Title: The Charge Gap exceeds the Neutral Gap in Fractional Quantum Hall Systems

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Abstract:

The so-called pseudo-potentials modeling fractional quantum Hall systems are quantum many-body Hamiltonians that are frustration free and have two symmetries, one related to the conservation of charge (particle number) and another to the conservation of dipole moment (angular momentum), in addition to translation invariance. We show that for such systems the minimum energy of charged excitations is bounded below by the minimum energy of neutral excitations. This property, which had been repeatedly observed in numerical simulations, has a surprisingly simple proof (joint work with Marius Lemm, Simone Warzel, and Amanda Young, arxiv:2410.11645).

Outline

- Introduction; FQHE, pseudopotentials, charge and neutral gap
- Setup and symmetries
- Assumptions on the Hamiltonian
- A combinatorial identity
- Spectral gap comparison
- Assumptions on the ground states
- The spectral gap inequality
- Generalizations
- Infinite system gaps, open problems

Introduction

This talk is about the energy spectrum of fractional quantum Hall systems, which are insulating materials on a surface (2D) subject to perpendicular magnetic field, with the number of electrons, $n \approx L/q$, where L is the number that would fill the first Landau level and $q \geq 3$ is an odd integer (or q even and bosonic particles).

For example, consider a systems on a torus (periodic b.c.):

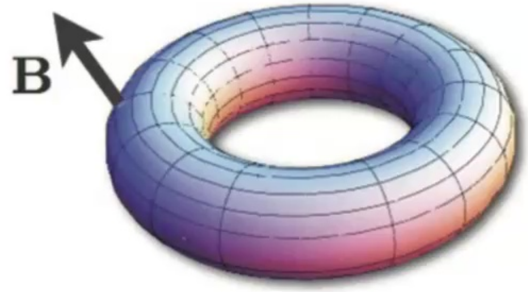


Figure: Torus geometry $[1, L] = \mathbb{Z}/L\mathbb{Z}$

We consider many-body Hamiltonians for fermions (or bosons) in 1D with a special symmetry: conservation of dipole moment. We focus on the fermionic case. $n_j = a_j^\dagger a_j$. For example, the Haldane pseudopotentials for Fractional Quantum Hall (FQH) systems have this structure. An example with $q = 3$ for which we know there is a gap above the ground states is:

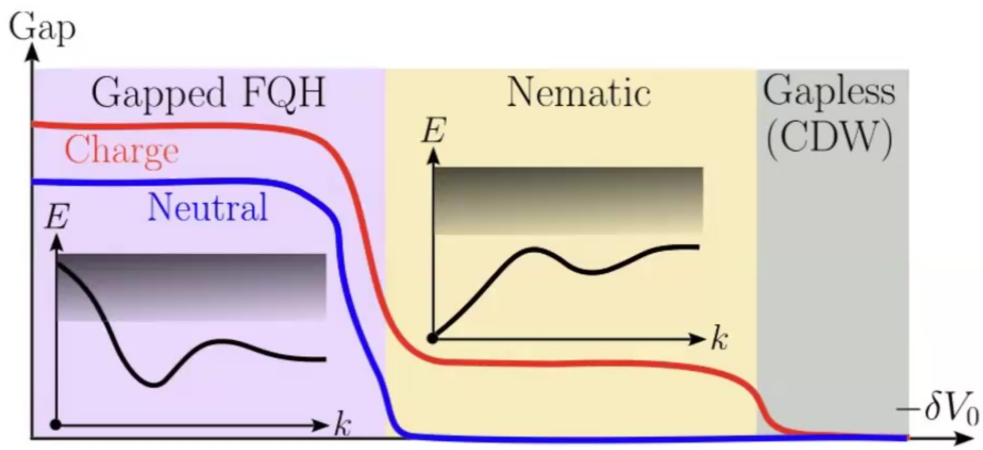
$$H = \sum_j \left(n_j n_{j+2} + \kappa \; q_j^* q_j \right), q_j := a_{j+1} a_{j+2} - \lambda \; a_j a_{j+3}, \; \kappa \geq 0, \lambda \in \mathbb{C},$$

Theorem (N-Young-Warzel 2020 & 2021, Young-Warzel 2022 & 2023)

For all $\lambda \in \mathbb{C}$, $0 \leq |\lambda| < 5.3548$, $\kappa \geq 0$ *there is a constant* $f\left(|\lambda|^2\right) < 1/3$ *for which*

$$\liminf_{L\to\infty} \mathrm{gap} H_{[1,L]} \geq \frac{\kappa}{3} \min \left\{ \frac{1}{2 + 2\kappa|\lambda|^2}, \frac{1}{1+\kappa}, \frac{1}{2(1+2|\lambda|^2)} \left(1 - \sqrt{3f(|\lambda|^2)}\right)^2 \right\}.$$

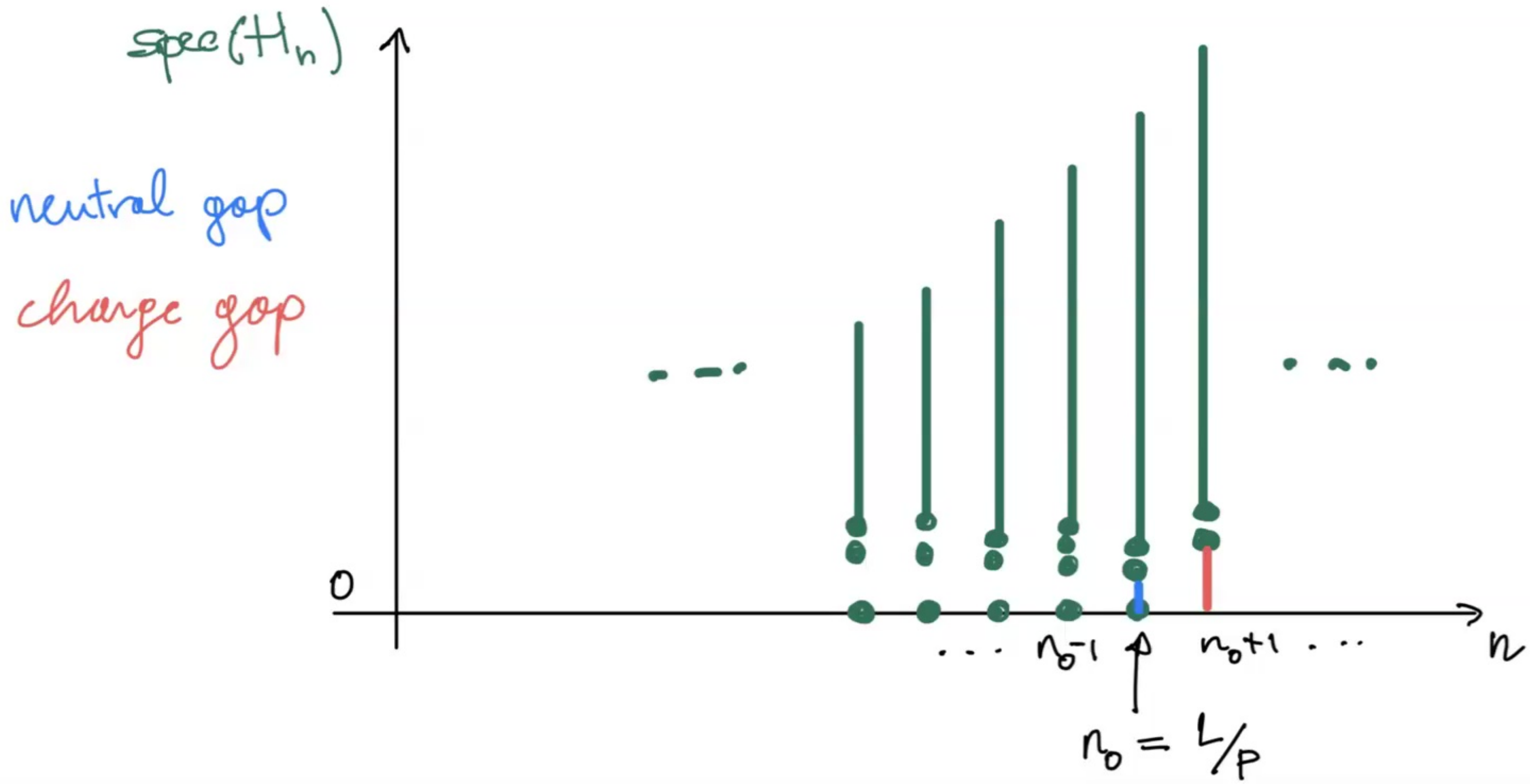
Variants of such models are also used to study phase transitions in FQH systems, some of which break a continuous symmetry (nematic order, skyrmions), leading to closing of the gap, as seen in the figure below.



From: Pu, Balram , Taylor, Fradkin , and Papić, Phys. Rev. Lett. **132**, 236503 (2024). Note that the charge gap always exceeds the neutral gap.

This relation between the two types of gap was also observed by Haldane and co-workers in numerical calculations for pseudopotential systems.

Observed qualitative feature of the spectrum: charge gap $\geq$ neutral gap for systems with n particles and $n \simeq L/q$. Energies are given by spectrum of the Hamiltonian H_n.



Setup and symmetries

We consider spinless fermions on a ring: $[1, L] = \mathbb{Z}/L\mathbb{Z}$. The fermionic Fock space with one-particle space $\mathcal{H} = \ell^2([1, L])$, is the finite direct sum of fixed particle number sectors:

$$\mathcal{F} = \bigoplus_{n\geq 0} \mathcal{F}_n, \quad N = \bigoplus_{n\geq 0} n\mathbb{1}\,|_{\mathcal{F}_n}$$

The systems we consider will have symmetries given by a group of unitaries on $\mathcal{F}$ generated by T, U, and V defined as follows:

1. translation: $T^* a_j T = a_{j-1,\ \mathrm{mod}\ L}$, $j = 1, \ldots, L$;
2. particle number conservation described by a gauge group generated by the particle number operator $N := \sum_{j=1}^{L} a_j^\dagger a_j$: $U = \exp\left(\frac{2\pi i}{L} N\right)$.
3. conservation of center of mass $D := \sum_{j=1}^{L} j\ a_j^\dagger a_j$, the dipole moment. This yields another group of unitaries generated by V: $V = \exp\left(\frac{2\pi i}{L} D\right)$.

The eigenspaces of N are the sectors $\mathcal{F}_n$.The eigenvalues of D are also a set of integers including 1. Hence, U and V each generate a finite group of order L.

It is straightforward to check the following relations among these symmetries:

$$VT = UTV, \qquad UT = TU, \qquad UV = VU. \tag{1}$$

Important: note that T and V do not commute. Here some useful implications of (1):

Suppose we have $\psi \in \mathcal{F}$ with $V\psi = \exp\left(2\pi i \frac{d}{L}\right)\psi$ for $d \in \mathbb{Z}$. Then
(i) If $\psi \in \mathcal{F}_n$, and $j \in \mathbb{N}$:

$$VT^j\psi = \exp\left(2\pi i\ \frac{d+jn}{L}\right) T^j\psi; \tag{2}$$

(ii) If $\psi \in \mathcal{F}$, and $j \in \{1, \ldots, L\}$:

$$Va_j\psi = \exp\left(2\pi i \frac{d-j}{L}\right) a_j\psi \tag{3}$$

Assumptions on the Hamiltonian

1. Hamiltonian H defined on $\mathcal{H}$ is a self-adjoint operator of the following form:

$$H = \sum_{k=1}^{M} C_k \, a^\dagger(f_1^{(k)}) \cdots a^\dagger(f_m^{(k)}) a(g_m^{(k)}) \cdots a(g_1^{(k)}) = \bigoplus_{n \geq 0} H_n,$$

   for some $m \geq 2, M \geq 1$, and $C_k \in \mathbb{C}$, $f_1^{(k)}, \ldots, f_m^{(k)}, g_1^{(k)}, \ldots, g_m^{(k)} \in \mathcal{H}$, $k = 1, \ldots, M$.

2. $H \geq 0$.
3. $\ker H_m \neq \{0\}$, and define $n_{\max} := \max\{n \geq m \mid \ker H_n \neq \{0\}\}$ Furthermore, we assume $n_{\max} < L$.
4. H commutes with T, U, and V, defined before.

Another way to express Assumption 3 is to require existence of $n_{\max} \geq m$ such that $\ker H_n \neq \{0\}$ for all $n \leq n_{\max}$, and $\ker H_n = \{0\}$ for all $n > n_{\max}$.

A combinatorial identity

$$
\begin{aligned}
Na^\dagger(f_1)\cdots a^\dagger(f_m)a(g_m)\cdots a(g_1) &= ma^\dagger(f_1)\cdots a^\dagger(f_m)a(g_m)\cdots a(g_1) \\
&\quad + \sum_j a_j^\dagger \left[a^\dagger(f_1)\cdots a^\dagger(f_m)a(g_m)\cdots a(g_1)\right] a_j.
\end{aligned}
$$

Let H_n denote the restriction H to $\mathcal{F}_n$. Then, the identity implies for all $n \geq m$:

$$
H_{n+1} = \frac{1}{n+1-m} \sum_{j=1}^{L} a_j^\dagger H_n a_j. \tag{4}
$$

This identity holds equally for bosonic particles and for spins.

An immediate implication of (4) is the following:

$$
\psi \in \ker H_{n+1} \quad \Rightarrow \quad \text{for all } j \in \{1,\ldots,L\}: \quad a_j\psi \in \ker H_n.
$$

Spectral gap comparison

Let $H_n, n \geq m$, be a family of Hamiltonians satisfying the above assumption, and let P_n denote the orthogonal projection onto $\ker H_n$. Define $\operatorname{gap} H_n$ by

$$\operatorname{gap} H_n := \inf_{\substack{\psi\in\mathcal{F}_n,\, \|\psi\|=1 \\ P_n\psi=0}} \langle \psi, H_n\psi \rangle. \tag{5}$$

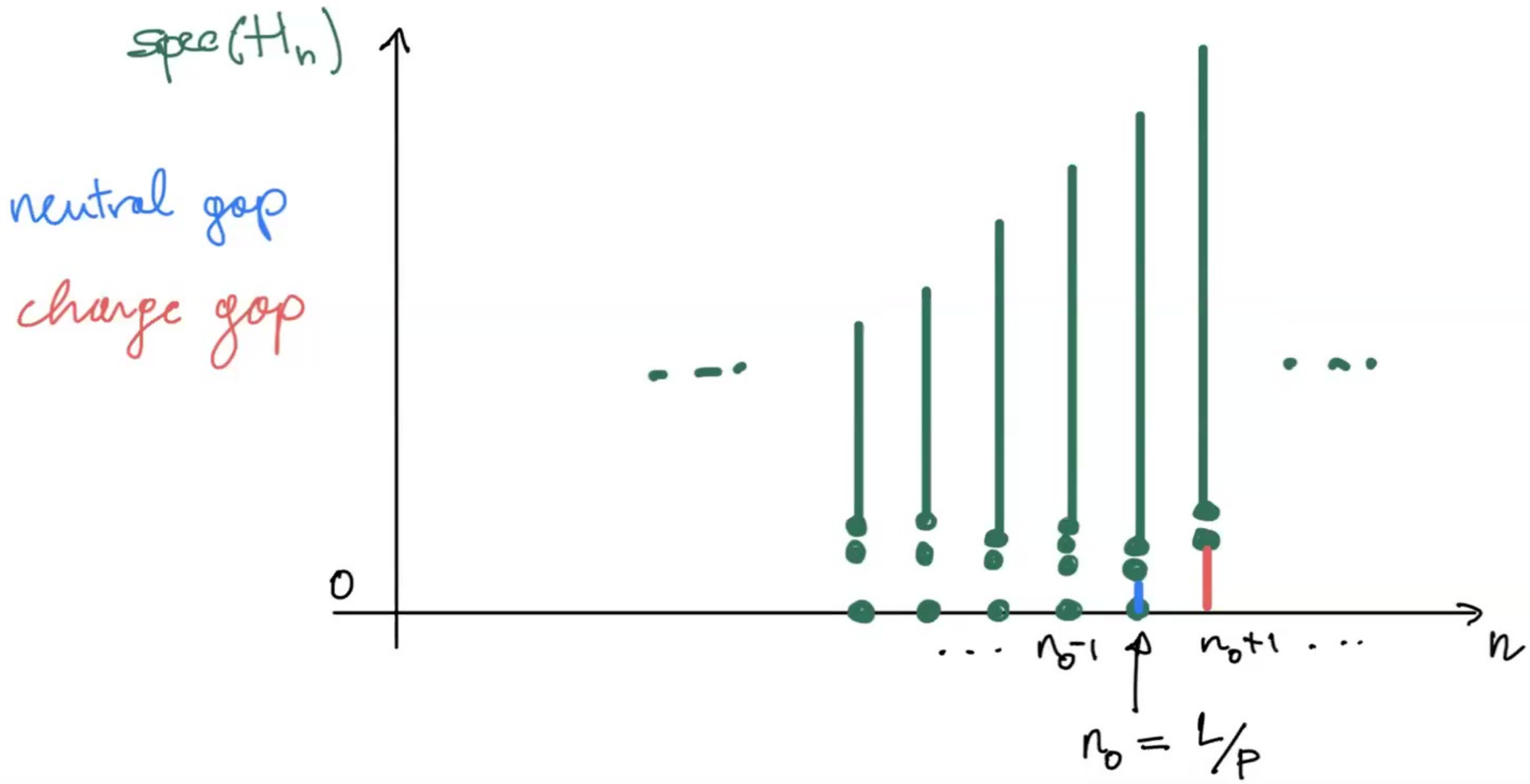
Note that for values of n for which $\ker H_n = \{0\}$, $\operatorname{gap} H_n$ is the ground state energy while if $\ker H_n \neq \{0\}$, the ground state energy vanishes and $\operatorname{gap} H_n$ is the spectral gap above 0. In general, we have $H_n \geq (\operatorname{gap} H_n)(\mathbb{1} - P_n)$.

Proposition (Inductive criterion)

For Hamiltonians satisfying (4) *one has for any* $n \geq m$*:*

$$\operatorname{gap} H_{n+1} \geq \frac{\operatorname{gap} H_n}{n+1-m} \left(n+1 - \left\| (1 - P_{n+1}) \sum_{j=1}^{L} a_j^\dagger P_n a_j (1 - P_{n+1}) \right\| \right). \tag{6}$$

Observed qualitative feature of the spectrum: charge gap $\geq$ neutral gap for systems with n particles and $n \simeq L/q$. Energies are given by spectrum of the Hamiltonian H_n.



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Moreover, if $\{\varphi_\alpha \mid \alpha = 1, \ldots, k\}$ denotes an orthonormal basis of $\mathrm{ran} P_n$, then

$$P_n = \sum_{\alpha=1}^{k} |\phi_\alpha\rangle\langle\phi_\alpha|$$

and

$$(1 - P_{n+1}) \sum_{j=1}^{L} a_j^\dagger P_n a_j (1 - P_{n+1}) = \sum_{j=1}^{L} \sum_{\alpha_1}^{k} |(1 - P_{n+1}) a_j^\dagger \phi_\alpha\rangle\langle(1 - P_{n+1}) a_j^\dagger \phi_\alpha|$$

the operator norm on the right side agrees with the operator norm of the $kL \times kL$ Gram matrix with entries

$$G_{j,\alpha;k,\beta} := \langle a_j^\dagger \varphi_\alpha, (1 - P_{n+1})\ a_k^\dagger \varphi_\beta\rangle. \tag{7}$$

Assumptions on the ground states of H_n

We focus on the situation where there is $q \in \mathbb{N}$, with $q \geq 2$, and q divides L, $L = qM$, with M a positive integer, and such that we have for $n = M$, a unit vector $\varphi \in \ker H_n$, with the properties:

1. φ is q periodic, i.e.. $T^q \varphi = \varphi$;
2. φ is an eigenvector of U and V;
3. the kernel of H_n is spanned by φ and its translates:

$$\ker H_n = \mathrm{span}\{\varphi, T\varphi, \ldots, T^{q-1}\varphi\}. \tag{8}$$

From the commutation relation between T and V, it follows that each $T^k\varphi$ is an eigenvector of V with a distinct eigenvalue. Hence the set in (8) is an orthonormal basis.

Remarkably, the assumption (8) for a fixed n, and $L = qn$, by itself, implies the existence of $n_{\max}$ assumed for the family of n-particle Hamiltonians,: $n_{\max} = L/q$.

Proposition

1. *If $qn = L$, any eigenvalue of H_n is at least q-fold degenerate.*
2. *The collection of vectors* (8) *are orthonormal and the 'orbital' occupation numbers $N_j := a_j^\dagger a_j$ satisfy*

$$\max_j \langle \varphi, N_j \varphi \rangle \leq \sum_{j=1}^{q} \langle \varphi, N_j \varphi \rangle = 1. \tag{9}$$

3. *If $n = L/q \geq q$, we have* $\ker H_{n+1} = \{0\}$.

Since $P_{n+1} = 0$, the matrix elements of the Gram matrix (7) become

$$G_{j,\alpha;k,\beta} = \langle a_j^\dagger \varphi_\alpha,\ a_k^\dagger \varphi_\beta \rangle$$

In other words: with $n = n_{\max}$, (6) becomes

$$\text{gap}\, H_{n+1} \geq \frac{1}{n+1-m}\left(n+1-\left\|G^{(n)}\right\|\right) \text{gap}\, H_n. \tag{10}$$

with

$$G^{(n)} = \left(\langle a_j^\dagger \varphi_\alpha,\ a_k^\dagger \varphi_\beta \rangle\right),$$

where $\{\phi_\alpha\}$ is an onb of $\ker H_n$.

Analysis of the Gram matrix

To turn the gap comparison inequality into some thing useful, we need to estimate the operator norm of the $qL \times qL$ Gram matrix:

$$G^{(n)}_{j,\alpha;k,\beta} = \langle a_j^\dagger \varphi_\alpha,\ a_k^\dagger \varphi_\beta \rangle = \delta_{j,k}\delta_{\alpha,\beta} - \langle a_k \varphi_\alpha,\ a_j \varphi_\beta \rangle = \delta_{j,k}\delta_{\alpha,\beta} - \langle \varphi_\alpha, a_k^\dagger a_j \varphi_\beta \rangle.$$

The diagonal elements are bounded by 1.We expect this matrix be diagonally dominated and we can show this in several cases. In general, it also has a block structure that allows for a useful norm bound in interesting cases.
In particular, if n itself is also a multiple of q, one observes that the Gram matrix decomposes as a direct sum of q $q \times q$ matrices.

Theorem (Lemm-N-Warzel-Young, arxiv:2410.11645)

For the case $L = nq$, $n = n_{\max}$, if $\|G\| \leq 2$, then

$$\text{gap}\, H_{n+1} \geq \frac{n-1}{n+1-m}\ \text{gap}\, H_n \geq \text{gap}\, H_n.$$

Generalization to p/q filling fractions for $L = \ell q^2$

We can consider more general Hamiltonians and more general filling fractions.

$$H = \sum_{m=m_0}^{m_1} H^{(m)}, \quad H^{(m)} = \sum_{\substack{1 \le j_1 \ldots j_m \le L \\ 1 \le k_1 \ldots k_m \le L}} W_{j_1 \ldots j_m}^{k_1 \ldots k_m} a_{j_1}^\dagger \ldots a_{j_m}^\dagger a_{k_m} \ldots a_{k_1}$$

where $m_0 \le m_1$ and the m-body coefficients $W_{j_1 \ldots j_m}^{k_1 \ldots k_m} \in \mathbb{C}$ are such that all $H^{(m)}$ are s.a. and satisfy $H^{(m)} \ge 0$, for $m > m_0$, and the symmetry assumptions.

Theorem (Lemm-N-Warzel-Young, arxiv:2410.11645)

Let $L = \ell q^2$ for some $\ell \ge 3$, and $n_{\max} = (p/q)L$, with $p < q$ relative prime. the charge gap dominates the neutral gap as follows: for any $n \ge n_{\max}$

$$\text{gap}\, H_{n_{\max}+1} \ge \frac{n_{\max}}{n_{\max} + 1 - m_0}\ \text{gap}\, H_{n_{\max}} \qquad \textit{(fermions)}$$

$$\text{gap}\, H_{n_{\max}+1} \ge \frac{n_{\max} - p}{n_{\max} + 1 - m_0}\ \text{gap}\, H_{n_{\max}} \qquad \textit{(bosons)}$$

Generalization to p/q filling fractions for $L = \ell q^2$

We can consider more general Hamiltonians and more general filling fractions.

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Infinite system gaps. A definition

Consider the CAR algebra $\mathcal{A}$ on $\ell^2(\mathbb{Z})$ and suppose ω is a thermodynamic limit of maximally filled states $\psi_n \in \ker H_n$ for finite system on intervals $[1, L]$ as discussed above. Suppose the Hamiltonians define a strongly continuous dynamics τ_t on $\mathcal{A}$ with generator δ. Recall, for a states ω for which GNS Hamiltonian has a one-dimensional kernel, the g.s. gap γ is the largest constant such that

$$\omega(A^*\delta(A)) \geq \gamma\omega(A^*A), \text{ for all } A \text{ with } \omega(A) = 0.$$

Define two subsets of $\operatorname{dom}\delta$ of as follows:

$$\begin{aligned} D_0 &= \{A \in \operatorname{dom}\delta \mid NA = AN, \omega(A) = 0, \omega(A^*A) = 1\} \\ D_1 &= \{A \in \operatorname{dom}\delta \mid NA = A(N + \mathbb{1}), \omega(A) = 0, \omega(A^*A) = 1\}. \end{aligned}$$

In this language, the gap comparison states the following:

$$\inf_{A \in D_1} \omega(A^*\delta(A)) \geq \inf_{A \in D_0} \omega(A^*\delta(A)).$$

Comments and open problems

- To prove charge gap $\geq$ neutral gap we used remarkably little information about the Hamiltonians.
- The inequality holds in the thermodynamic limit.
- Next: learn more about $\|G\|$, using its relation to correlation functions of the ground states.
- Next: use the inductive criterion to prove an absolute lower bound on the gap (as has been done for the Kac master equation in Carlen-Carvalho-Loss 2003).

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