Title: Quasi-Einstein equations and a Myers-Perry rigidity problem

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Abstract:

Quasi-Einstein equations are generalizations of the Einstein equation. They arise from warped product Einstein metrics (Kaluza-Klein reductions), Ricci solitons, cosmology, near-horizon geometries, and smooth measured Lorentzian length spaces. Despite their apparent generality, they often have a surprising rigidity. I will review some recent developments in the area, focusing on near-horizon geometries, including Dunajski and Lucietti's near-horizon version of the Hawking rigidity theorem. I will discuss an application to 5-dimensional extreme (Myers-Perry type) black holes whose horizons admit the structure of the group SU(2).

Quasi-Einstein equations: A Myers-Perry rigidity problem

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The Einstein equation

$$\mathrm{Ric}(g) = \lambda g$$

where $\mathrm{Ric}(g)$ is the Ricci curvature tensor of g, and

either

- g is a Riemannian metric,
- (M, g) is a Riemannian manifold,
- $\lambda \in \mathbb{R}$ is the Einstein constant, and
- the Einstein equation is a degenerate elliptic second-order PDE system for g.

or

- g is a Lorentzian metric,
- (M, g) is a spacetime manifold,
- $\lambda \in \mathbb{R}$ is the cosmological constant, and
- the Einstein equation is a degenerate hyperbolic second-order PDE system for g.

Quasi-Einstein metrics

Consider the triple (M, g, X), with g a Riemannian metrics and X a 1-form such that

$$\mathrm{Ric}(g) + \frac{1}{2}\pounds_{X^\sharp} g - \frac{1}{m} X \otimes X = \lambda g.$$

Here $X^\sharp = g^{-1}(X, \cdot)$ and $m \neq 0$ and λ are constants. In index notation:

$$R_{ij} + \frac{1}{2}(\nabla_i X_j + \nabla_j X_i) - \frac{1}{m} X_i X_j = \lambda g_{ij}.$$

- $m = 0$ denotes the Einstein case $\mathrm{Ric} = \lambda g$, $X \equiv 0$.
- The $m = \pm\infty$ case denotes Ricci solitons.
- $m =$ positive integer, $X = df$, get Ricci curvature restricted to the base of a warped product (e.g., Kaluza-Klein, etc)
- $m = 2$: Near horizon geometry equation for extreme black hole.

Killing vectors

- Continuous isometries: Curves in a manifold such that the manifold is unchanged under transport along the curves.
- Tangents to these curves are *Killing vectors*.
- The corresponding covectors obey Killing's equation $(\pounds_K g)_{ij} = \nabla_i K_j + \nabla_j K_i = 0$.
- Example: Lines (translation isometries) and circles (rotation isometries) in $\mathbb{R}^n$.

Wilhelm Karl Joseph Killing
1847–1923

Horizons in spacetime

Relevant types for today:

- Black hole event horizons,
- Killing horizons,
  - degenerate (extreme), and
  - nondegenerate (bifurcate),
- MOTSs and apparent horizons,

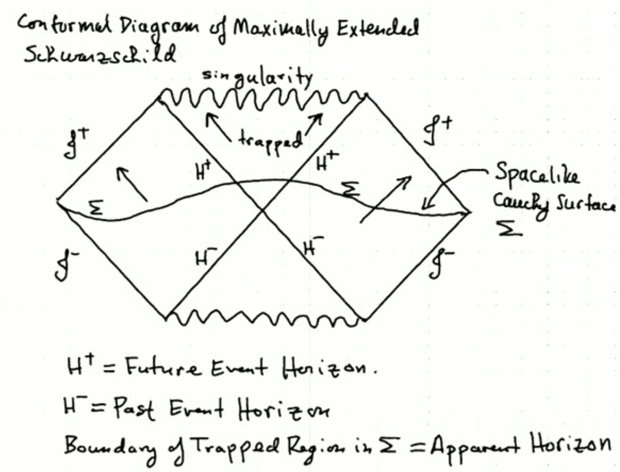
For a stationary black hole, event horizons are Killing horizons, and are foliated by apparent horizons, so we can just say *horizon*.

Others (not relevant today's talk):

- Cauchy horizons,
- Cosmological horizons (particle horizons),
- ...

Event horizons

- Consider a future-timelike curve γ that which has future-infinite Lorentzian length (proper time).
- Let $I^-(\gamma)$ denote its past (called a *TIP*).
- Take the union of all the $I^-(\gamma)$ over all such curves.
- The complement of the closure of this set is the *black hole*. The boundary is the *event horizon*.



Killing horizons I

- Killing's equation: $\nabla_i K_j + \nabla_j K_i = 0$.
- Then the flow of the *Killing vector field* K (whose components are $K^i = g^{ij} K_j$) is a family of isometries.
- Example: Schwarzschild. $\frac{\partial}{\partial t}$ is a Killing vector field, orthogonal to the hypersurfaces of constant t-coordinate. The isometry is that the Schwarzschild solution appears frozen in time.
- We say the Schwarzschild metric is *static*.
- The Kerr black hole has a Killing vector field that is *not* orthogonal to spacelike surfaces (and is timelike only near infinty). Its *twist* defines the (constant) rotation rate of Kerr.
- The Kerr black hole is not "frozen". It rotates, but it rotates at a constant rate so it has the same appearance at all times. It is called *stationary*.

Killing horizons II

- If there is a null hypersurface and a Killing vector field that is null on the hypersurface and timelike immediately outside it, the hypersurface is a *Killing horizon*.
- Fact: If K is the Killing vector field of a Killing horizon, then $\nabla_K K = \kappa K$ for a constant κ called the *surface gravity*.
- If $\kappa = 0$, the Killing horizon is *degenerate* and the black hole is *extreme*.
- The extreme Kerr metric is a black hole that is spinning as fast as possible: if it spins any faster, the horizon disappears and the singularity inside the black hole becomes visible.

Degenerate Killing horizon: NHG equation

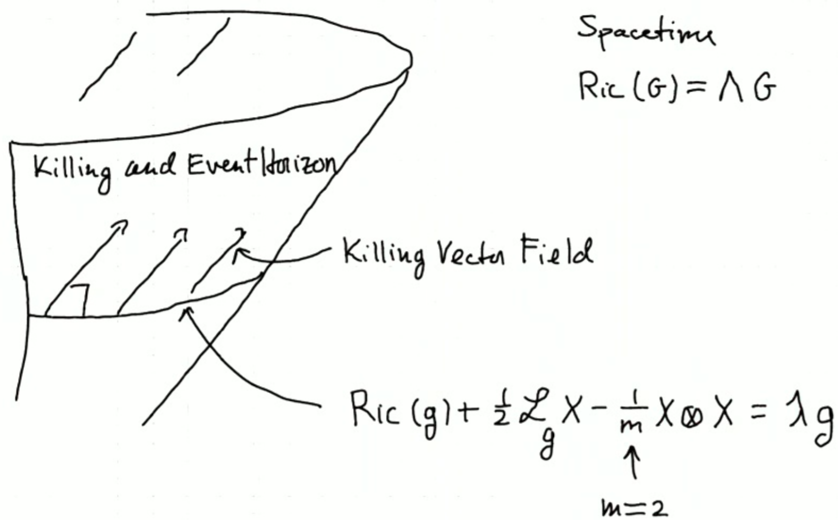


Figure: Degenerate horizon $\kappa = 0$: Cross-sections obey a quasi-Einstein equation with $m = 2$.

Kerr NHG: Non-closed X

The Kerr NHG has $n = m = 2$ and

$$ds^2 = \left(1 + \cos^2\theta\right) d\theta^2 + \frac{4\sin^2\theta}{(1+\cos^2\theta)} d\phi^2,$$

$$X = \frac{2\sin\theta\cos\theta}{(1+\cos^2\theta)} d\theta - \frac{4\sin^2\theta}{(1+\cos^2\theta)^2} d\phi.$$

Notice that $dX \neq 0$, so X is not a closed 1-form.

- Lucietti-Kunduri: $\exists$ an analogue of Kerr for each $m > 0$, $\lambda = 0$.
- For $m = 2$ and $\lambda > 0$ ($\lambda < 0$), there are the Kerr de Sitter (Kerr anti-de Sitter) NHGs.

Questions:

- Is Kerr the only $\lambda = 0$, $m = 2$ quasi-Einstein metric on $\mathbb{S}^2$?
- Is the AdS-Kerr NHG the only $\lambda < 0$ quasi-Einstein metric on $\mathbb{S}^2$?

Rigidty of the closed case of the NHG equation

- Take X to be a closed 1-form: $0 = dX$.
- The closed case with $m = 2$ is the near horizon geometry equation $\mathrm{Ric} + \frac{1}{2}\pounds_X g - \frac{1}{m} X \otimes X = \lambda g$ arising from a *static* extreme black hole.

Theorem (Bahuaud-Gunasekharan-Kunduri-EW 2023a; Wylie 2023)

Let (M, g) be a compact quasi-Einstein manifold with $m > 0$ and $dX = 0$. Then X is a Killing vector field. Indeed:

- *X is parallel (i.e., $\nabla X = 0$), and*
- *either $X = 0$ and then (M, g) is an Einstein manifold, or (M, g) is the product of a negative Einstein manifold and a circle $\mathbb{S}^1$.*

Rigidity of the incompressible case of the NHG equation

- Now take $0 = \delta X \equiv \operatorname{div} X$ and let M be a closed manifold.

Theorem (Bahuaud-Gunasekharan-Kunduri-EW 2023b)

If $\operatorname{Ric} + \frac{1}{2}\pounds_X g - \frac{1}{m} X \otimes X = \lambda g$ *and* $\operatorname{div} X = 0$ *then* $\pounds_X g = 0$, $|X| = const$, *and* $\nabla_X X = 0$. *That is, either* (M, g) *is Einstein or it possesses a global Killing vector field whose integral curves are complete geodesics.*

Consequences: If $X \neq 0$ then

- Incompressibility implies symmetry!
- M must have Euler characteristic 0 (since $|X| = const$).
  - Rules out $\mathbb{S}^{2n}$.
- QE equation reduces to $\operatorname{Ric} = \lambda g + \frac{1}{m} X \otimes X$.
- Ricci has exactly two distinct eigenvalues, multiplicies 1 and $n-1$ respectively, and the former is nonnegative (by the Bochner identity: $\operatorname{Ric} < 0$ implies there are no global isometries).

Hawking rigidity

- A stationary spacetime is one with a Killing vector field that is *timelike near infinity*.
- It must be null or spacelike on any black hole event horizon. If it is null, the event horizon is a Killing horizon.
- But it may become spacelike between infinity and the event horizon, if a black hole is present.
- Then spacetime has an *ergosphere region* (e.g., Kerr has one).
- A theorem of Hawking then establishes that spacetimes with an ergosphere region immediately outside the event horizon have a second Killing vector field that is null on the event horizon and timelike immediately outside the horizon.
- Hence the event horizon in a stationary spacetime is a Killing horizon.
- Then spacetime is axisymmetric (if analytic, or if the KVF is small: Alexakis, Ionescu, Klainerman; GAFA 2012).

Rigidity theorem whenever $m = 2$

Theorem (Dunajski-Lucietti)

Let (M, g) be a compact quasi-Einstein manifold with $m = 2$. If the one-form X in the quasi-Einstein equation is not closed, there exists a nontrivial solution of Killing's equation on (M, g).

- If X is closed (so $dX = 0$) but nonzero, the near horizon geometry is static and X itself is Killing (Bahuaud-Gunasekaran-Kunduri-EW 2022a; Wylie 2023).
- Corollary: For $n = 2$, the unique solution of the quasi-Einstein equation on $\mathbb{S}^2$ is the Kerr near horizon geometry.
- This can be viewed as Hawking rigidity of the near horizon geometry for extreme black holes.
- Remarkably, only works for $m = 2$ in $\text{Ric} + \frac{1}{2}\pounds_X g - \frac{1}{m} X \otimes X = \lambda g$.

Higher-dimensional black holes

- Frank Tangherlini 1963 extended the Schwarzschild static black hole solution to spacetimes of dimension ≥ 5.
- He was motivated by an anthropic principle question: could bound orbits about stars exist if spacetime were not 4-dimensional?
- Does the effective potential for geodesic motion about a Schwarzschild black hole admit bound orbits?
- In Newtonian theory this question had been asked by Ehrenfest (1917, 1920), building on ideas of Paley (1802).
- In his PhD thesis directed by Malcolm Perry, Robert Myers found analogues of the Kerr rotating black holes in all higher dimensions: Myers-Perry 1986.
- Later extended to black holes in the background of a nonzero cosmological constant (Gibbons-Lu-Page-Pope for $\Lambda > 0$).

Myers-Perry black holes in 5 spacetime dimensions

- 5-dimensional Myers-Perry black holes have (Killing and event) horizons whose spatial cross-sections are 3-spheres.
- The 3-sphere is the manifold of the Lie group SU(2).
- Some extreme (and non-extreme) Myers-Perry black hole horizons carry a left-invariant SU(2) metric.
- Main point: SU(2) acts on itself, so left-invariant SU(2) metrics admit an isometry group of at least 3-dimensions.
- The isometry group of a left-invariant SU(2) metric can be 3-dimensional (generic case), 4-dimensional (Berger spheres), or 6-dimensional (round sphere).
- Myers-Perry SU(2) horizons always have 4-dimensional isometry group.
- Puzzle: Why does the generic case not occur?

Quasi-Einstein metrics on SU(2)

- Alice Lim (2022) studied left-invariant quasi-Einstein metrics on 3-dimensional Lie groups.
- Milnor (1976) famously studied left-invariant Einstein metrics on 3-dimensional Lie groups.
- The answer to our question can be found in
  - Lim 2022 (though a key lemma has an erroneous proof), and
  - Chen-Liang-Zhu 2016 (employing variational arguments, which we can avoid).
- Along the way, we will encounter a 3-dimensional manifold whose Ricci tensor *does not determine* its geometry:
  - See Kulkarni, Annals of Math 1970,
  - Berger, *A Panoramic View ...* (2003) pp 213–216.

NHG Equation for degenerate Killing horizon

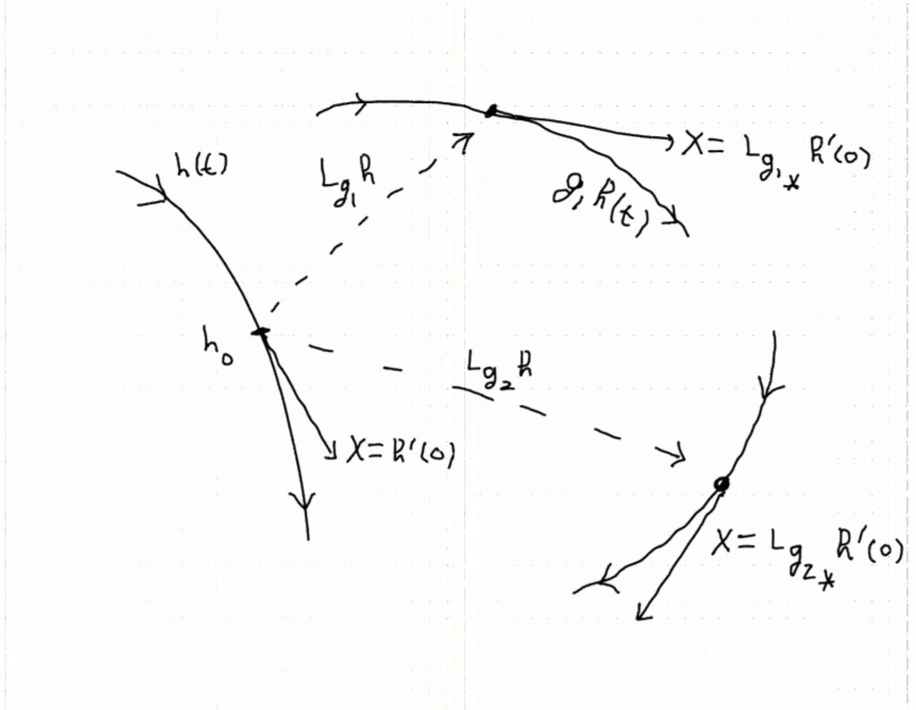


Figure: Building a left-invariant vector field on a group

SU(2) metrics and Berger spheres

- Generic left-invariant SU(2) metric:

$$ds^2 = \varepsilon^2\sigma^1 \otimes \sigma^1 + \beta^2\sigma^2 \otimes \sigma^2 + \sigma^3 \otimes \sigma^3,$$

  where ε and β are numbers between 0 and 1. The left-invariant 1-forms σ^i obey $[\sigma^1, \sigma^2] = 2\sigma^3$, and cyclic (e.g., Pauli matrices).
- The maximal isometry group is 3-dimensional.
- Berger spheres have $\varepsilon = \beta$, yielding an additional U(1) isometry. Maximal isometry group contains SU(2) $\times$ U(1) and is 4-dimensional.
- The Ricci endomorphism of the generic SU(2) metric has eigenvalues

$$\rho_1 = 2\frac{(\varepsilon^4 - (1-\beta^2)^2)}{\varepsilon^2\beta^2},$$

$$\rho_2 = 2\frac{(\beta^4 - (1-\varepsilon^2)^2)}{\varepsilon^2\beta^2},$$

$$\rho_3 = 2\frac{(1 - (\varepsilon^2-\beta^2)^2)}{\varepsilon^2\beta^2}.$$

SU(2) metrics with the same Ricci endomorphism I

If two Ricci eigenvalues are equal, WLOG set $\rho_1 = \rho_2$.

$$\begin{aligned} & \rho_1 = \rho_2 \\ \Longrightarrow\ & \varepsilon^4 - \left(1-\beta^2\right)^2 = \beta^4 - \left(1-\varepsilon^2\right)^2 \\ \Longrightarrow\ & \varepsilon^4 - \beta^4 - 1 + 2\beta^2 = \beta^4 - \varepsilon^4 - 1 + 2\varepsilon^2 \\ \Longrightarrow\ & \varepsilon^4 - \beta^4 + \beta^2 - \varepsilon^2 = 0 \\ \Longrightarrow\ & \left(\varepsilon^2 - \beta^2\right)\left(\varepsilon^2 + \beta^2 - 1\right) = 0. \end{aligned}$$

Since $\varepsilon, \beta \in (0,1)$, then either

$$\varepsilon = \beta$$

or

$$\varepsilon^2 + \beta^2 = 1.$$

SU(2) metrics with the same Ricci endomorphism II

- If $\varepsilon^2+\beta^2=1$, set $\beta=\sin\theta$, $\varepsilon=\cos\theta$, $\theta\in\left(0,\frac{\pi}{2}\right)$. Then

$$ds^2=\cos^2\theta\sigma^1\otimes\sigma^1+\sin^2\theta\sigma^2\otimes\sigma^2+\sigma^3\otimes\sigma^3.$$

- The Ricci eigenvalues
  - $\rho_1=2\frac{\left(\varepsilon^4-(1-\beta^2)^2\right)}{\varepsilon^2\beta^2}$,
  - $\rho_2=2\frac{\left(\beta^4-(1-\varepsilon^2)^2\right)}{\varepsilon^2\beta^2}$,
  - $\rho_3=2\frac{\left(1-(\varepsilon^2-\beta^2)^2\right)}{\varepsilon^2\beta^2}$.

  become

$$\boxed{\rho_1=\rho_2=0,\ \rho_3=8,}$$

  for any θ. For quasi-Einstein metrics, this arises only when $\lambda=0$.
- Only $\theta=\frac{\pi}{4}$ is a Berger sphere: $ds^2=\frac{1}{2}\sigma^1\otimes\sigma^1+\frac{1}{2}\sigma^2\otimes\sigma^2+\sigma^3\otimes\sigma^3$.

Extreme Myers-Perry near horizon geometries

- The horizon metric obeys $\mathrm{Ric} + \frac{1}{2}\pounds_X g - \frac{1}{m} X \otimes X = \lambda g$.
- Since g is left-invariant and Ric is natural, Ric is left-invariant.
- Then $\frac{1}{2}\pounds_X g - \frac{1}{m} X \otimes X$ is left-invariant.
- But then X is left-invariant (Lim; Chen-Liang-Zhu; BGKW2024).
- Fact: The divergence of a left-invariant vector field must vanish.
- Then BGKW2023b implies that $\pounds_X g = 0$.
- Then $\mathrm{Ric} = \lambda g + \frac{1}{m} X \otimes X$.
- Hence λ is a multiplicity-two eigenvalue of Ric and $\lambda + \frac{|X|^2}{m}$ is a multiplicity-one eigenvalue.
- From the eigenvalue formulas with $\rho_1 = \rho_2$, get either
  - $\beta = \varepsilon$ (Berger sphere), or
  - $\beta^2 = 1 - \varepsilon^2$ but then Killing equation implies that $\beta = \varepsilon = \frac{1}{\sqrt{2}}$.

Theorem

We have proved the following:

Theorem

Let (M, g) be the near horizon geometry of an extreme Myers-Perry black hole, with arbitrary cosmological constant. If g is a left-invariant metric on M, it's a Berger sphere.

- Furthermore, the Ricci endomorphism has the following signature:
  - $\lambda < 0 \implies (-,-,0)$ or $(-,-,+)$,
  - $\lambda = 0 \implies (0,0,+)$,
  - $\lambda > 0 \implies (+,+,+)$.
- As things stand, the theorem does not apply to non-extreme Myers-Perry black holes.