Title: Noninvertible Gauge Symmetry in (2+1)d Topological Orders: A String-Net Model Realization

Speakers: Yidun Wan

Collection/Series: Quantum Fields and Strings

Subject: Quantum Fields and Strings

Date: February 11, 2025 - 2:00 PM

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Abstract:

In this talk, we develop a systematic framework for understanding symmetries in topological phases in \(2+1\) dimensions using the string-net model, encompassing both gauge symmetries that preserve anyon types and global symmetries permuting anyon types, including both invertible symmetries describable by groups and noninvertible symmetries described by categories. As an archetypal example, we reveal the first noninvertible categorical gauge symmetry of topological orders in \(2+1\) dimensions: the Fibonacci gauge symmetry of the doubled Fibonacci topological order, described by the Fibonacci fusion \(2\)-category. Our approach involves two steps: first, establishing duality between different string-net models with Morita equivalent input UFCs that describe the same topological order; and second, constructing symmetry transformations within the same string-net model when the dual models have isomorphic input UFCs, achieved by composing duality maps with isomorphisms of degrees of freedom between the dual models.

GAUGE SYMMETRY OF TOPOLOGICAL ORDERS IN 2+1 DIMENSIONS: A STRING-NET MODEL REALIZATION

based on *arXiv:2408.02664*, *under review in PRL*

Yidun Wan

PI QFS Seminar
Feb 11, 2025

Yu Zhao

He's looking for a postdoc position

COLLABORATOR

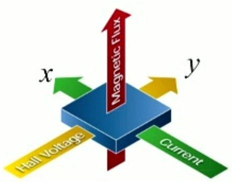
DISCLAIMER

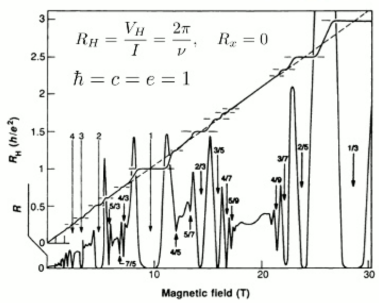
Gauge symmetry is a misnomer; it is a redundancy, not a symmetry.

But we compromise and still use the term gauge symmetry.

TOPOLOGICALLY ORDERED MATTER PHASES

Fractional quantum Hall states





$$\nu = \frac{1}{3}, \frac{2}{3}, \frac{2}{5}, \frac{2}{7}, \frac{3}{7}, \dots$$

Very particular fractions

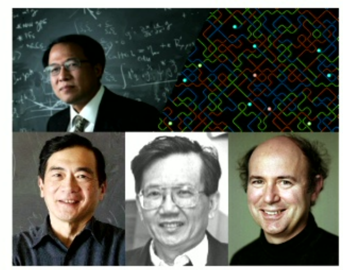
Distinct FQHS share the same symmetry

↓

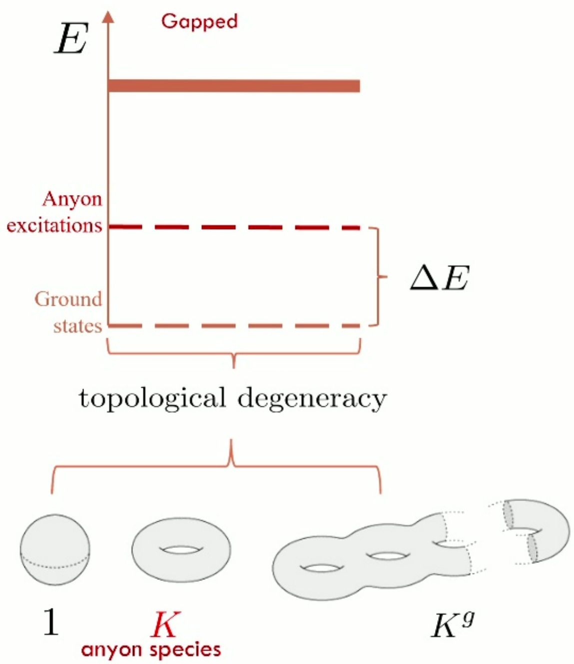
Landau-Ginzburg symmetry breaking fails

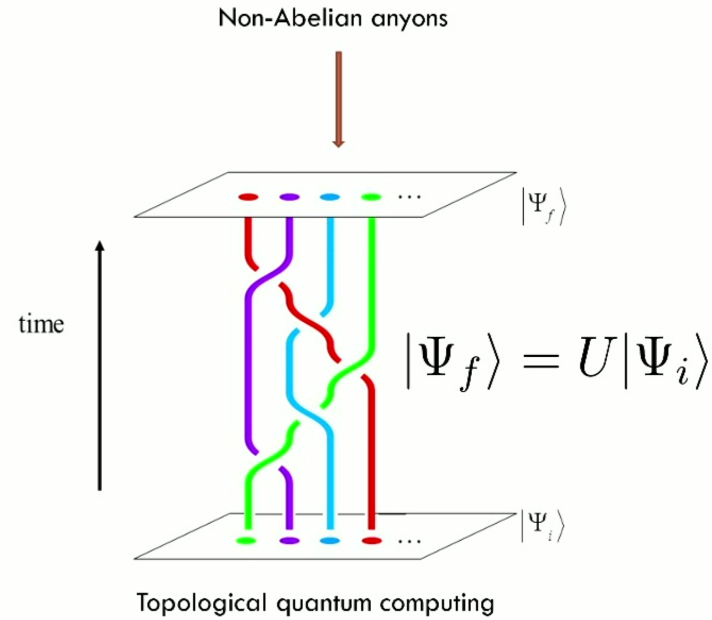
↓

Topological orders



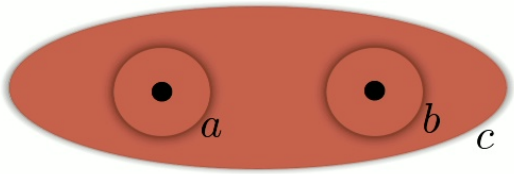
TOPOLOGICAL ORDERS IN 2D





TOPOLOGICAL DATA OF TOPOLOGICAL ORDER $\mathcal{C}$

Fusion



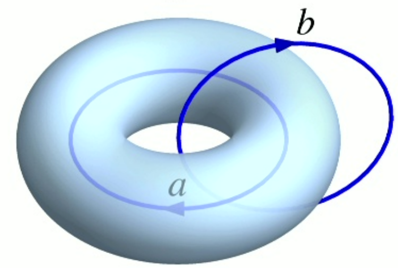
$$a \times b = \sum_c \left(N^{\mathcal{C}}\right)_{ab}^{c} c$$

Natural numbers

Verlinde formula $$\left(N^{\mathcal{C}}\right)_{ab}^{c} = \sum_{e \in \mathcal{L}_{\mathcal{C}}} \frac{S_{ae}^{\mathcal{C}} S_{be}^{\mathcal{C}} \left(S^{\mathcal{C}}\right)_{ec}^{-1}}{S_{1e}^{\mathcal{C}}}$$

Topological data captured by a unitary modular tensor category (UMTC) $\mathcal{C}$

Braiding: S matrix



$$\frac{S_{ab}^{\mathcal{C}}}{S_{11}^{\mathcal{C}}}$$

Quantum dimension $$d_a := \frac{S_{a1}^{\mathcal{C}}}{S_{11}^{\mathcal{C}}}$$

The asymptotic dimension of a in the thermodynamical limit

THEORETICAL LANGUAGE OF TOPOLOGICAL ORDERS

- Topological data is given by the low energy effective theory of topological orders: topological gauge field theory (TGFT)

  but the gauge structure is often vague

- Dijkgraaf-Witten theory & its lattice Hamiltonian model---twisted quantum double model

  gauge symmetry is described by a gauge group

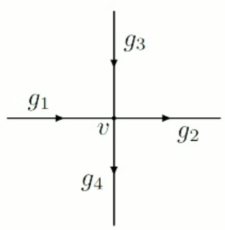
- Turaev-Viro theory & its lattice Hamiltonian model---string-net model

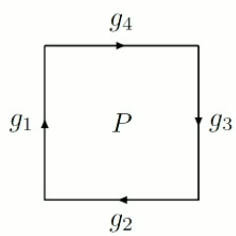
  gauge symmetry is often unclear, likely not describable by groups

(TWISTED) QUANTUM DOUBLE

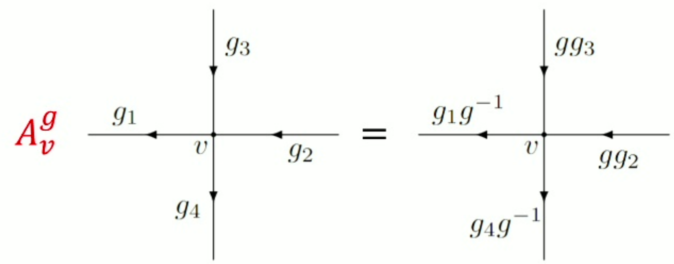
PRB 87, 125114 (2013), Yidun Wan, et al

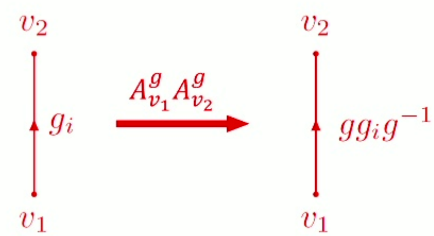
$$H = -\sum_{\text{Vertices } v} A_v - \sum_{\text{Plaquettes } P} B_P$$





$$A_v = \frac{1}{|G|}\sum_{g \in G} A_v^g$$



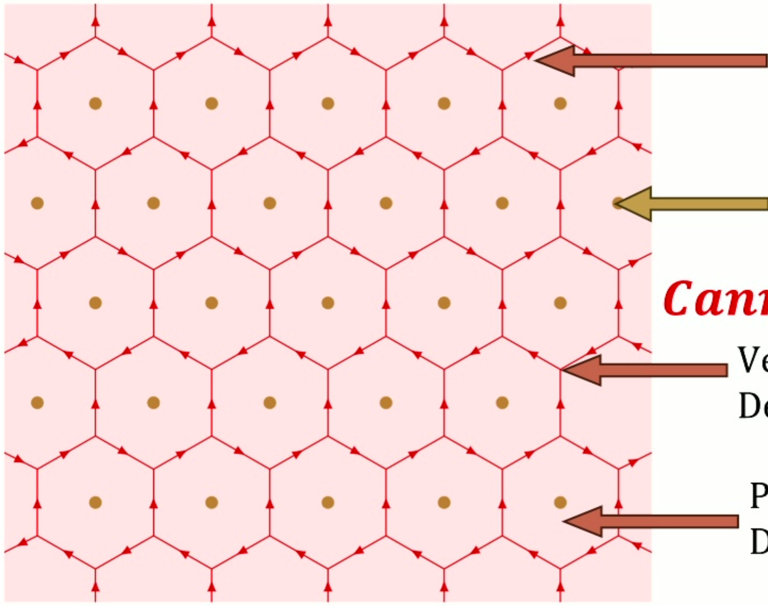


$$A_v^g\!: g_i \to g g_i g^{-1}$$

A_v^g: *Gauge Transformation*!

$\mathrm{Inn}(G)$: *Gauge Group*!

THE STRING-NET MODEL, BUT OUR VERSION



Edges: Carrying Basic Dofs
Dofs ∈ Simple objects of $\mathcal{F}$, gauge field dof

Charges: Carrying Basic Dofs

Cannot fully describe charge excitations!

Vertices: A_v Operators Detecting Charge Existence

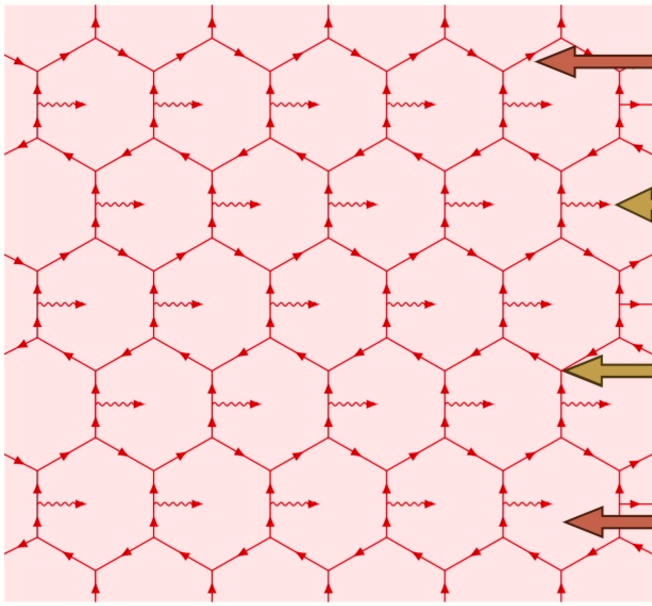
Plaquettes: B_P Operators Detecting Flux Existence

Original String-Net Model; Input Fusion Category $\mathcal{F}$

$$H = -\sum_{\text{Vertices } V} A_V - \sum_{\text{Plaquettes } P} B_P$$

Sum of Commuting Projectors! Exactly Solvable!

THE STRING-NET MODEL, BUT OUR VERSION



Edges: Carrying Basic Dofs
Dofs $\in$ Simple objects of $\mathcal{F}$, gauge field dof

Tails: Carrying Basic Dofs
How charges and gauge fields couple

Vertices: In Hilbert space, $A_V = 1$ Always Satisfied

Plaquettes: B_P Operators
Detecting Flux Existence

String-Net Model, Our Version; Input Fusion Category $\mathcal{F}$

$$H = -\sum_{\text{Vertices } V} A_V - \sum_{\text{Plaquettes } P} B_P$$

***Sum of Commuting Projectors*!**
***Exactly Solvable*!**

THE STRING-NET MODEL: EXCITATIONS

Ground States: $Q_P = 1$ for all P

Excited States: $Q_P = 0$ *for certain* P

Anyon $J \in \mathrm{Irr}\big(\mathcal{Z}(\mathcal{F})\big)$ in P, where $Q_P = 0$.

Creating anyons by $W_E^{J;pq}$

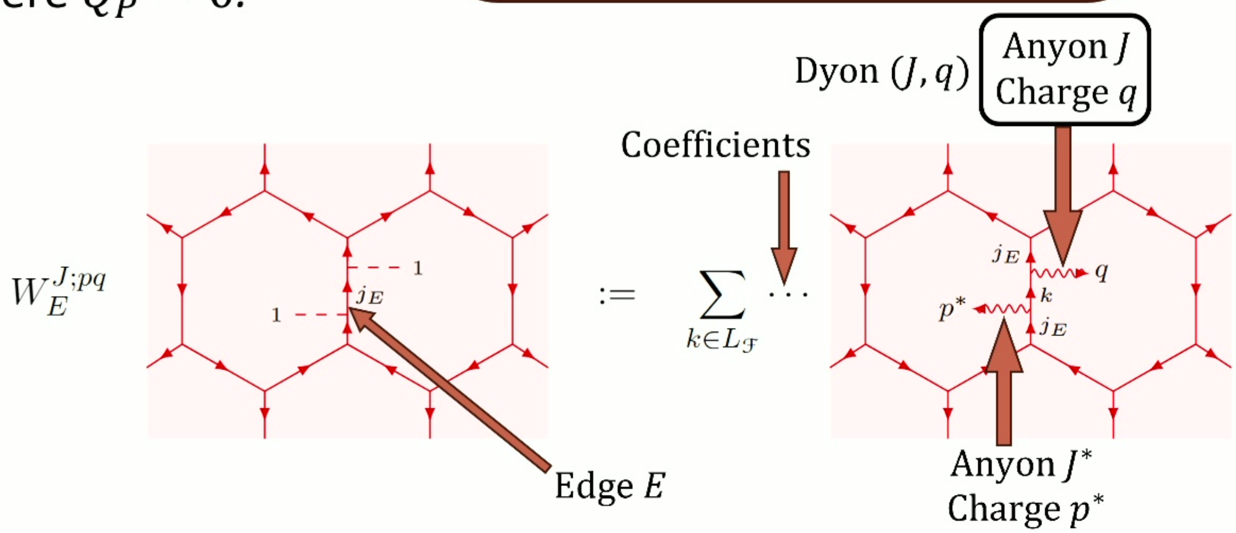
Represent *anyons* as *dyons*

Manifest *Internal Spaces*

- Example: Fibonacci Model
- Input UFC Fibo: $\mathrm{Irr} = \{1, \tau\}$
- Anyon Types: $\mathrm{Irr}(\mathcal{Z}(\mathrm{Fibo}))$:

  $1\bar{1}, \quad 1\bar{\tau}, \quad \tau\bar{1}, \quad \tau\bar{\tau}$
- Dyon Type:

  $(1\bar{1}, 1), (\tau\bar{1}, \tau), (1\bar{\tau}, \tau), (\tau\bar{\tau}, 1), (\tau\bar{\tau}, \tau)$



DUAL STRING-NET MODEL

Do two apparently different input UFC $\mathcal{F}$ and $\mathcal{F}'$ always yield different topological orders?

No. If $\mathcal{F}$ and $\mathcal{F}'$ are categorical Morita equivalent, i.e., $\mathcal{Z}(\mathcal{F}) \cong \mathcal{Z}(\mathcal{F}')$, they yield the same topological order

DUAL STRING-NET MODELS: FROBENIUS ALGEBRA

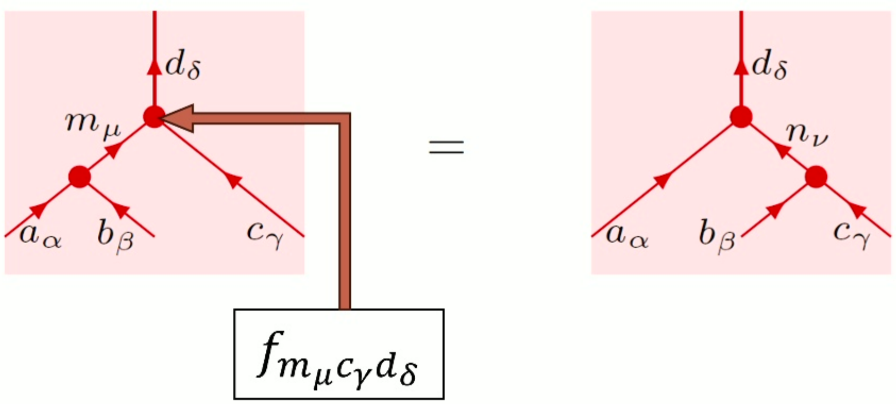
$\mathcal{Z}(\mathcal{F}) \cong \mathcal{Z}(\mathcal{F}') \quad \Leftrightarrow \quad \mathcal{F}' = \mathrm{Bimod}_{\mathcal{F}}(\mathcal{A}) \subseteq \mathcal{F}$

Etingof, et al. Tensor categories. Vol. 205. American Mathematical Soc., 2015.

Frobenius algebra: $\mathcal{A} = (L_{\mathcal{A}}, f)$:

$$L_{\mathcal{A}} = \{a_\alpha | a \in \mathrm{Irr}(\mathcal{F}), 1 \leq \alpha \leq n_a^{\mathcal{A}}\},$$

$$f: L_{\mathcal{A}}^3 \to \mathbb{C}.$$



Example: Fibonacci UFC:

$$L_{\mathcal{A}} = \{1, \tau\},$$

$$f_{111} = f_{11\tau} = f_{1\tau 1} = f_{\tau\tau 1} = 1,$$

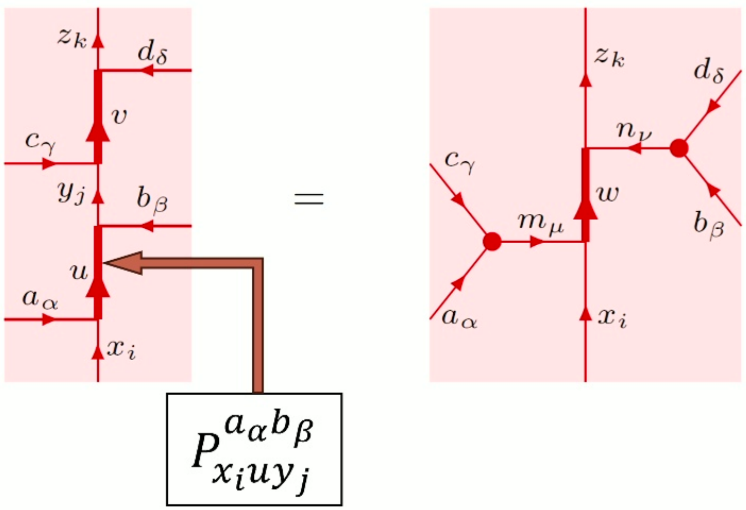
$$f_{\tau\tau\tau} = \phi^{-\frac{3}{4}}.$$

DUAL STRING-NET MODELS: BIMODULES

Bimodules: $M = (L_M, P)$:

$$L_M = \{x_i | x \in \mathrm{Irr}(\mathcal{F}), 1 \le i \le n_x^M\},$$

$$P: L_{\mathcal{A}}^2 \otimes L_M \otimes L_{\mathcal{F}} \otimes L_M \to \mathbb{C}.$$



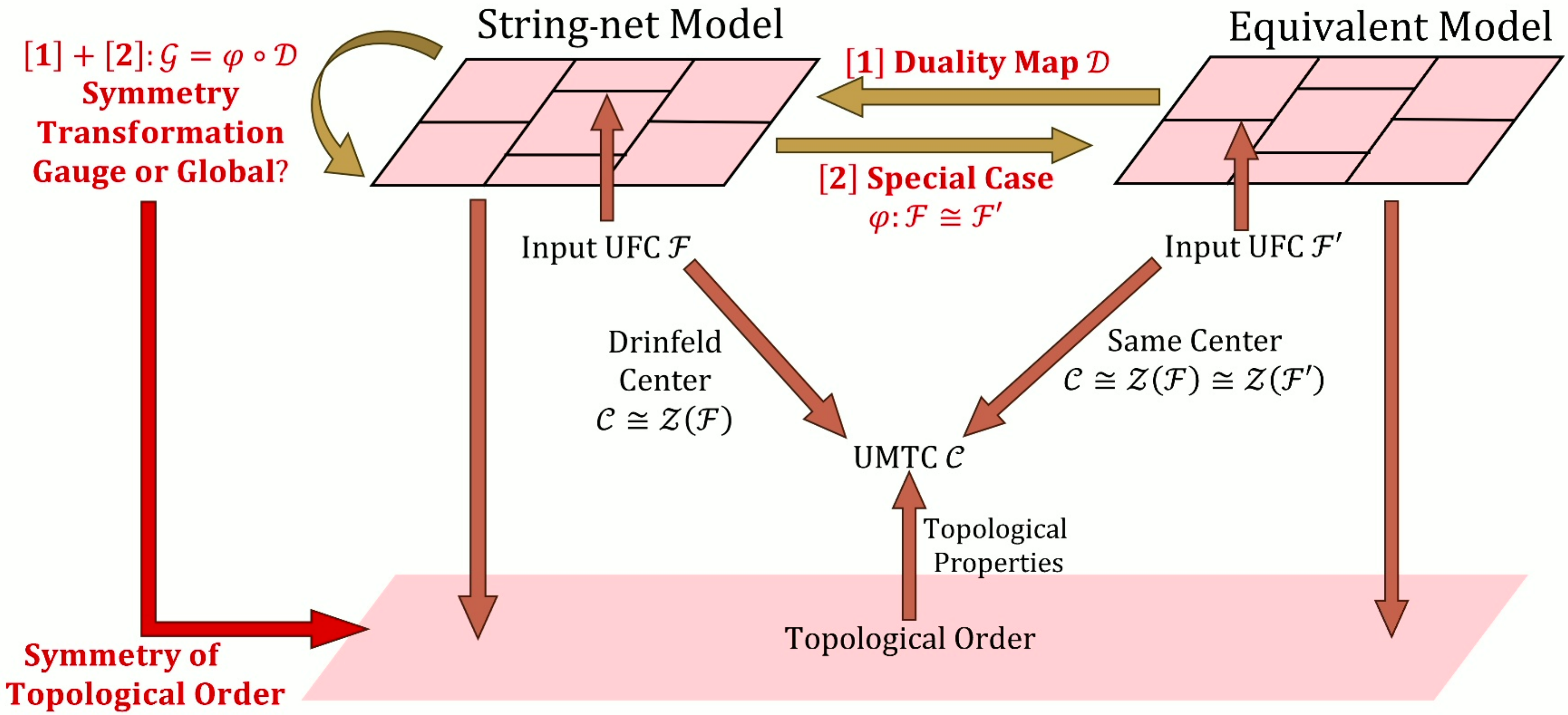
Example: Fibonacci UFC:

$L_{\mathcal{A}} = \{1, \tau\}$

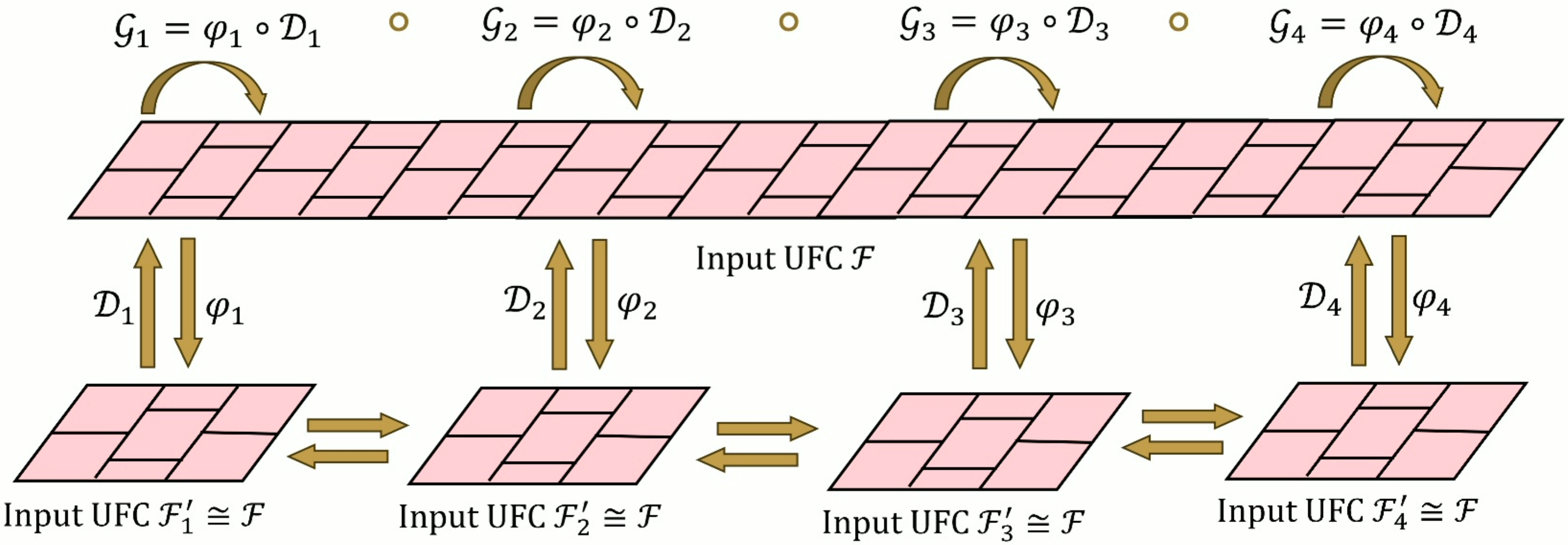
Two simple Bimodules over $\mathcal{A}$:

- Trivial Bimodule M_1:
  $L_{M_1} = \{1, \tau\}$
- Nontrivial Bimodule M_τ:
  $L_{M_\tau} = \{1, \tau_1, \tau_2\}$

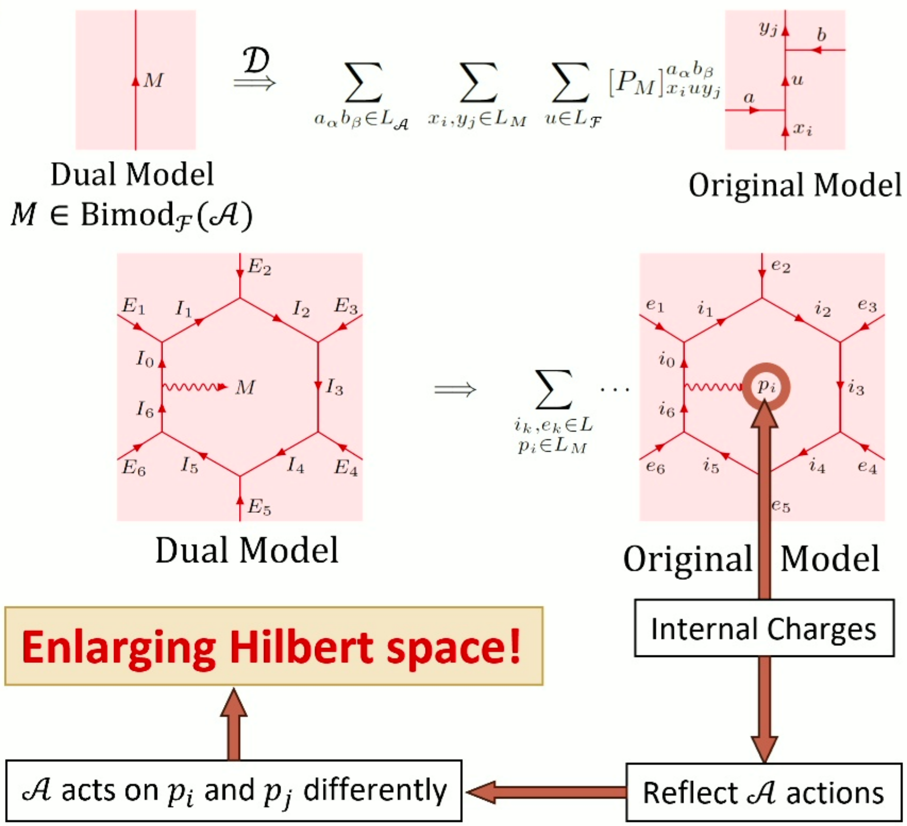
CONSTRUCTING SYMMETRY TRANSFORMATIONS: THE IDEA



CONSTRUCTING SYMMETRY TRANSFORMATIONS: THE IDEA



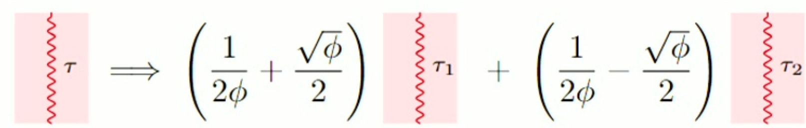
DUALITY MAPS: CONSTRUCTION AND HILBERT SPACE ENLARGEMENT

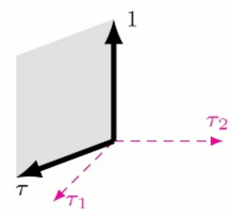


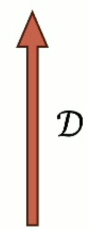
Example: Fibonacci UFC:

$L_{M_1} = \{1, \tau\}, \qquad L_{M_\tau} = \{1, \tau_1, \tau_2\}$

Tail Dof Enlarging: $\{1, \tau\} \rightarrow \{1, \tau_1, \tau_2\}$







Orthonormality:

$\langle M_1 | M_\tau \rangle = 0$ in dual model

WHY ENLARGING HILBERT SPACE

Electromagnetic Field

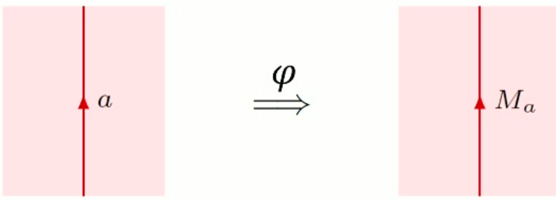
- Field configurations: A_μ
- Physical field dof: 2 transverse polarizations
- Gauge Transformation: $A_\mu \to A_\mu + \mathrm{d}\alpha_\mu$
- Enlarged dof: Longitudinal Modes: 2 → 3
- The longitudinal modes of massless photons are reduced by gauge fixing

Fibonacci String-Net Model

- Field Configurations: j_E, label on edge E
- Physical field dof: $1, \tau$
- Gauge Transformations: $1 \to M_1, \tau \to M_\tau$
- Enlarged dof: $\{1, \tau\} \to \{1, \tau, M_\tau\} = \{1, \tau_1, \tau_2\}$
- The enlarged dofs $\{\tau_1, \tau_2\}$ are reduced to $\{\tau\}$ by gauge fixing

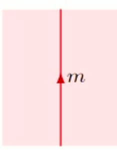
SYMMETRY TRANSFORMATION

When $\mathcal{F} \cong \mathrm{Bimod}_{\mathcal{F}}(\mathcal{A}), \exists \varphi$:

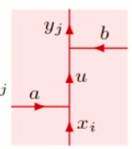


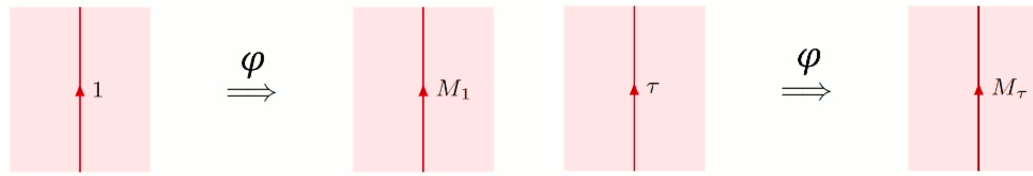
Original Model Equivalent Model

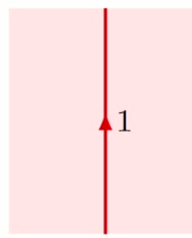
$\mathcal{G} = \varphi \circ \mathcal{D}$: Original $\rightarrow$ Original



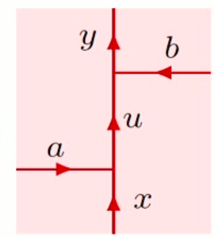
$\xrightarrow{\mathcal{G}} \sum_{a_\alpha b_\beta \in L_{\mathcal{A}}} \sum_{x_i, y_j \in L_{M_m}} \sum_{u \in L_{\mathcal{F}}} [P_{M_m}]^{a_\alpha b_\beta}_{x_i u y_j}$

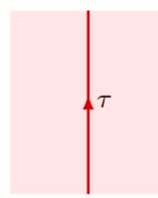




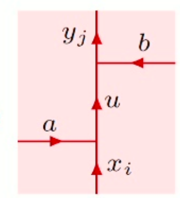


$\Longrightarrow \sum_{a,b,u,x,y \in \{1,\tau\}} [P_{M_1}]^{ab}_{xuy}$

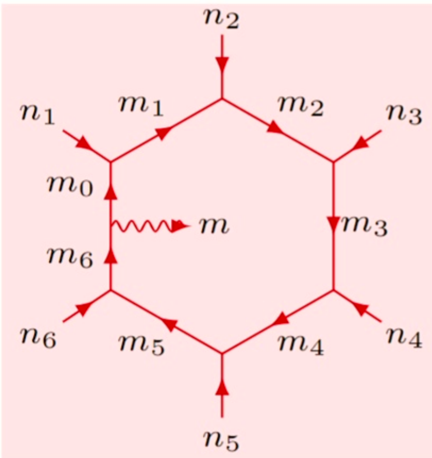




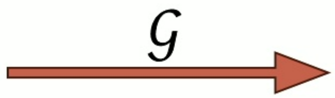
$\Longrightarrow \sum_{a,b,u \in \{1,\tau\}} \sum_{x_i, y_j \in \{1,\tau_1,\tau_2\}} [P_{M_\tau}]^{ab}_{x_i u y_j}$

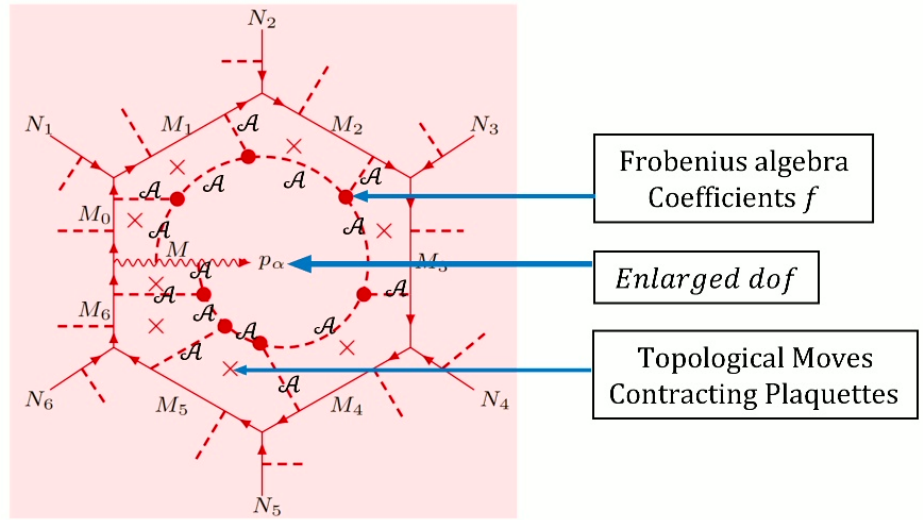


SYMMETRY TRANSFORMATION ON PLAQUETTES



Original Model: $m_i, n_i, m \in L_\mathcal{F}$



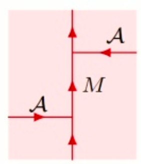


Original Model: Dof ∈ L_ℱ

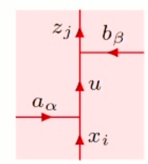
Isomorphism:

$L_\mathcal{F} \to L_{\mathrm{Bimod}_\mathcal{F}(\mathcal{A})}$

$m_i \mapsto M_i, n_i \mapsto N_i, m \mapsto M$



$$:= \sum_{a_\alpha, b_\beta \in L_\mathcal{A}} \sum_{u \in L_\mathcal{F}} \sum_{x_i, y_j \in L_M} [P_M]^{a_\alpha b_\beta}_{x_i u y_j}$$



GAUGE TRANSFORMATION

Gauge Symmetry $\Longleftrightarrow$

$\mathcal{G}$ preserves anyon types &

$\mathcal{G}$ transforms internal charges

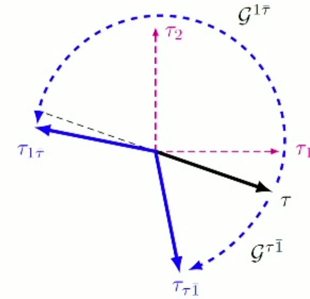
Global Symmetry (Otherwise)

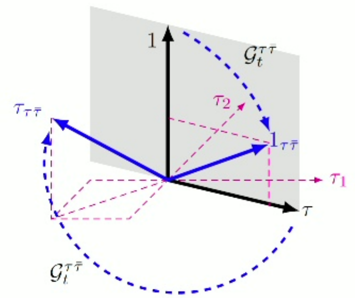
Example: Fibonacci UFC:

Gauge Symmetry

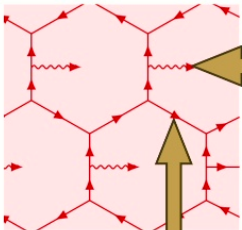
$$\mathcal{G} = \sum_{\text{Anyon Type } J} \mathcal{G}^J \Pi_J$$

$$\mathcal{G}^{1\bar{1}} = \mathbb{I}$$





GAUGE TRANSFORMATION OF FIBONACCI GAUGE FIELD



Tail

Dofs: $1, \tau_1, \tau_2$

Boundary Condition:
How charges and gauge fields couple

Pure Gauge Field
+
Boundary (Matter) Charges

Edge

Dofs: $1, \tau$

Pure Gauge Field

Pure gauge dof on a tail: $1, \tau$

Transformation of ***Pure Gauge Field***:

$$\tilde{\mathcal{G}} = \mathcal{P}\mathcal{G}\mathcal{P}$$

$\mathcal{P}$: Projector: $\{1, \tau_1, \tau_2\} \to \{1, \tau\}$

$\tilde{\mathcal{G}}$: ***Noninvertible Gauge Symmetry!***

Example: Fibonacci UFC:

Gauge Symmetry

$$\tilde{\mathcal{G}}^{1\bar{1}} = \mathbb{I}$$

$$\tilde{\mathfrak{G}}^{\tau\bar{1}} = \begin{pmatrix} \frac{(\phi\sqrt{D}-1)(\sqrt{\phi}+1)}{4\phi^2} & \frac{1-\phi\sqrt{D}}{4\phi^2\sqrt{\phi}} \\ \frac{1-\phi\sqrt{D}}{4\phi^2\sqrt{\phi}} & \frac{(\phi\sqrt{D}-1)(\sqrt{\phi}-1)}{4\phi^2} \end{pmatrix}$$

$$\tilde{\mathfrak{G}}^{\tau\bar{1}} = \begin{pmatrix} -\frac{(\phi\sqrt{D}+1)(\sqrt{\phi}+1)}{4\phi^2} & \frac{1+\phi\sqrt{D}}{4\phi^2\sqrt{\phi}} \\ \frac{1+\phi\sqrt{D}}{4\phi^2\sqrt{\phi}} & -\frac{(\phi\sqrt{D}+1)(\sqrt{\phi}-1)}{4\phi^2} \end{pmatrix}$$

$$\tilde{\mathfrak{G}}^{\tau\bar{\tau}} = \begin{pmatrix} \frac{1}{\phi^2} & \frac{\sqrt[4]{5}+\phi\sqrt{D}}{2\phi^2\sqrt{\phi}} & \frac{\sqrt[4]{5}-\phi\sqrt{D}}{2\phi^2\sqrt{\phi}} \\ \frac{\sqrt[4]{5}+\phi\sqrt{D}}{2\phi^2} & -\frac{1+\sqrt{\phi}}{2\phi^3} & \frac{1}{2\phi^3\sqrt{\phi}} \\ \frac{\sqrt[4]{5}-\phi\sqrt{D}}{2\phi^2} & \frac{1}{2\phi^3\sqrt{\phi}} & \frac{1-\sqrt{\phi}}{2\phi^3} \end{pmatrix}$$

$\mathrm{rank}(\tilde{\mathcal{G}}^{\tau\bar{1}}) = \mathrm{rank}(\tilde{\mathcal{G}}^{1\bar{\tau}}) = 1$
$\mathrm{rank}(\tilde{\mathcal{G}}^{\tau\bar{\tau}}) = 2$

FUSION 2-CATEGORICAL SYMMETRY

Our Categorical Gauge Structure

- Objects: all $\mathrm{Bimod}_{\mathcal{F}}(\mathcal{A}) \subseteq \mathcal{F}$
- 1-Morphisms:

$$\varphi_{\mathcal{A}}: \mathcal{F} \to \mathrm{Bimod}_{\mathcal{F}}(\mathcal{A}) \subseteq \mathcal{F}$$

- 2-Morphisms: Compositions, etc.

$$\varphi_{\mathcal{B}} \circ \varphi_{\mathcal{A}} = \mathcal{F} \to \mathrm{Bimod}_{\mathcal{F}}(\mathcal{A}) \to \mathrm{Bimod}_{\mathrm{Bimod}_{\mathcal{F}}(\mathcal{A})}(\mathcal{B})$$

- Phase Space (Fusion 2-category)

  $=$ Configuration Space ($\mathcal{F}$)

  $+$ Conjugate Momenta (1-morphisms)

- Fusion 2-category: ``(Co)Adjoint representation'' of $\mathcal{F}$

Traditional Gauge Structure

- Objects: Only one: Gauge Group G
- 1-Morphisms: Adjoint Automorphism:

$$A_g: G \to G, \qquad h \mapsto ghg^{-1}$$

- 2-Morphisms: Group Multiplication:

$$A_g A_{g'} = A_{gg'}$$

- Phase Space (lattice): **Cotangent Bundle** over G.
- Group G itself suffices to describe the symmetry.

LOCAL TRANSFORMATION

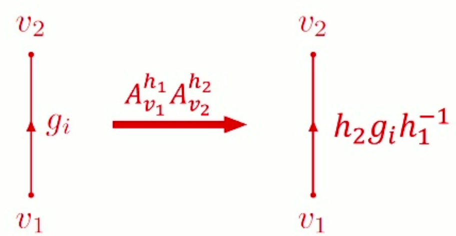
We've defined the symmetry transformation, but what we defined is a GLOBAL Transformation!

Gauge Transformation should be local: Different positions experience different transformations!

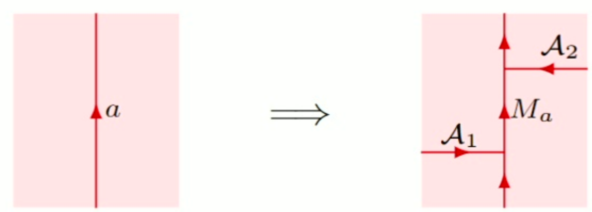
Gauge Transformation of dof: $g_i \to g g_i g^{-1}$

Gauge Transformation of dof: $a \to M_a$: Bimodule over $\mathcal{A}$

Local Transformation: Depended on Position



Gauge Transformations: $\mathcal{A}_1 - \mathcal{A}_2$ bimodules



$\mathcal{A}_1 \quad \mathcal{A}_2 \quad \mathcal{A}_3$

$\mathcal{A}_4 \quad \mathcal{A}_5 \quad \mathcal{A}_6$