Title: Uniqueness of bipartite and multipartite quantum state over time

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Abstract:

Recent efforts to formulate a unified, causally neutral approach to quantum theory have highlighted the need for a framework treating spatial and temporal correlations on an equal footing. Building on this motivation, we propose operationally inspired axioms for quantum states over time, demonstrating that, unlike earlier approaches, these axioms yield a unique quantum state over time that is valid across both bipartite and multipartite spacetime scenarios. In particular, we show that the Fullwood-Parzygnat state over time uniquely satisfies these axioms, thus unifying bipartite temporal correlations and extending seamlessly to any number of temporal points. In particular, we identify two simple assumptions—linearity in the initial state and a quantum analog of conditionability—that single out a multipartite extension of bipartite quantum states over time, giving rise to a canonical generalization of Kirkwood-Dirac type quasi-probability distributions. This result provides a new characterization of quantum Markovianity, advancing our understanding of quantum correlations across both space and time.

Characterizing the bipartite quantum state over time function

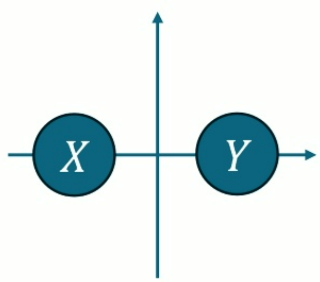
Seok Hyung Lie UNIST
& **Nelly Ng** NTU
Seminar @ Perimeter Institute

Symmetry of space and time in probabilistic theories?

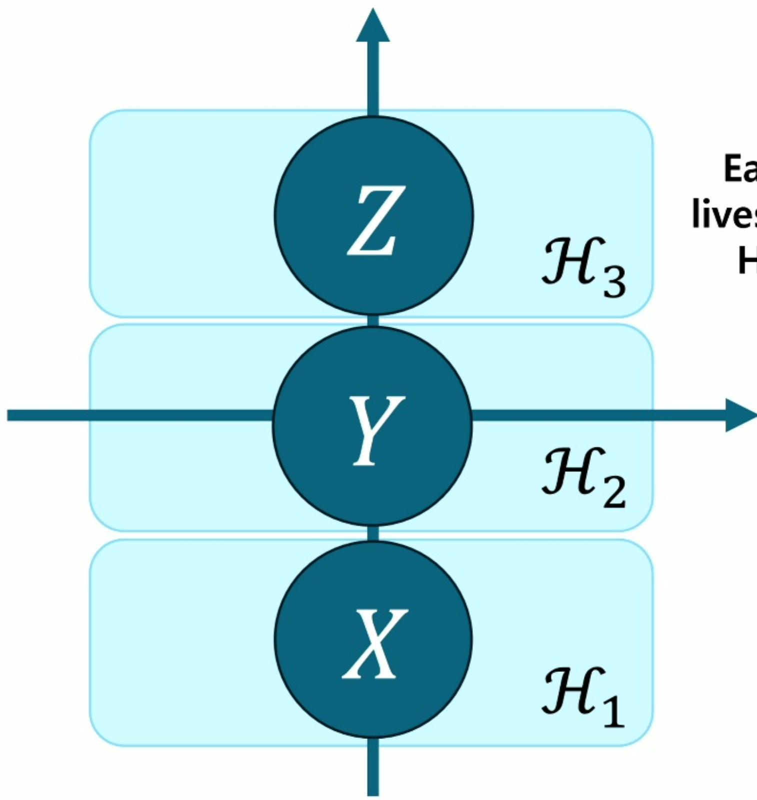
In classical theories…

Positions of events X and Y in spacetime?

$$P(X,Y)$$



Space-like?



Each time slice lives in a different Hilbert space

Is sensitivity to spatiotemporal structure
an inherent property of quantum theory?

Can we construct quantum state over time?

$$(\rho_A, \Phi_{\mathrm{B|A}}) \Rightarrow \Phi_{B|A} \star \rho_A$$

"Quantum state over time function"

$$\Phi_{B|A} \star \rho_A \overset{?}{\geq} O$$

PROCEEDINGS OF THE ROYAL SOCIETY A

MATHEMATICAL, PHYSICAL AND ENGINEERING SCIENCES

Can a quantum state over time resemble a quantum state at a single time?

Dominic Horsman ✉, Chris Heunen, Matthew F. Pusey, Jonathan Barrett and Robert W. Spekkens

Published: **20 September 2017** https://doi.org/10.1098/rspa.2017.0395

(a) Hermiticity

$$(\Phi_{B|A} \star \rho_A)^{\dagger} = (\Phi_{B|A} \star \rho_A)$$

(b) Preservation of probabilistic mixtures

$$\Phi_{B|A} \star (p\rho_A + (1-p)\sigma_A)$$
$$= p\Phi_{B|A} \star \rho_A + (1-p)\Phi_{B|A} \star \sigma_A$$

(c) Preservation of classical limit

$$\Phi_{B|A}(\rho_A) = \sum_{i,j} p_{j|i} |j\rangle\langle j| \langle i|\rho|i\rangle$$

$$\Rightarrow \Phi_{B|A} \star \rho_A = \sum_{i,j} p_{ij} |j\rangle\langle j| \otimes |i\rangle\langle i|$$

(Actually the exact form of the axiom assumed is different from this)

(e) Compositionality

$$\Psi_{C|B} \star (\Phi_{B|A} \star \rho_A)$$
$$= (\Psi_{C|B} \star \Phi_{B|A}) \star \rho_A$$

It turned out that the criteria were translated into too strong mathematical conditions

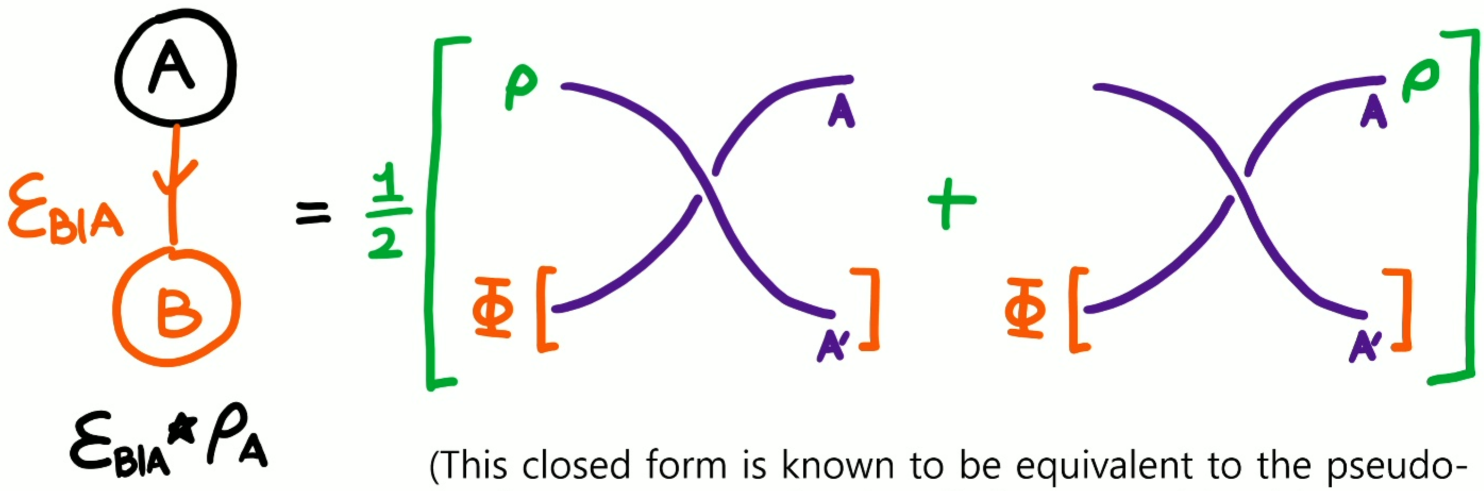
$$\Phi_{B|A} \star \rho_A = \frac{1}{2}\{\rho_A \otimes I_B, D[\Phi_{B|A}]\}$$

where $D[\Phi_{B|A}] = \mathrm{id}_A \otimes \Phi_{\mathrm{B|A'}}(\mathrm{F_{AA'}})$

Jamiołkowski isomorphism

$$\Phi_{B|A} \star \rho_A = \frac{1}{2}\{\rho_A \otimes I_B, D[\Phi_{B|A}]\}$$

where $D[\Phi_{B|A}] = \mathrm{id}_A \otimes \Phi_{\mathrm{B|A'}}(\mathrm{F}_{\mathrm{AA'}})$



(This closed form is known to be equivalent to the pseudo-density operator (PDO) when limited to multi-qubit systems)

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From Time-Reversal Symmetry to Quantum Bayes' Rules

Arthur J. Parzygnat and James Fullwood
PRX Quantum **4**, 020334 – Published 2 June 2023

TABLE II. The many state-over-time functions appearing in this work, along with their formulas, properties satisfied, and associated Bayes maps. The axioms are Hermiticity (P1), block-positivity (P2), positivity (P3), state linearity (P4), process linearity (P5), the classical limit (P7), and associativity A (bilinearity has been removed from the table to avoid redundancy). The $*$ for Ohya's compound state over time is because the classical limit is satisfied for density matrices with no repeating eigenvalues. Note that we do not fully define Ohya's compound state over time for arbitrary CPTP maps between multimatrix algebras (this will be addressed in future work, along with additional examples of state-over-time functions). The question mark represents the fact that we have not yet determined whether the given axiom is satisfied.

Name (page ref.)	State over time $\mathcal{E} \star \rho$	P1	P2	P3	P4	P5	P7	A	Bayes map $\mathcal{E}^{\star}_{\rho}$
Uncorrelated (7)	$\rho \otimes \mathcal{E}(\rho)$	✓	✓	✓	✗	✓	✗	✗	Any CPTP such that $\mathcal{E}^{\star}_{\rho}(\mathcal{E}(\rho)) = \rho$
Ohya compound (7)	$\sum_{\alpha} \lambda_{\alpha} P_{\alpha} \otimes \mathcal{E}\left(\frac{P_{\alpha}}{\mathrm{tr}(P_{\alpha})}\right)$	✓	✓	✓	✗	✓	$*$	?	Not computed here
Leifer-Spekkens (7)	$(\sqrt{\rho} \otimes 1_{\mathcal{B}})\mathscr{D}[\mathcal{E}](\sqrt{\rho} \otimes 1_{\mathcal{B}})$	✓	✓	✗	✗	✓	✓	✗	Petz map $\mathscr{R}_{\rho,\mathcal{E}} := \mathrm{Ad}_{\rho^{1/2}} \circ \mathcal{E}^* \circ \mathrm{Ad}_{\mathcal{E}(\rho)^{-1/2}}$
t-rotated (8)	$(\rho^{1/2-it} \otimes 1_{\mathcal{B}})\mathscr{D}[\mathcal{E}](\rho^{1/2+it} \otimes 1_{\mathcal{B}})$	✓	✓	✗	✗	✓	✓	✗	Rotated Petz map $\mathrm{Ad}_{\rho^{-it}} \circ \mathscr{R}_{\rho,\mathcal{E}} \circ \mathrm{Ad}_{\mathcal{E}(\rho)^{it}}$
STH (8)	$(U_{\rho}^{\dagger}\rho^{1/2} \otimes 1_{\mathcal{B}})\mathscr{D}[\mathcal{E}](\rho^{1/2}U_{\rho} \otimes 1_{\mathcal{B}})$	✓	✓	✗	✗	✓	✓	✗	$\mathrm{Ad}_{U_{\rho}^{\dagger}} \circ \mathscr{R}_{\rho,\mathcal{E}} \circ \mathrm{Ad}_{U_{\mathcal{E}(\rho)}}$
Symmetric bloom (11)	$\frac{1}{2}\{\rho \otimes 1_{\mathcal{B}}, \mathscr{D}[\mathcal{E}]\}$	✓	✗	✗	✓	✓	✓	✓	$\lvert w_k\rangle\langle w_l\rvert \mapsto (q_k + q_l)^{-1}\{\rho, \mathcal{E}^*(\lvert w_k\rangle\langle w_l\rvert)\}$
Right bloom (13)	$(\rho \otimes 1_{\mathcal{B}})\mathscr{D}[\mathcal{E}]$ (e.g., two-state)	✗	✗	✗	✓	✓	✓	✓	$B \mapsto \rho\mathcal{E}^*(\mathcal{E}(\rho)^{-1}B)$ (e.g., weak values)
Left bloom (15)	$\mathscr{D}[\mathcal{E}](\rho \otimes 1_{\mathcal{B}})$ (e.g., correlator)	✗	✗	✗	✓	✓	✓	✓	$B \mapsto \mathcal{E}^*(B\mathcal{E}(\rho)^{-1})\rho$

Quantum stat over time (QSOT) function

A function $\star : \mathcal{C}(A,B) \times \mathcal{S}(A) \to A \otimes B$ that maps $\left(\Phi_{\mathrm{B|A}}, \rho_A\right)$ to $\Phi_{B|A} \star \rho_A$ is a QSOT function if

$$\mathrm{Tr}_B \Phi_{B|A} \star \rho_A = \rho_A$$
$$\mathrm{Tr}_A \Phi_{B|A} \star \rho_A = \Phi(\rho)_B$$

Axiom (E): Completeness
For any quantum state over spacetime ρ_{AE} with two arbitrary regions A and E in spacetime, and any quantum channel $\mathcal{E}_{B|A}$, the action of QSOT function on a subsystem $\mathcal{E}_{B|A} \star \rho_{AE}$ can be defined and has the following properties: For any completely positive trace non-increasing operation $\mathcal{I}_E$ on system E,

$$\mathcal{I}_E\left[\mathcal{E}_{B|A} \star \rho_{AE}\right] = \mathcal{E}_{B|A} \star \mathcal{I}_E(\rho_{AE}), \quad (3)$$

Axiom (P): Compositionality, [24]
A QSOT function should be compatible with composition of quantum channels. In other words, for any two quantum channels $\mathcal{E}_{B|A}$ and $\mathcal{F}_{C|B}$, we have

$$\mathrm{Tr}_B\left[\mathcal{F}_{C|B} \star \left(\mathcal{E}_{B|A} \star \rho_A\right)\right] = (\mathcal{F} \circ \mathcal{E})_{C|A} \star \rho_A. \quad (5)$$

Axiom (T): Time reversal symmetry
A state over time corresponding to the trivial evolution should be symmetric under the time reversal transformation, i.e. $F_{AB}(\mathrm{id}_{B|A} \star \rho_A)F_{AB} = \mathrm{id}_{B|A} \star \rho_A$, for all $\rho_A \in \mathfrak{S}(A)$, where F_{AB} denotes the swap gate between systems A and B.

Axiom (CC): Classical Conditionability (informal)
When the input state and the channel are prepared in an ensemble $\{\lambda_i, \pi_{A_i}, \mathcal{E}_{B|A_i}\}$, then the corresponding QSOT is given as $\mathcal{E}_{B|A} \star (\sum_i \lambda_i \pi_{A_i}) = \sum_i \lambda_i \mathcal{E}_{B|A_i} \star \pi_{A_i}$, where $\mathcal{E}_{B|A} = \sum_i \mathcal{E}_{B|A_i}$.

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Axiom (H) (Hermiticity) For any quantum channel and state $\mathcal{E}_{B|A}, \rho_A$, the state over time $\mathcal{E}_{B|A} \star \rho_A$ must be Hermitian.

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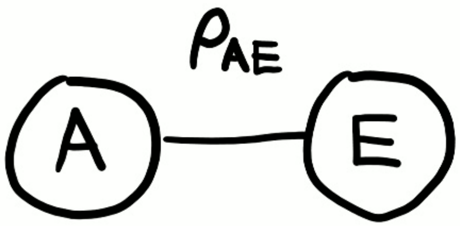

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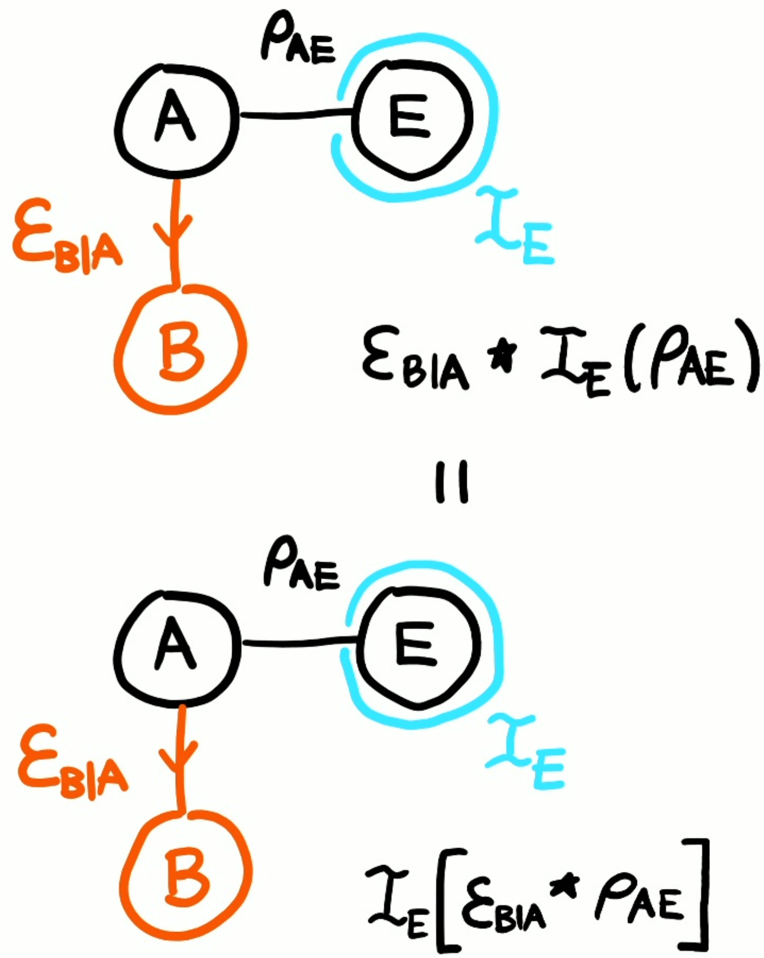
(Maybe we should've named it Axiom (L) for **L**ocality and **L**inearity...)

It is equivalent to state-linearity of

$\mathcal{E}_{B|A} \star \rho_A$



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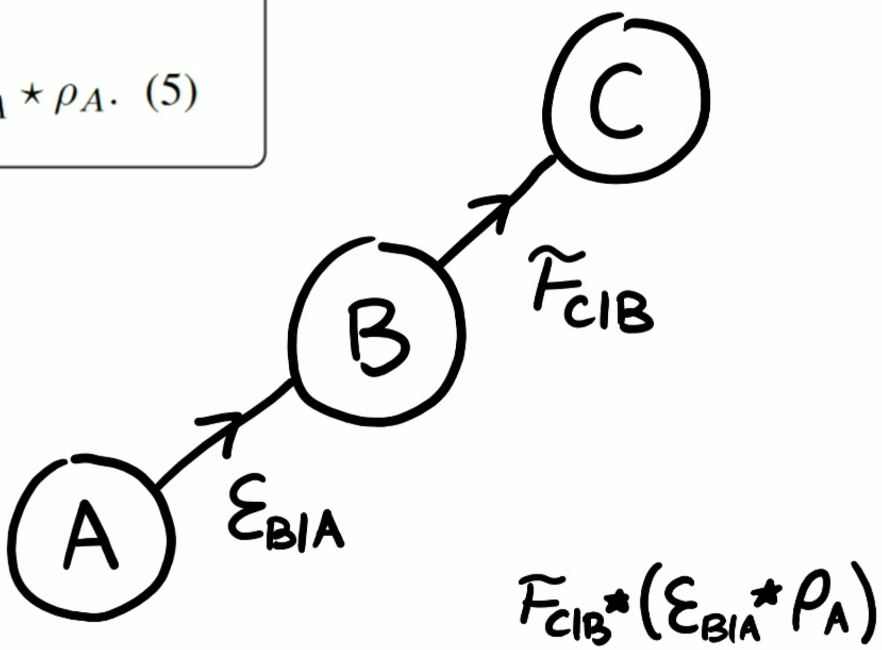

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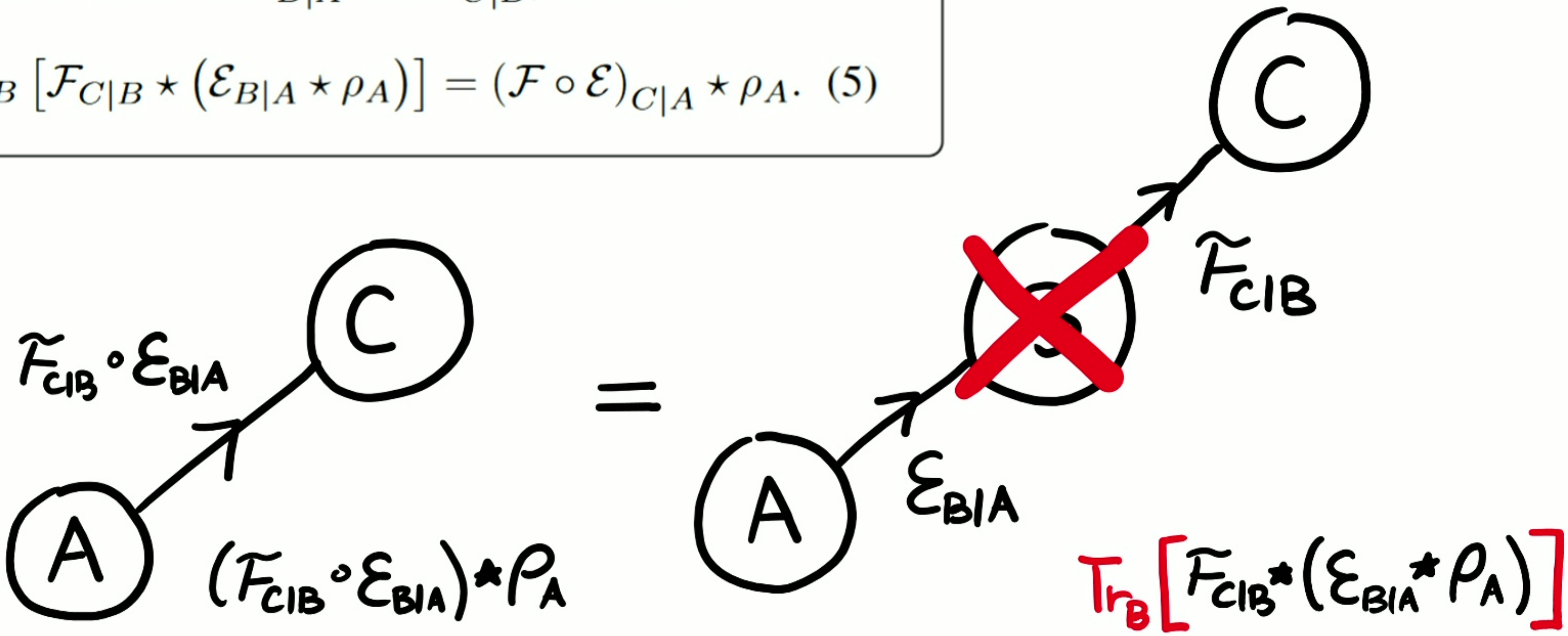

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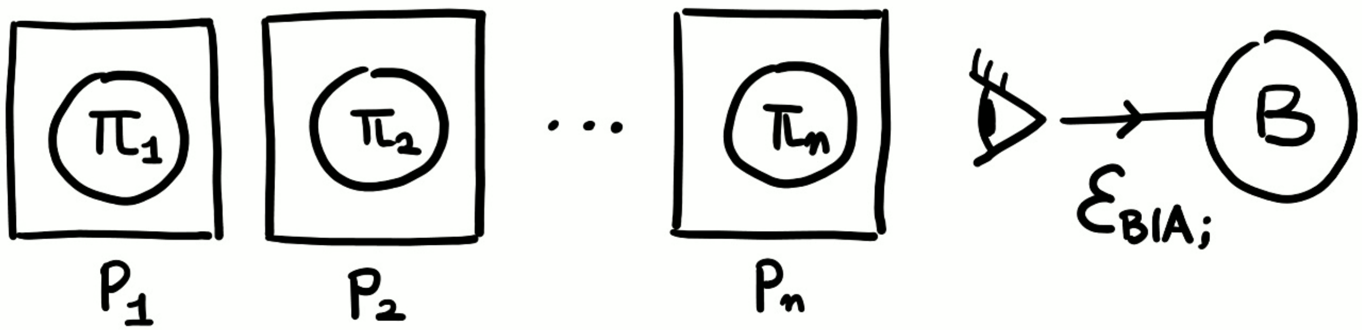

Ensemble of maximally mixed states on orthogonal subspaces

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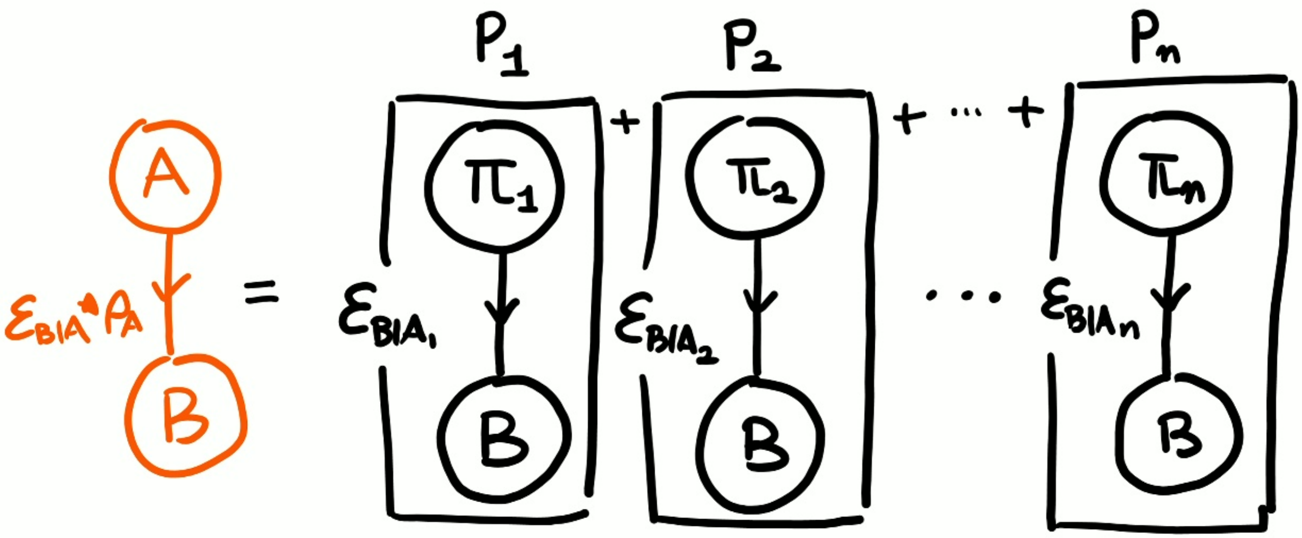

Ensemble of QSOTs for each orthogonal maximally mixed input state

Axiom (T): Time reversal symmetry
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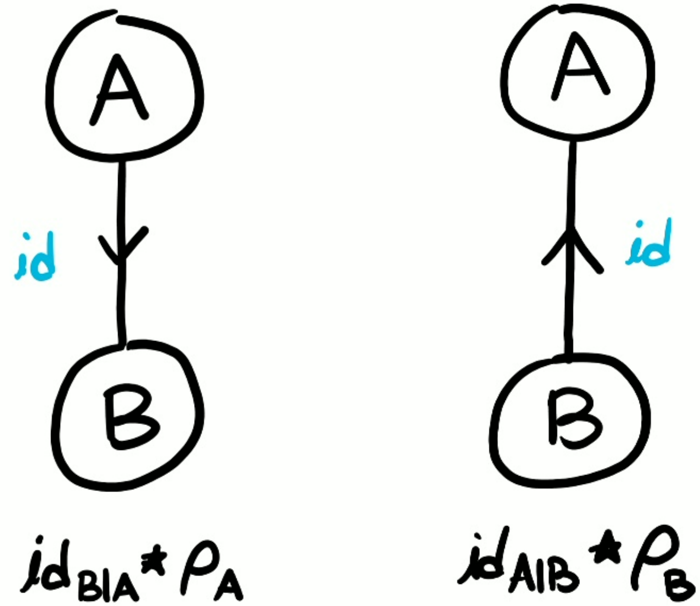

Axiom (QC): Quantum conditionability

For every state $\rho \in \mathfrak{S}(A)$, there exists a state-rendering function Θ_ρ [17,40,41] on $\mathfrak{B}(A)$ such that

$$\mathcal{E}_{B|A} \star \rho_A = (\Theta_\rho \otimes \mathrm{id}_B)(\mathcal{E}_{B|A} \star \mathbb{1}_A) \qquad (13)$$

for all $\mathcal{E} \in \mathfrak{C}(A, B)$, where Θ_ρ is linear, and for any $M \in \mathfrak{B}(A)$, whenever $[\rho, M] = 0$, we have $\Theta_\rho(M) = \rho M$.

$$P(X,Y) = P(Y|X)P(X)$$

"state-rendering"

Axiom (QC): Quantum conditionability

For every state $\rho \in \mathfrak{S}(A)$, there exists a state-rendering function Θ_ρ [17,40,41] on $\mathfrak{B}(A)$ such that

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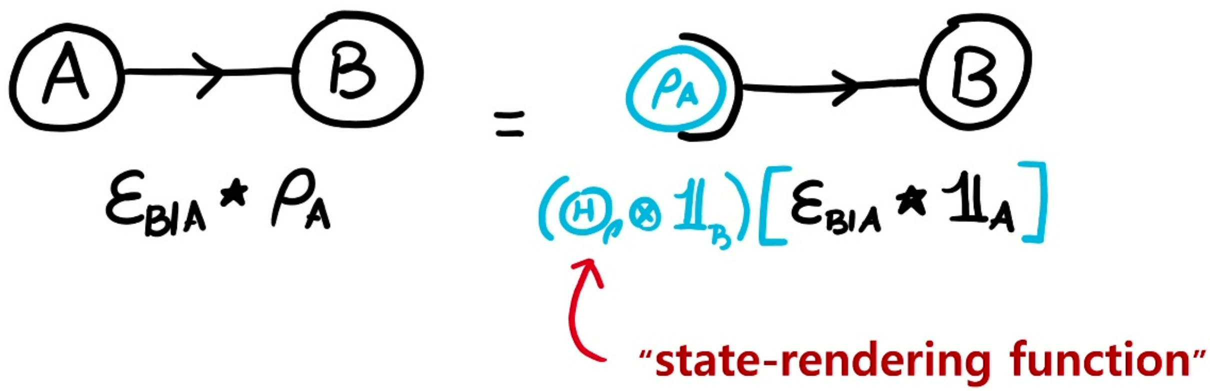
(This part requires reduction to classical state rendering function, which may warrant a separate axiom.)



$$[E_{B|A}, \rho_A] = 0 \implies E_{B|A} \star \rho_A = E_{B|A} \rho_A.$$

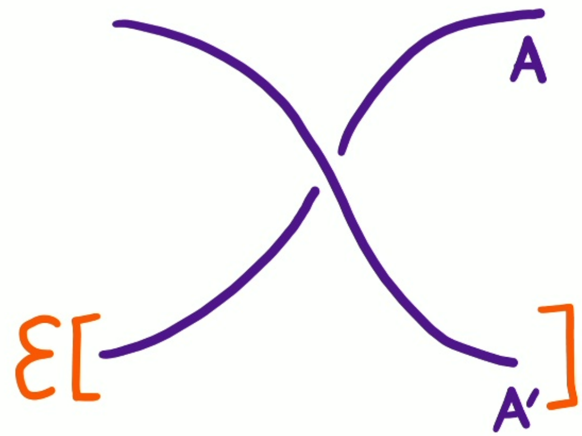
(Horsman et al. , Proceedings A, 2017)

Axiom (J): Jamiołkowski

For a system A with dimension $|A|$, the state over time associated with the maximally mixed state π_A is

$$\mathcal{E}_{B|A} \star \pi_A = \frac{1}{|A|}(\mathrm{id}_A \otimes \mathcal{E}_{B|A'})(F_{AA'}). \qquad (11)$$

$$\mathcal{E}_{B|A} \star \pi_A = \frac{1}{|A|}$$



=Jamiolkowski isomorphism is the "canonical" isomorphism

Theorem 2. The following sets of axioms are equivalent, and satisfied only by the FP function:

$$(\mathbf{E}) + (\mathbf{P}) + (\mathbf{CC}) + (\mathbf{T}) \quad \textbf{or} \quad (\mathbf{E}) + (\mathbf{P}) + (\mathbf{J}) + (\mathbf{T})$$

$$\textbf{or } (\mathbf{E}) + (\mathbf{QC}) + (\mathbf{T}) \quad \textbf{or} \quad (\mathbf{E}) + (\mathbf{QC} + \mathbf{SA}) + (\mathbf{H}).$$

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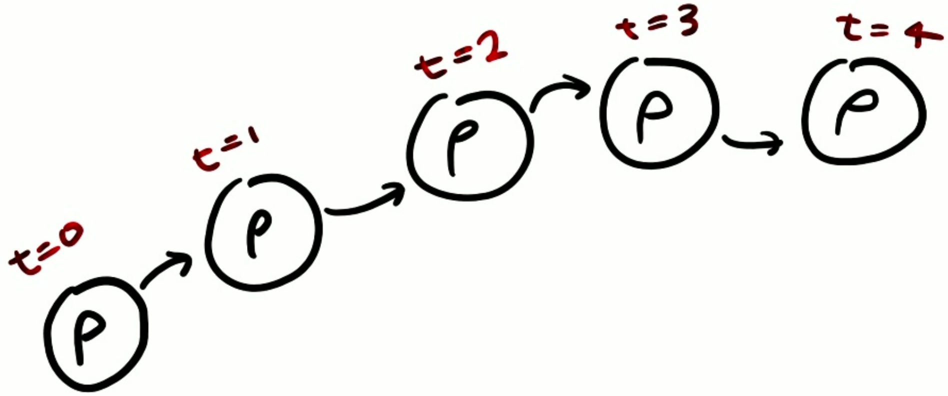
$$\textbf{or} \; (\mathbf{E}) + (\mathbf{QC}) + (\mathbf{T}) \quad \textbf{or} \quad (\mathbf{E}) + (\mathbf{QC} + \mathbf{SA}) + (\mathbf{H}).$$

Personal favorite

Cheat sheet

(E) : Completeness (State-Linearity) **(P)** : Compositionality
(CC) : Classical Conditionability **(QC)** : Classical Conditionability
(J) : Jamolkowski **(T)**: Time Reversal Symmetry **(H)** : Hermiticity


t=0
ρ
t=1
ρ
t=2
ρ
t=3
ρ
t=4
ρ

(The axioms are
(i) reduction to classical limit
(ii) local unitary covariance
(iii) exchange symmetry)

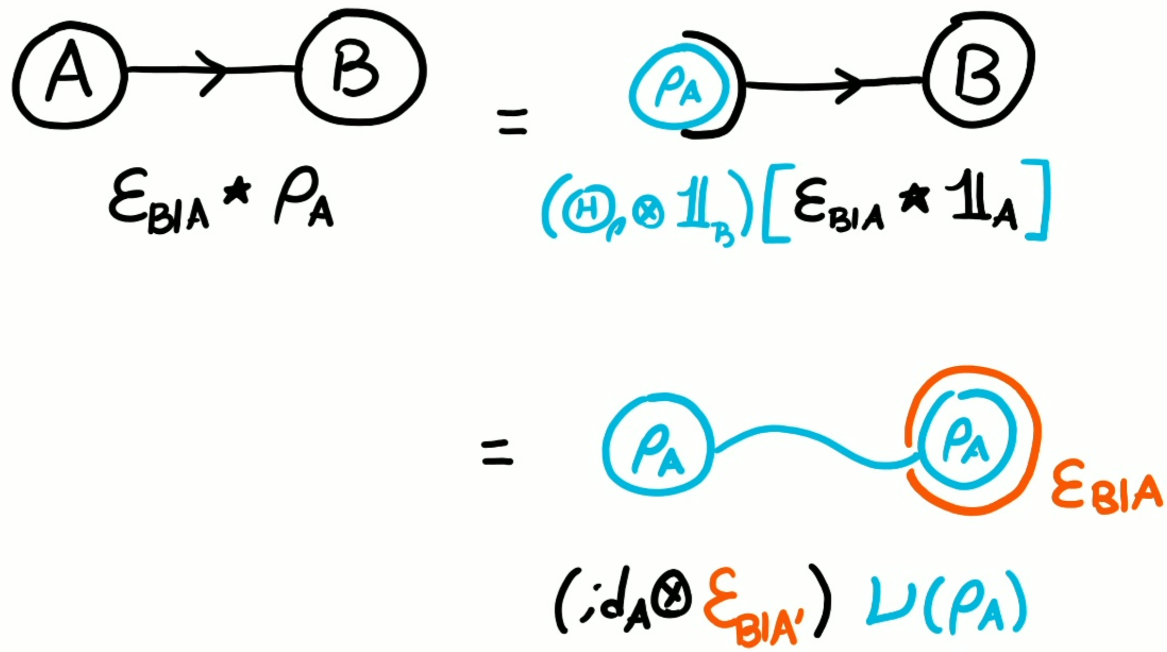
In this paper, they used a different set of axioms to uniquely characterize...

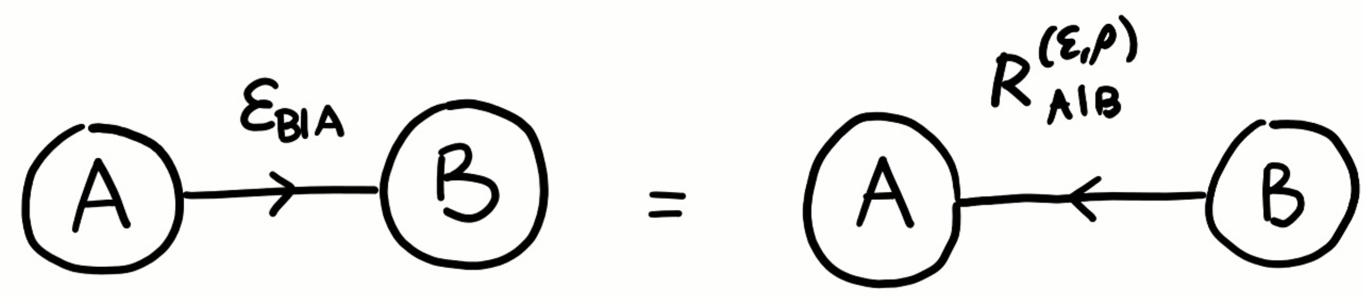
The virtual broadcasting map

$$\mathrm{id}_{B|A} \star \rho_A = \frac{1}{2}\{\rho_A \otimes I_B, F_{AB}\}$$

Simulatable with post-processing!

There are mainly **two ways** to understand QSOTs:




A
$\mathcal{E}_{B|A}$
B
=
A
$R^{(\mathcal{E},\rho)}_{A|B}$
B

✗ Any CPTP such that $\mathcal{E}^\star_\rho\big(\mathcal{E}(\rho)\big) = \rho$

? Not computed here

✗ Petz map $\mathscr{R}_{\rho,\mathcal{E}} := \mathrm{Ad}_{\rho^{1/2}} \circ \mathcal{E}^* \circ \mathrm{Ad}_{\mathcal{E}(\rho)^{-1/2}}$

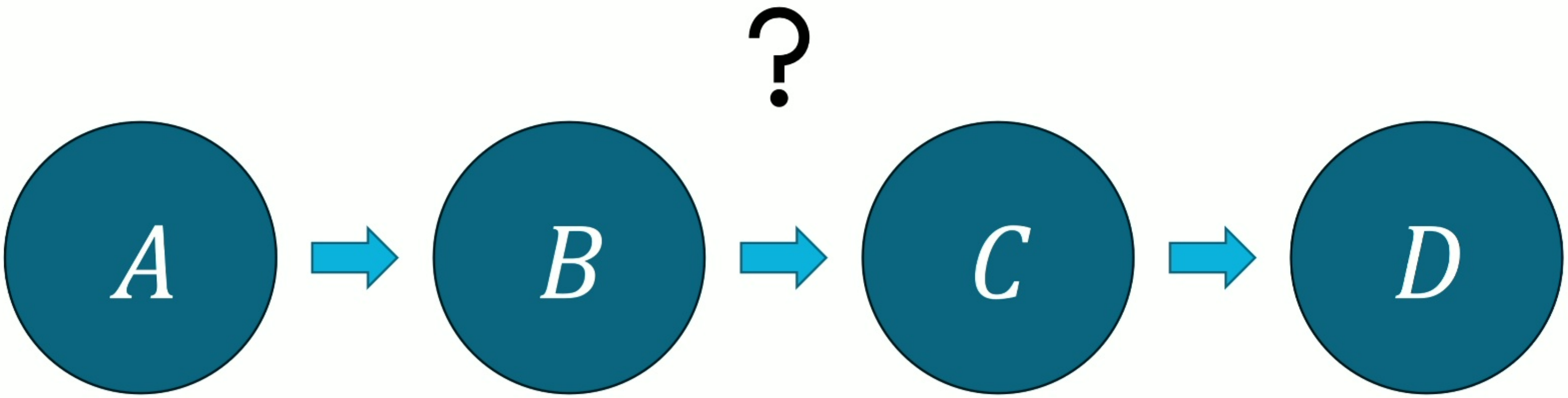
✗ Rotated Petz map $\mathrm{Ad}_{\rho^{-it}} \circ \mathscr{R}_{\rho,\mathcal{E}} \circ \mathrm{Ad}_{\mathcal{E}(\rho)^{it}}$

✗ $\mathrm{Ad}_{U_\rho^\dagger} \circ \mathscr{R}_{\rho,\mathcal{E}} \circ \mathrm{Ad}_{U_{\mathcal{E}(\rho)}}$

✓ $|w_k\rangle\langle w_l| \mapsto (q_k + q_l)^{-1}\big\{\rho, \mathcal{E}^*\big(|w_k\rangle\langle w_l|\big)\big\}$

✓ $B \mapsto \rho\mathcal{E}^*\big(\mathcal{E}(\rho)^{-1}B\big)$ (e.g., weak values)

✓ $B \mapsto \mathcal{E}^*\big(B\mathcal{E}(\rho)^{-1}\big)\rho$


?
A
B
C
D

QSOT function, QSOT product, spatiotemporal product, start product... all same

Definition 1 (Quantum state over time)**.** A *spatiotemporal product* (or $\star$-*product*) is a binary operation that maps every pair $(\boldsymbol{\mathcal{E}}, \rho) \in \mathbf{CPTP}(A_0, \ldots, A_n) \times \mathfrak{S}(A_0)$ to an $(n+1)$-partite operator $\boldsymbol{\mathcal{E}} \star \rho$ on $A_0 \cdots A_n$ such that

$$\mathrm{Tr}_{A_0}[\boldsymbol{\mathcal{E}} \star \rho] = \underline{\boldsymbol{\mathcal{E}}} \star \mathcal{E}_1(\rho) \quad \text{and} \quad \mathrm{Tr}_{A_n}[\boldsymbol{\mathcal{E}} \star \rho] = \overline{\boldsymbol{\mathcal{E}}} \star \rho\,.$$

Truncations of n-chains

$$\underline{\boldsymbol{\mathcal{E}}} := (\mathcal{E}_2, \mathcal{E}_3, \ldots, \mathcal{E}_n) \quad \text{and} \quad \overline{\boldsymbol{\mathcal{E}}} := (\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_{n-1})\,.$$

Common question:

We have a uniqueness result for bipartite QSOTs;

Why can't we just extend it to the multipartite setting?

$$\mathcal{E} \star \rho = \mathcal{E}_n \star (\mathcal{E}_{n-1} \star (\cdots \star (\mathcal{E}_1 \star \rho)))$$

Well, yeah, you **CAN** do that.

1. State-linearity

We want our QSOT function $\mathcal{E} \star \rho$ to be linear in ρ.

2. Conditionability

We want our QSOT function $\mathcal{E} \star \rho$ to behave similarly with classical probability distributions.

Especially: We want '**conditioning**'

$$P(x_0, \ldots, x_n) = P(x_0) \cdot P(x_1, \ldots, x_n | x_0)$$

Theorem 1 (Unique multi-partite extension of QSOTs). If a $\star$-product is conditionable and convex-linear in ρ, then it satisfies the iterative formula (3) for every $(\boldsymbol{\mathcal{E}}, \rho) \in \mathbf{CPTP}(A_0, \ldots, A_n) \times \mathfrak{S}(A_0)$.

$$\boldsymbol{\mathcal{E}} \star \rho = \mathcal{E}_n \star (\mathcal{E}_{n-1} \star (\cdots \star (\mathcal{E}_1 \star \rho))) \qquad (3)$$

So what?

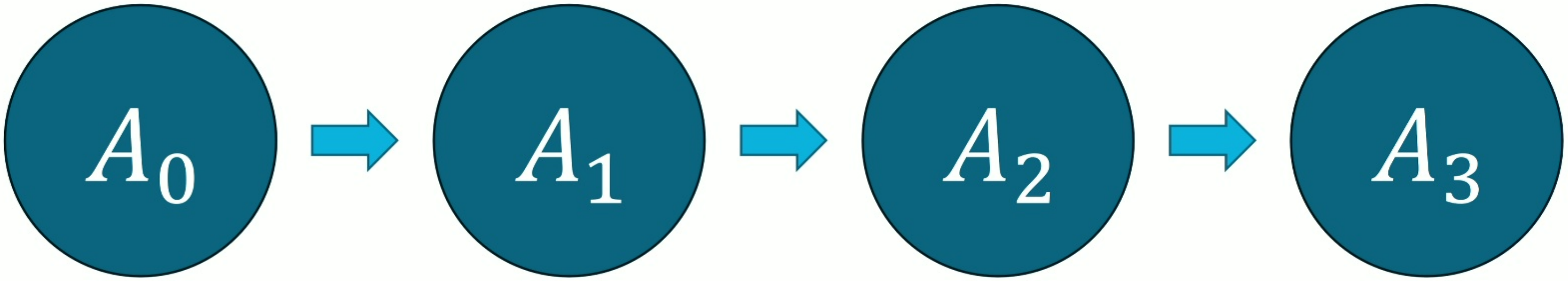
Is this result important out of the QSOT context?

QSOTs have a one-to-one correspondence with **quasi-probability distributions.**

Especially, the **Kirkwood-Dirac distribution** and its variants have a deep connection with QSOTs.

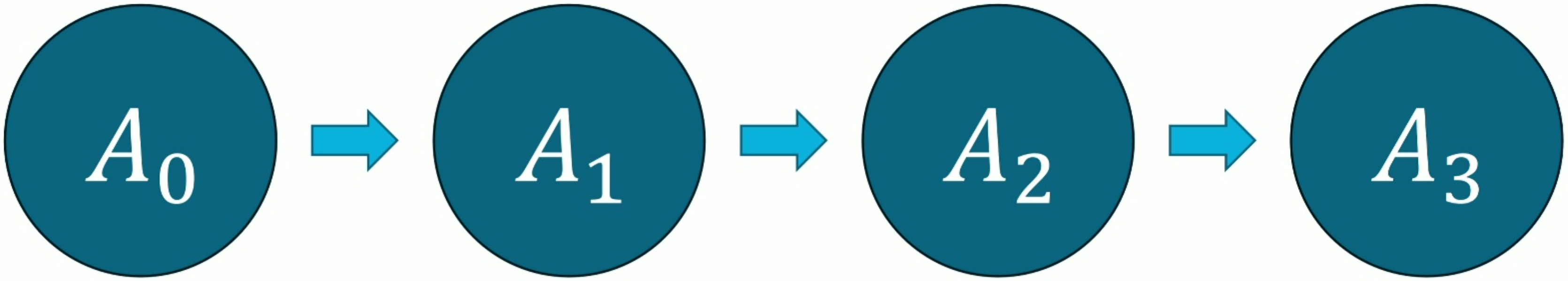
Let $\{M_i\} \subset A$ and $\{N_j\} \subset B$ be POVMs. It then follows that for every $\star$-product on 1-chains the elements $Q_{AB}(i,j) \in \mathbb{C}$ given by

$$Q_{AB}(i,j) = \mathrm{Tr}\left[\mathcal{E} \star \rho \, (M_i \otimes N_j)\right] \tag{7}$$



Dynamics is given in terms of an n-chain
=It is Markovian

However, in general for a given QSOT ρ_{ABC}, the corresponding quasi-probability distribution Q_{ABC} does NOT satisfy Markvianity, i.e. $Q_{C|BA} \neq Q_{C|B}$.



The QSOT describing the dynamics has iterativity

Quasi-probability distributions are accessible with classical post-processing

(Quantum process snapshotting)

QSOT is the ‘generating function’ of quasi-probability distributions

Then why do we need QSOT?

When we already have quasi-probability distributions?

Advantages of QSOT:

1) It **unifies quasi-distributions**; A QSOT can give quasi-distributions for every set of observables.
2) It represents the **Markovian nature** of quantum dynamics better