Title: Lecture - Strong Gravity, PHYS 777

Speakers: William East

Collection/Series: Strong Gravity (Elective), PHYS 777, February 24 - March 28, 2025

Subject: Strong Gravity

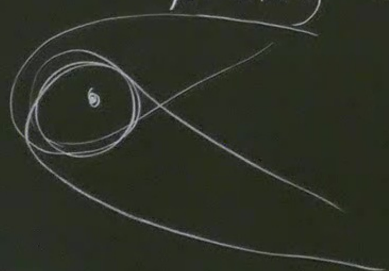
Date: February 27, 2025 - 10:15 AM

URL: https://pirsa.org/25020020

Geodesics on Kerr

Radial potential $-\frac{1}{2}(u^r)^2 = (1-\tilde{E}^2) - 2M\left(\frac{1}{r}\right) + \left[a^2(1-\tilde{E}^2) + \tilde{J}^2\right]\Big($

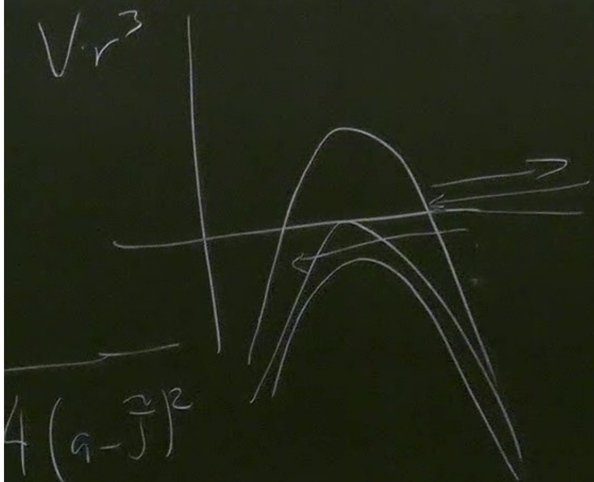
Marginally bound (parabolic) orbit $\tilde{E} = 1$



$$V = -M\left(\frac{1}{r^3}\right)\left[r^2 - \frac{\tilde{J}^2}{2M} r + (a-\tilde{J})^2\right]$$

Turning points $r_p = \dfrac{\frac{\tilde{J}^2}{2M} \pm \sqrt{\left(\tilde{J}^2/2M\right)^2 - 4(a-\tilde{J})^2}}{2}$

$$\tilde{E}^2) + \tilde{J}^2] \left(\frac{1}{r^2}\right) - 2M(a\tilde{E} - \tilde{J})^2 \left(\frac{1}{r^3}\right) := V(\tilde{E}, \tilde{J}, r)$$



$$\tilde{J}_m = 2M\left(1 + \sqrt{1 - \bar{a}}\right)$$

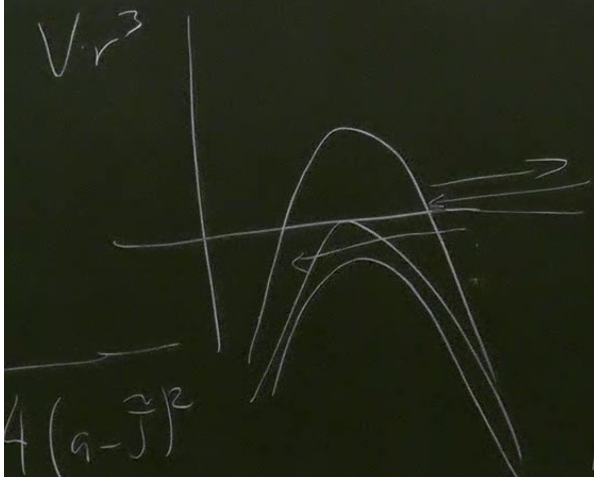
$$r_p = \frac{\tilde{J}^2}{4M} = M\left(2 - \bar{a} + 2\sqrt{1 - \bar{a}}\right)$$

For $\bar{a} = -1$ $\quad r_p = M(3 + 2\sqrt{2}) \approx 5.8M$

$\bar{a} = 0$ $\quad r_p = 4M$

$\bar{a} = 1$ $\quad r_p = M$

$$\ldots \tilde{E}^2) + \tilde{J}^2\big]\left(\frac{1}{r^2}\right) - 2M(a\tilde{E} - \tilde{J})^2\left(\frac{1}{r^3}\right) := V(\tilde{E}, \tilde{J}, r)$$



$$\tilde{J}_m = 2M\left(1 + \sqrt{1 - \bar{a}}\right)$$

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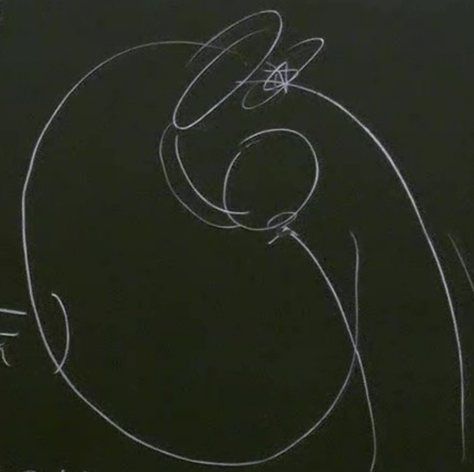


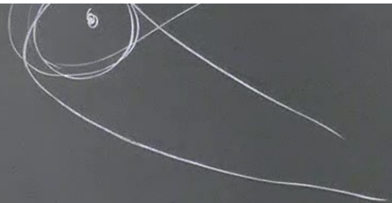
For $\bar{a} = -1$ $\quad r_p = M(3 + 2\sqrt{2}) \approx 5.8M$

$r_E = 2M$ $\quad \bar{a} = 0 \quad r_p = 4M$

$\bar{a} = 1 \quad r_p = M$

$r_p < 2M \quad \bar{a} > 2\sqrt{1 - \bar{a}}$

$\bar{a} \gtrsim 0.91$



$$V = -M\left(\frac{1}{r^3}\right)\left[r^2 - \frac{\tilde{J}^2}{2M} r + (a-\tilde{J})^2\right]$$

Turning points $r_p = \dfrac{\dfrac{\tilde{J}^2}{2M} \pm \sqrt{\left(\tilde{J}^2/2M\right)^2 - 4(a-\tilde{J})^2}}{2}$

(A) Penrose Process

$$p^a = m v^a$$

Obs n^a

timelike

$$E_n = -n_a p^a > 0$$

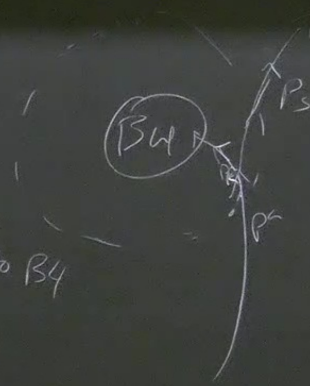
$$E_K = -\xi^a p_a \quad \text{can be} < 0$$

(1) Initial momentum p^a

(2) $p^a = p_s^a + p_R^a$ falls into BH

(3) $E_R = -\xi_a p^a < 0$

(4) $E_s = E - E_R > E$



$m=1$

$$p^a = \frac{dt}{d\tau}\left(1\ ,\ 0, 0, \Omega\right)$$

$$\Omega := \frac{d\phi/d\tau}{dt/d\tau}$$

$$p_a p^a = -m^2 = -1$$

$$= \left(\frac{dt}{d\tau}\right)^2 \left(g_{tt} + g_{\phi\phi}\Omega^2 + 2\Omega g_{t\phi}\right) = -1 \qquad (*)$$

$\tilde{E} = 1$

$$\frac{d\tau}{dt} = -\left(g_{tt} + g_{t\phi}\Omega\right) \qquad (**)$$

$$\left(g_{\phi\phi} + g_{t\phi}^2\right)\Omega^2 + 2 g_{t\phi}\left(1 + g_{tt}\right)\Omega +$$

$-E_R > E$

$(g_{\phi\phi} + g_{t\phi}^2)\Omega^2 + 2g_{t\phi}(1+g_{tt})\Omega + g_{tt}(1+g_{tt}) = 0$

$$\rho_R^a = C_R(1, 0, 0, \Omega_R)$$

$$\rho_S^a = C_S(1, 0, 0, \Omega_S)$$

② $\Rightarrow$ $\frac{dt}{d\tau} = C_R + C_S$ t-comp.

$\frac{dt}{d\tau}\Omega = C_R\Omega_R + C_S\Omega_S$ ϕ-comp.

$$\Rightarrow \quad C_S = \frac{dt}{d\tau}\frac{(\Omega - \Omega_R)}{(\Omega_S - \Omega_R)}$$

$$E_S = -g_{ab}\hat{t}^a\hat{\rho}_S^b = -C_S(g_{tt} + g_{t\phi}\Omega_S)$$

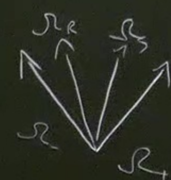
$$= \left(\frac{g_{tt} + g_{t\phi}\Omega_S}{g_{tt} + g_{t\phi}\Omega}\right)\left(\frac{\Omega - \Omega_R}{\Omega_S - \Omega_R}\right)$$

$-E_R > E$

$$(g_{\phi\phi} + g_{t\phi}^2)\Omega^2 + 2g_{t\phi}(1+g_{tt})\Omega + g_{tt}(1+g_{tt}) = 0$$

$$\rho_R^a = C_R(1, 0, 0, \Omega_R)$$

$$\rho_S^a = C_S(1, 0, 0, \Omega_S)$$



② $\Rightarrow$ $\frac{dt}{d\tau} = C_R + C_S$ t-comp.

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$$\Rightarrow \quad C_S = \frac{dt}{d\tau}\frac{(\Omega - \Omega_R)}{(\Omega_S - \Omega_R)}$$

$$E_S = -g_{ab}\hat{t}^a \hat{\rho}_S^b = -C_S(g_{tt} + g_{t\phi}\Omega_S)$$

$$= \left(\frac{g_{tt} + g_{t\phi}\Omega_S}{g_{tt} + g_{t\phi}\Omega}\right)\left(\frac{\Omega - \Omega_R}{\Omega_S - \Omega_R}\right)$$

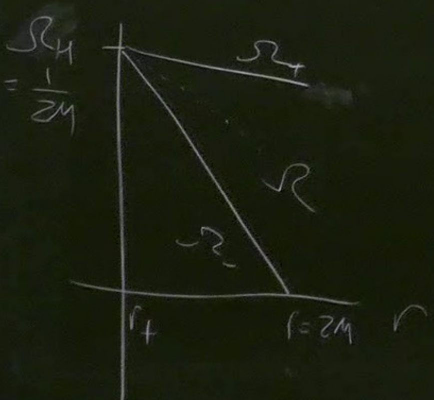
$$\Omega_\pm = \frac{-g_{t\phi} \pm \sqrt{g_{t\phi}^2 - g_{tt}}}{g_{\phi\phi}}$$

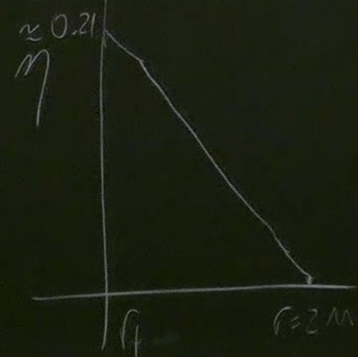
$$\Omega_- < \Omega_R < \Omega_S < \Omega_+$$

$\Omega_H = \frac{\bar a}{2 r_+}$

$$\eta = \frac{E_s - E}{E}$$

$\bar a \to 1 \;,\; \Omega_S \to \Omega_+ \;,\; \Omega_R \to \Omega_-$





$\chi_a = \hat t_a + \Omega_H \hat\phi_a$

$\Omega_H > 0$

$p^a \chi_a < 0$

$= \underbrace{-\delta E}_{<0} + \Omega_H \delta J < 0$

$\delta E < 0 \to \delta J < 0$

BH Thermodynamics

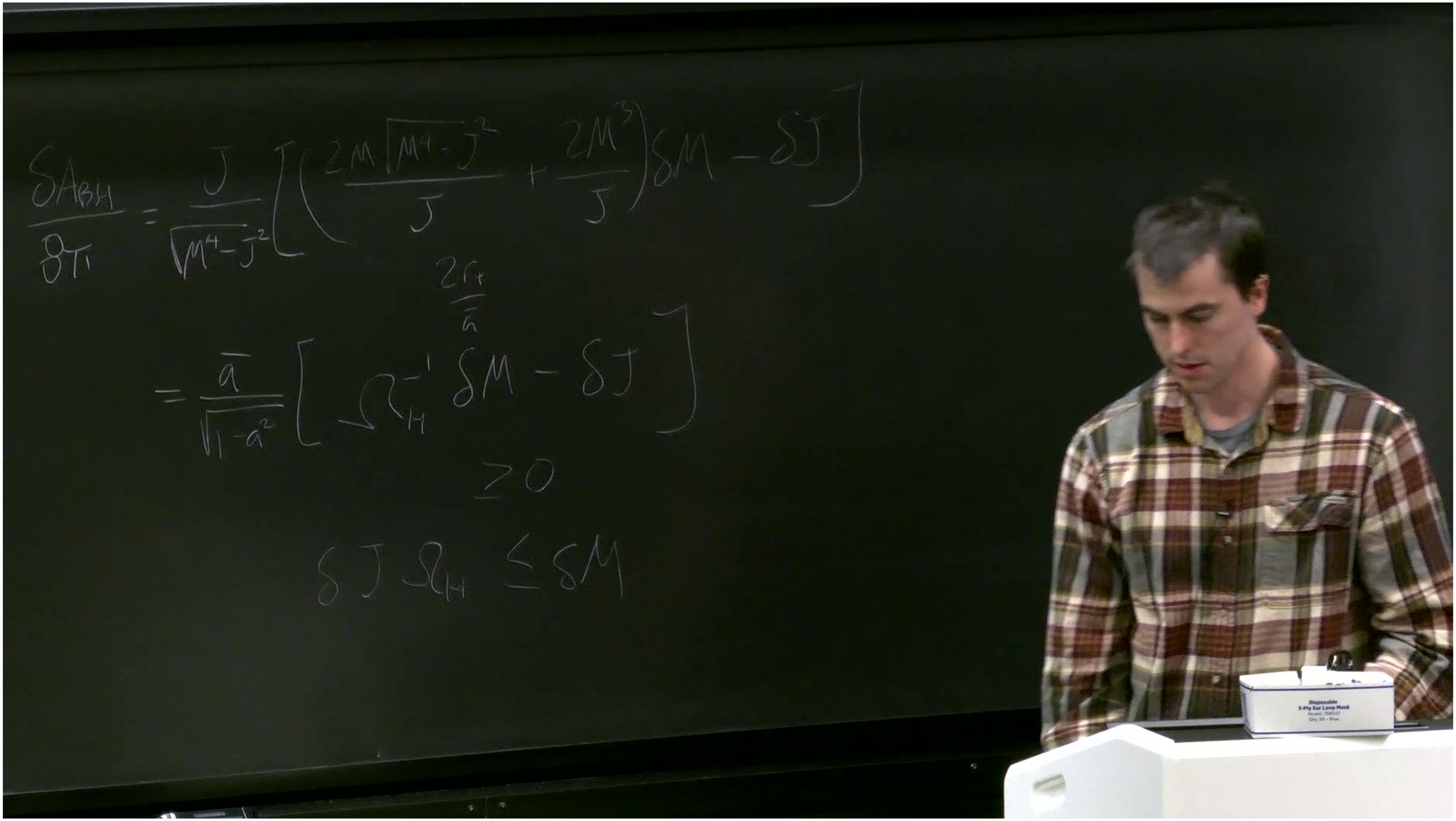
Hawking (1971) BH Area Thm. $\delta A_{BH} \geq 0$

assuming CC, Null energy condition

$$A_{BH} = \int_{r=r_+} \sqrt{|\underbrace{g_{\theta\theta} g_{\phi\phi}}_{(r_+^2+a^2)\sin^2\theta}|}\, d\theta d\phi$$

$$= 4\pi(r_+^2 + a^2) = 8\pi\left[M^2 + \sqrt{M^4 - J^2}\right]$$

$$\frac{\delta A_{BH}}{8\pi} = \frac{J}{\sqrt{M^4 - J^2}}\Big[$$



$\frac{\delta A_{BH}}{8\pi} = \frac{J}{\sqrt{M^4 - J^2}} \left[\left(\frac{2M\sqrt{M^4 - J^2}}{J} + \frac{2M^3}{J} \right) \delta M - \delta J \right]$
$= \frac{\bar{a}}{\sqrt{1-\bar{a}^2}} \left[\overset{\frac{2r_+}{\bar{a}}}{\Omega_H^{-1}} \delta M - \delta J \right]$
≥ 0
$\delta J \, \Omega_H \leq \delta M$

$(r_+^2 + a^2)\sin^2\theta$

$$= 4\pi(r_+^2 + a^2) = 8\pi\left[M^2 + \sqrt{M^4 - J^2}\right]$$

$$M_{ir} = \sqrt{\frac{A_{BH}}{16\pi}}$$

$a = 0, \quad M_{ir} = M$

$a = M \quad A_{BH} = 8\pi M_{BH}^2$

$$M_{ir} = \frac{1}{\sqrt{2}} M_{BH}$$

$$E_{rot} = M_{BH} - M_{ir}$$

$$(a=1) \quad = \left(1 - \frac{1}{\sqrt{2}}\right) M_{BH} \approx 0.29\, M_{BH}$$

2nd Law: $\delta A_{BH} \geq 0 \iff \delta S \geq 0$

$$S_{BH} = \frac{A_{BH}}{4\hbar}$$