1}) \cdots \chi(\delta R_{x_n}) \rangle = \langle \chi(\delta R_{x_1}) \rangle \cdots \langle \chi(\delta R_{x_n}) \rangle$

For coinciding points: $\langle \chi(\delta \mathscr{R}_x)^n \rangle = \langle \chi(\delta \mathscr{R}_x) \rangle \neq \langle \chi(\delta \mathscr{R}_x) \rangle^n$

arXiv:2407.03395

$\zeta - \zeta$ Correlations

$P_N(V) = \frac{(\rho V)^N}{N!} e^{-\rho V}$

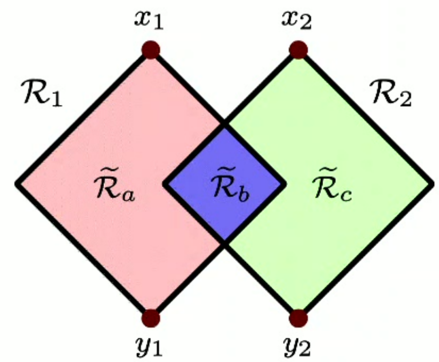
$\langle \zeta_{i_1}(x_1, y_1) \cdots \zeta_{i_n}(x_n, y_n) \rangle$ or the probability of having i_1 elements in $\mathscr{R}_1$, i_2 elements in $\mathscr{R}_2$, and so on. For non-overlapping regions:

$\langle \zeta_i(x_j, y_j) \rangle = P_i(V_{I(x_j, y_j)})$

$$\langle \zeta_{i_1}(x_1, y_1) \cdots \zeta_{i_n}(x_n, y_n) \rangle = \langle \zeta_{i_1}(x_1, y_1) \rangle \cdots \langle \zeta_{i_n}(x_n, y_n) \rangle$$

For overlapping regions, use **Disjoint Decomposition Property**:

$$\begin{aligned}
\langle \zeta_i(x_1, y_1) \zeta_j(x_2, y_2) \rangle &= \sum_{\substack{\alpha, \beta, \gamma \geq 0 \\ \alpha + \beta = i \\ \beta + \gamma = j}} \langle \zeta_\alpha(\widetilde{\mathscr{R}}_a)\, \zeta_\beta(\widetilde{\mathscr{R}}_b)\, \zeta_\gamma(\widetilde{\mathscr{R}}_c) \rangle \\
&= \sum_{\substack{\alpha, \beta, \gamma \geq 0 \\ \alpha + \beta = i \\ \beta + \gamma = j}} \langle \zeta_\alpha(\widetilde{\mathscr{R}}_a) \rangle \langle \zeta_\beta(\widetilde{\mathscr{R}}_b) \rangle \langle \zeta_\gamma(\widetilde{\mathscr{R}}_c) \rangle \\
&= \sum_{\substack{\alpha, \beta, \gamma \geq 0 \\ \alpha + \beta = i \\ \beta + \gamma = j}} P_\alpha(V_a) P_\beta(V_b) P_\gamma(V_c)\,.
\end{aligned}$$



$$\mathscr{R}_1 \cup \mathscr{R}_2 = \widetilde{\mathscr{R}}_a \sqcup \widetilde{\mathscr{R}}_b \sqcup \widetilde{\mathscr{R}}_c$$

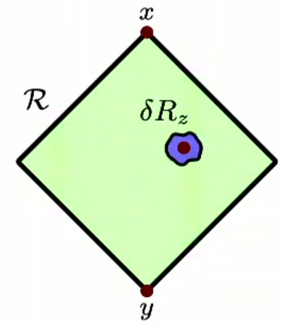
$\widetilde{\mathscr{R}}_a = \mathscr{R}_1 \backslash \widetilde{\mathscr{R}}_b$ $\quad \widetilde{\mathscr{R}}_b = \mathscr{R}_1 \cap \mathscr{R}_2$

$\widetilde{\mathscr{R}}_c = \mathscr{R}_2 \backslash \widetilde{\mathscr{R}}_b$ **arXiv:2407.03395**

$\zeta - \chi$ Correlations

$$P_N(V) = \frac{(\rho V)^N}{N!} e^{-\rho V}$$

$$\langle \chi(\delta \mathcal{R}_x) \rangle = P_1(V_{\delta \mathcal{R}_x}) = \rho dV \qquad \langle \zeta_i(x_j, y_j) \rangle = P_i(V_{I(x_j, y_j)})$$



$$\langle \zeta_i(x,y)\chi(\delta\mathcal{R}_z)\rangle = \begin{cases} \langle \zeta_i(x,y)\rangle\langle \chi(\delta\mathcal{R}_z)\rangle & \text{if } z \notin I(x,y), \\ \langle \zeta_{i-1}(x,y)\rangle\langle \chi(\delta\mathcal{R}_z)\rangle & \text{if } z \in I(x,y)\,. \end{cases}$$

Since $\chi(\delta\mathcal{R}_z) = \zeta_1(\delta\mathcal{R}_z)$, use **Disjoint Decomposition Property** again:

$$\begin{aligned} \langle \zeta_i(x,y)\chi(\delta\mathcal{R}_z)\rangle &= \langle \zeta_i(x,y)\zeta_1(\delta\mathcal{R}_z)\rangle \\ &= \sum_{\substack{\alpha,\beta,\gamma \geq 0 \\ \alpha+\beta=i \\ \beta+\gamma=1}} \langle \zeta_\alpha(\widetilde{\mathcal{R}}_a)\rangle\, \langle \zeta_\beta(\widetilde{\mathcal{R}}_b)\rangle\, \langle \zeta_\gamma(\widetilde{\mathcal{R}}_c)\rangle \xrightarrow{} 1 \\ &= \langle \zeta_{i-1}(\mathcal{R}\backslash\delta\mathcal{R}_z)\rangle P_1(V_{\delta\mathcal{R}_z}), \end{aligned}$$

$$\mathcal{R} \cup \delta\mathcal{R}_z = \widetilde{\mathcal{R}}_a \sqcup \widetilde{\mathcal{R}}_b \sqcup \widetilde{\mathcal{R}}_c$$

$$\widetilde{\mathcal{R}}_a = \mathcal{R}\backslash\delta\mathcal{R}_z$$

$$\widetilde{\mathcal{R}}_b = \delta\mathcal{R}_z$$

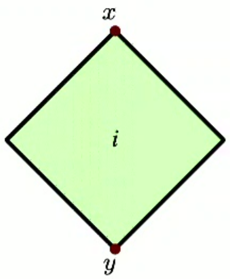
$$\widetilde{\mathcal{R}}_c = \varnothing$$

$$P_1(V_{\delta\mathcal{R}}) = \rho dV + O(\rho^2 dV^2) \qquad \langle \zeta_{i-1}(\mathcal{R}\backslash\delta\mathcal{R}_z)\rangle = P_{i-1}(V_{\mathcal{R}} - V_{\delta\mathcal{R}_z}) = P_{i-1}(V_{\mathcal{R}}) + O(\rho dV)$$

arXiv:2407.03395

Causal Set Action

$$R(x) = \sum_{i=0}^{n_d} C_i^{(d)} \sum_{y \in \Diamond_i(x)} 1 \qquad S = \sum_{x \in \mathscr{C}} \sum_{i=0}^{n_d} C_i^{(d)} \sum_{y \in \Diamond_i(x)} 1$$



$$(\Delta S)^2 = \langle S^2 \rangle - \langle S \rangle^2$$

$$\langle S \rangle = \sum_{i=0}^{n_d} C_i^{(d)} \langle \sum_{x \in \mathscr{C}} \sum_{y \in \Diamond_i(x)} \rangle = \sum_{i=0}^{n_d} C_i^{(d)} \mathscr{L}_{i-1}$$

$$\mathscr{L}_i \equiv \langle \sum_{x \in \mathscr{C}} \sum_{y \in \Diamond_{i+1}(x)} \rangle$$

$$i = -1, \ldots, n_d - 1$$

$$\mathscr{L}_i = \begin{cases} \int_{\mathscr{M}} \langle \chi(\delta R_x) \rangle & i = -1, \\ \int_{\mathscr{M}} \int_{\mathscr{J}^-(x)} \langle \zeta_i(x,y) \chi(\delta R_y) \chi(\delta R_x) \rangle & i \geq 0, \end{cases}$$

arXiv:gr-qc/0703099
arXiv:1403.1622
arXiv:1001.2725
arXiv:1305.2588
arXiv:2412.14036
arXiv:2407.03395

Causal Set Action

$$\langle S^2 \rangle = \sum_{i,j=0}^{n_d} C_i^{(d)} C_j^{(d)} \langle \sum_{x_1,x_2 \in \mathscr{C}} \sum_{y_1 \in \Diamond_i(x_1)} \sum_{y_2 \in \Diamond_j(x_2)} \rangle,$$

$$= \sum_{i,j=0}^{n_d} C_i^{(d)} C_j^{(d)} \mathscr{K}_{i-1\, j-1},$$

$$\mathscr{K}_{ij} \equiv \langle \sum_{x_1,x_2 \in \mathscr{C}} \sum_{y_1 \in \Diamond_{i+1}(x_1)} \sum_{y_2 \in \Diamond_{j+1}(x_2)} \rangle$$

$$\mathscr{K}_{ij} = \begin{cases} \int_{\mathscr{M}} \int_{\mathscr{M}} \langle \chi(\delta R_{x_1}) \chi(\delta R_{x_2}) \rangle & i = j = -1, \\ \int_{\mathscr{M}} \int_{\mathscr{M}} \int_{\mathscr{J}^-(x_2)} \langle \zeta_j(x_2, y_2)\, \chi(\delta R_{y_2}) \chi(\delta R_{x_1}) \chi(\delta R_{x_2}) \rangle & i = -1, j \geq 0, \\ \int_{\mathscr{M}} \int_{\mathscr{M}} \int_{\mathscr{J}^-(x_2)} \int_{\mathscr{J}^-(x_1)} \langle \zeta_i(x_1, y_1) \zeta_j(x_2, y_2)\, \chi(\delta R_{y_1}) \chi(\delta R_{y_2}) \chi(\delta R_{x_1}) \chi(\delta R_{x_2}) \rangle & i, j \geq 0 \end{cases}$$

Summary & Outlook

Statistical fluctuations in the causal set-continuum correspondence are important and interesting to study.

- **Tools and techniques have been developed to streamline these calculations.**
- **These studies are necessary in order to understand how close some causal set functions and their continuum analogues are. Deviations can lead to interesting phenomenology.**
- **Complementary numerical studies of these fluctuations can also be carried out:** $\langle f \rangle = \frac{1}{|\mathbf{Sp}[\mathscr{M}]|} \sum_{\mathscr{C} \in \mathbf{Sp}[\mathscr{M}]} f(\mathscr{C})$**, where** $\mathbf{Sp}[\mathscr{M}] = \{\mathscr{C}_1, \mathscr{C}_2, \ldots\}$**.**