Title: Lecture - Strong Gravity, PHYS 777

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Subject: Strong Gravity

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Kerr Metric in Boyer-Lindquist Coördinates

$$ds^2 = -\left(\frac{\Delta - a^2\sin^2\theta}{\Sigma}\right)dt^2 - \frac{2a\sin^2\theta\,(r^2+a^2-\Delta)}{\Sigma}\,dt\,d\phi + \left[\frac{(r^2+a^2) - \Delta a^2 \sin^2}{\Sigma}\right.$$

$$\Sigma := r^2 + a^2\cos^2\theta \qquad \Delta = r^2 + a^2 - 2Mr$$

Kerr-Schild Cartesian Coördinates

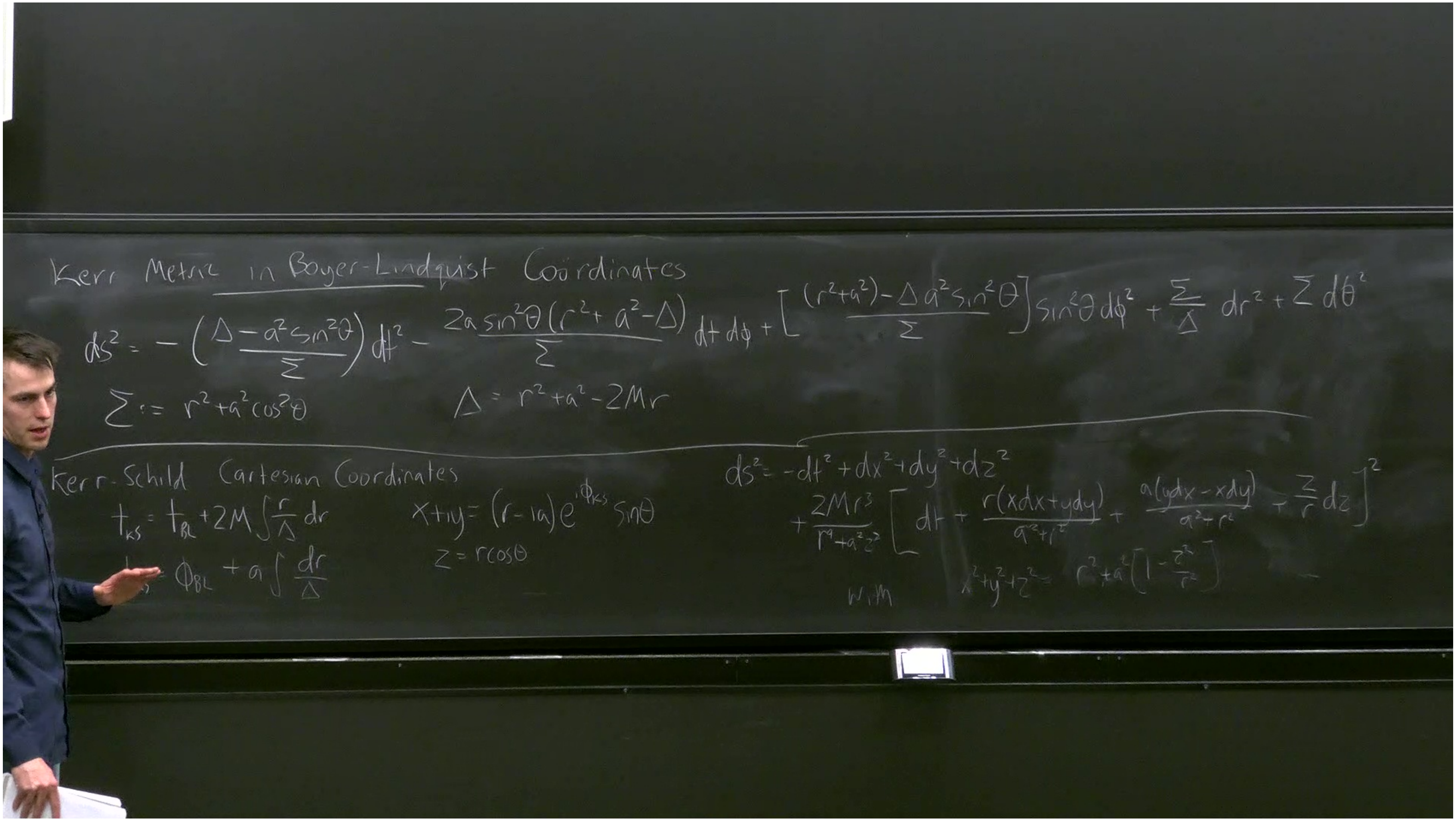
$$t_{KS} = t_{BL} + 2M\int \frac{r}{\Delta}\,dr \qquad x + iy = (r - ia)e^{i\phi_{KS}}\sin\theta$$

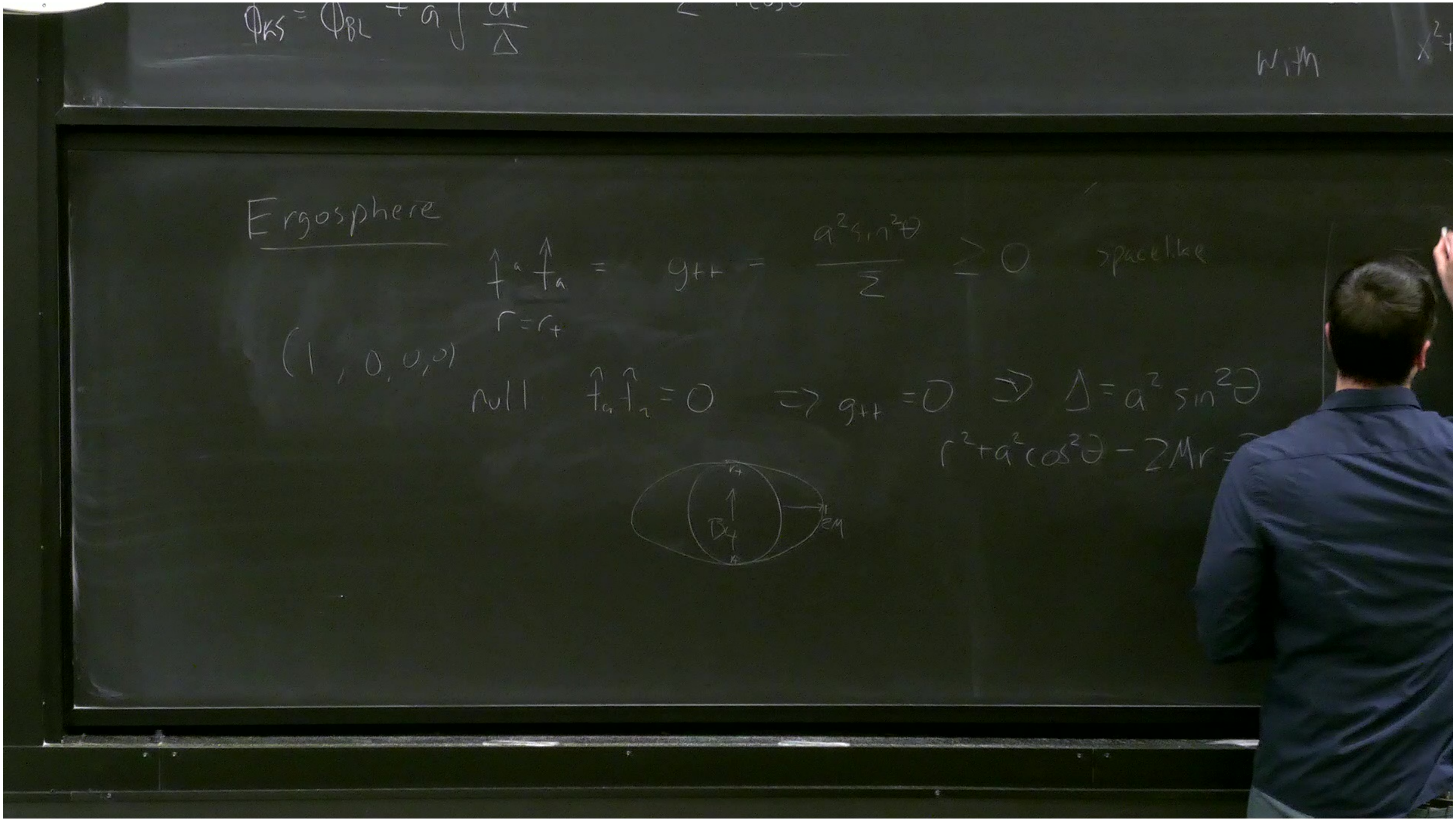
$$\phi_{KS} = \phi_{BL} + a\int\frac{dr}{\Delta} \qquad z = r\cos\theta$$

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz$$

$$+\frac{2Mr^3}{r^4 + a^2z^2}\left[dt + \right.$$

With $x^2 +$


Kerr Metric in Boyer-Lindquist Coördinates
$ds^2 = -\left(\frac{\Delta - a^2\sin^2\theta}{\Sigma}\right)dt^2 - \frac{2a\sin^2\theta(r^2+a^2-\Delta)}{\Sigma}dt\,d\phi + \left[\frac{(r^2+a^2) - \Delta a^2\sin^2\theta}{\Sigma}\right]\sin^2\theta\, d\phi^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma\, d\theta^2$
$\Sigma := r^2 + a^2\cos^2\theta$
$\Delta = r^2 + a^2 - 2Mr$
Kerr-Schild Cartesian Coordinates
$t_{KS} = t_{BL} + 2M\int \frac{r}{\Delta}dr$
$\phi_{KS} = \phi_{BL} + a\int \frac{dr}{\Delta}$
$x + iy = (r - ia)e^{i\phi_{KS}}\sin\theta$
$z = r\cos\theta$
$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$
$+ \frac{2Mr^3}{r^4+a^2z^2}\left[dt + \frac{r(xdx+ydy)}{a^2+r^2} + \frac{a(ydx-xdy)}{a^2+r^2} + \frac{z}{r}dz\right]^2$
with
$x^2+y^2+z^2 = r^2 + a^2\left[1 - \frac{z^2}{r^2}\right]$


$\phi_{KS} = \phi_{BL} + a\int \frac{dr}{\Delta}$
With
Ergosphere
$\hat{t}^a \hat{t}_a = g_{tt} = \frac{a^2 \sin^2\theta}{\Sigma} \geq 0$ spacelike
$r = r_+$
$(1, 0, 0, 0)$
null $\hat{t}_a \hat{t}_a = 0 \Rightarrow g_{tt} = 0 \Rightarrow \Delta = a^2 \sin^2\theta$
$r^2 + a^2\cos^2\theta - 2Mr =$
r_+
$2M$

$$\frac{2Mr^3}{r^4+a^2z^2}\left[dt + \frac{r(xdx+ydy)}{a^2+r^2} + \frac{\ldots}{a^2+r^2} + \frac{z}{r}dz\right]$$

with $x^2+y^2+z^2 = r^2 + a^2\left[1 - \frac{z^2}{r^2}\right]$

$$-1 = g_{ab}u^a u^b = \underbrace{g_{tt}(u^t)^2 + g_{rr}(u^r)^2 + g_{\theta\theta}(u^\theta)^2 + g_{\phi\phi}(u^\phi)^2}_{\geq 0}$$

$$+ 2 g_{t\phi} u^t u^\phi$$
$$< 0$$

$g_{t\phi} < 0 \qquad u^t = \frac{dt}{d\tau} > 0 \qquad u^\phi = \frac{d\phi}{d\tau} > 0$

$\chi_a = \alpha\left(\hat{t}^a + \Omega\hat{\phi}^a\right)$

ZAMO: $\hat{\phi}^a \chi_a = 0$

$g_{t\phi} + g_{\phi\phi}\Omega = 0$

$\Omega = \frac{d\phi}{dt} = \frac{d\phi/d\tau}{dt/d\tau}$

$\Omega = \frac{-g_{t\phi}}{g_{\phi\phi}}$

as $r \to r_+$ $\quad \Omega \to \frac{a}{2r_+} := \Omega_H$

$\chi_a = \hat{t}^a + \Omega_H\hat{\phi}^a$ Killing vector, $\chi_a\chi^a = 0$ at $r = r_+$

For $\bar{a} << 1$

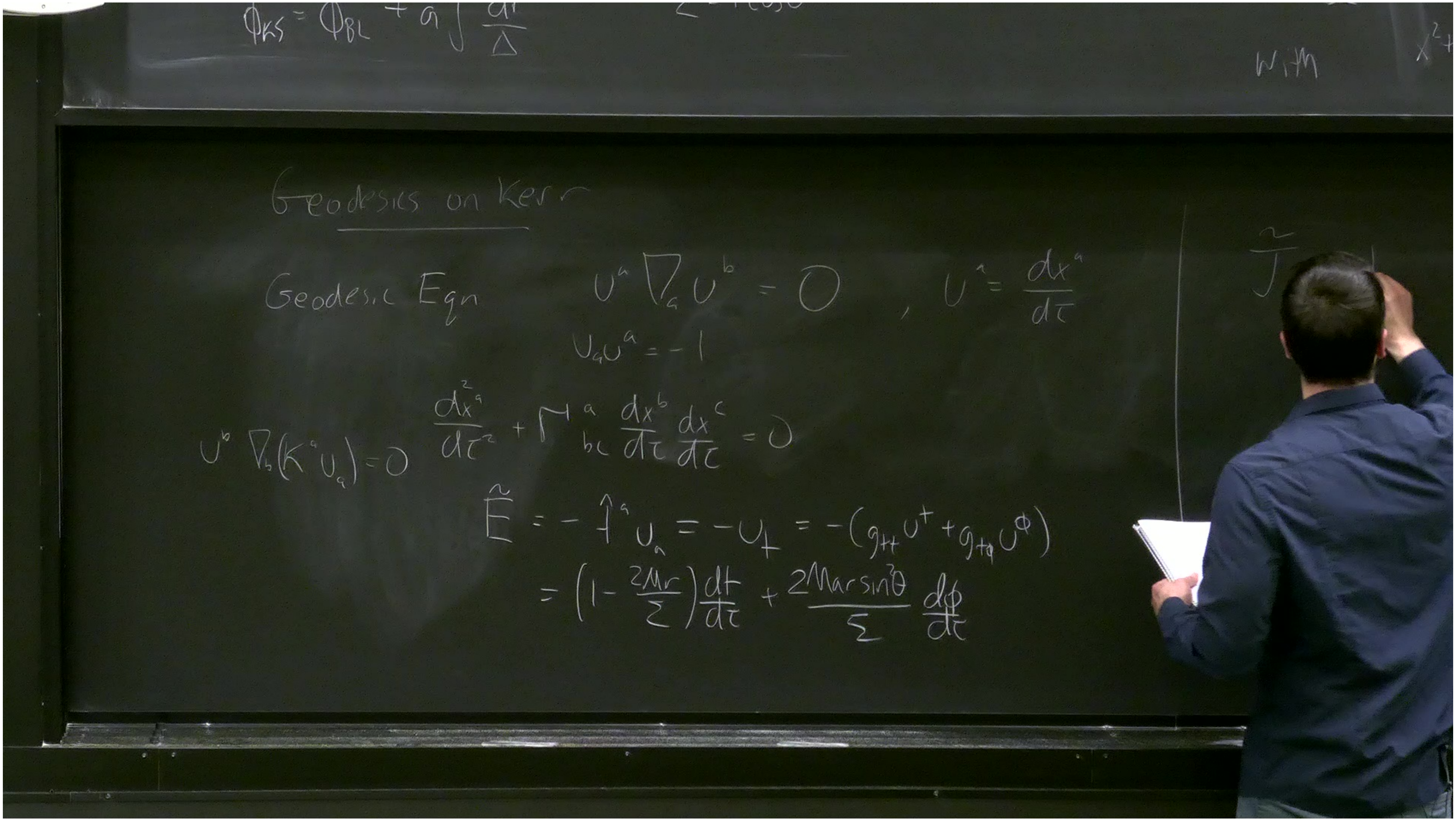
$$r_+ = \left(2 - \frac{1}{2}\bar{a}^2\right)M$$

$$\Omega_H = \frac{\bar{a}}{4M}$$

$\bar{a} = 1 - \epsilon \qquad \epsilon << 1$

$$r_+ = M\left(1 + \sqrt{2\epsilon}\right)M$$

$$\Omega_H = \frac{1 - \sqrt{2\epsilon}}{2M}$$


$\phi_{KS} = \phi_{BL} + a\int \frac{dr}{\Delta}$
With
Geodesics on Kerr
Geodesic Eqn $\quad u^a \nabla_a u^b = 0 \quad , \quad u^a = \frac{dx^a}{d\tau}$
$u_a u^a = -1$
$u^b \nabla_b (K^a u_a) = 0 \qquad \frac{d^2x^a}{d\tau^2} + \Gamma^a{}_{bc} \frac{dx^b}{d\tau}\frac{dx^c}{d\tau} = 0$
$\tilde{E} = -\hat{t}^a u_a = -u_t = -(g_{tt}u^t + g_{t\phi}u^\phi)$
$= \left(1 - \frac{2Mr}{\Sigma}\right)\frac{dt}{d\tau} + \frac{2Mar\sin^2\theta}{\Sigma}\frac{d\phi}{d\tau}$
$\tilde{J}$

$$+\frac{2Mr^3}{r^4+a^2z^2}\left[dt+\frac{r(xdx+ydy)}{a^2+r^2}+\frac{a(ydx-xdy)}{a^2+r^2}+\frac{z}{r}dz\right]$$

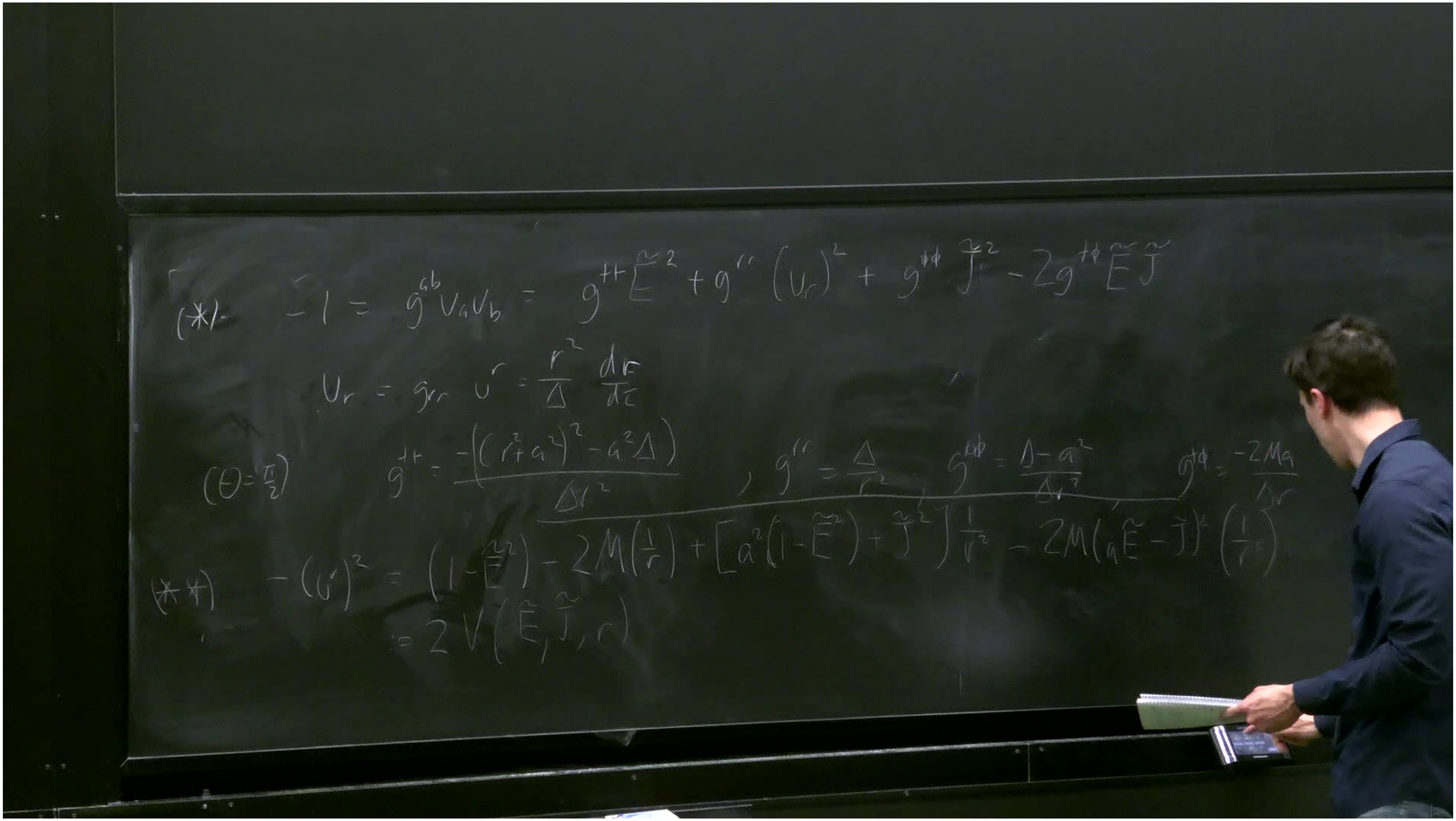
with $x^2+y^2+z^2=r^2+a^2\left[1-\frac{z^2}{r^2}\right]$

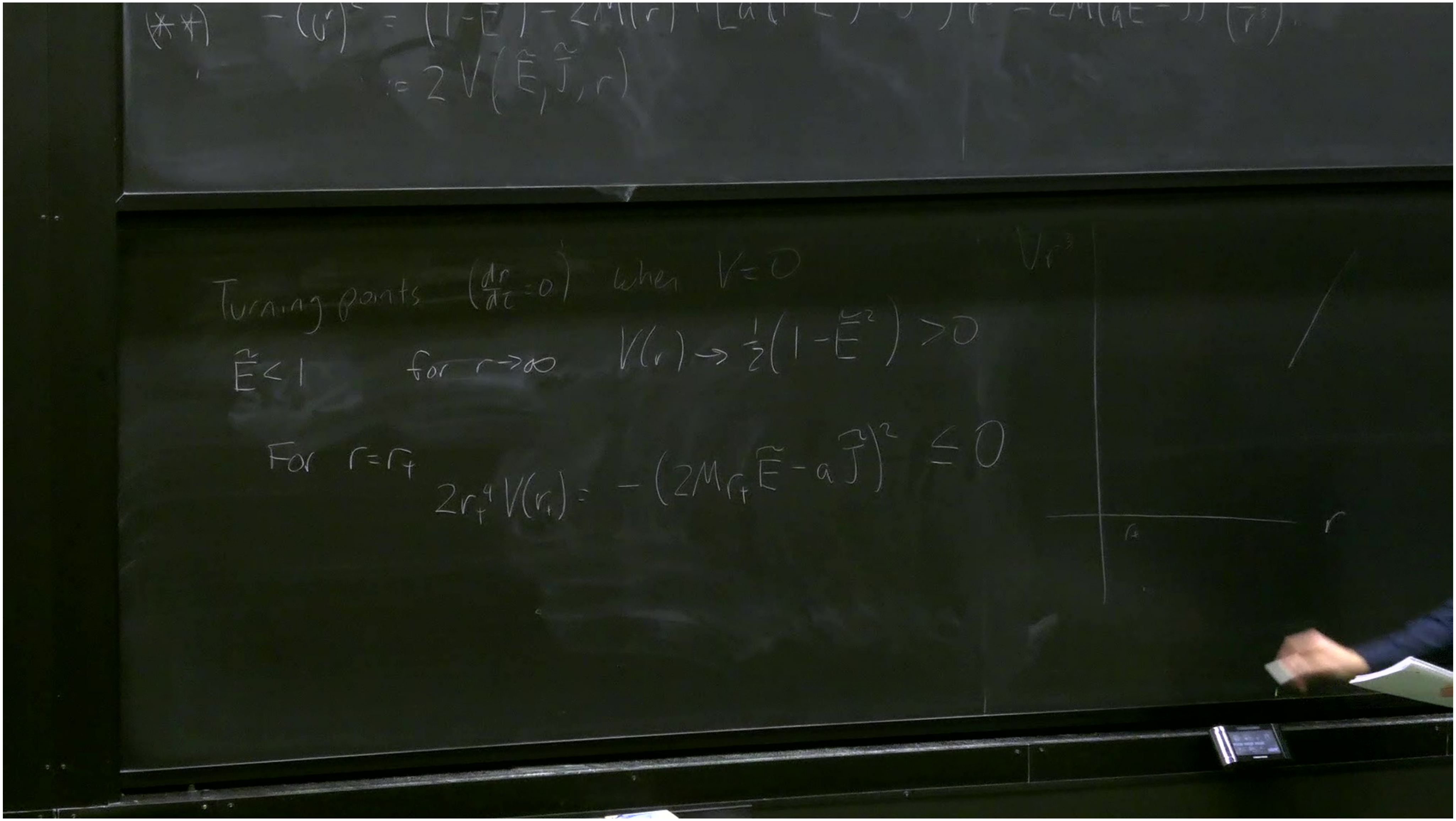
$$\tilde{J}=\hat{\phi}^a u_a=u_\phi=\frac{(r^2+a^2)^2-\Delta a^2\sin^2\theta}{\Sigma}\sin^2\theta\frac{d\phi}{d\tau}-\frac{2Mar\sin^2\theta}{\Sigma}\frac{dt}{d\tau}$$

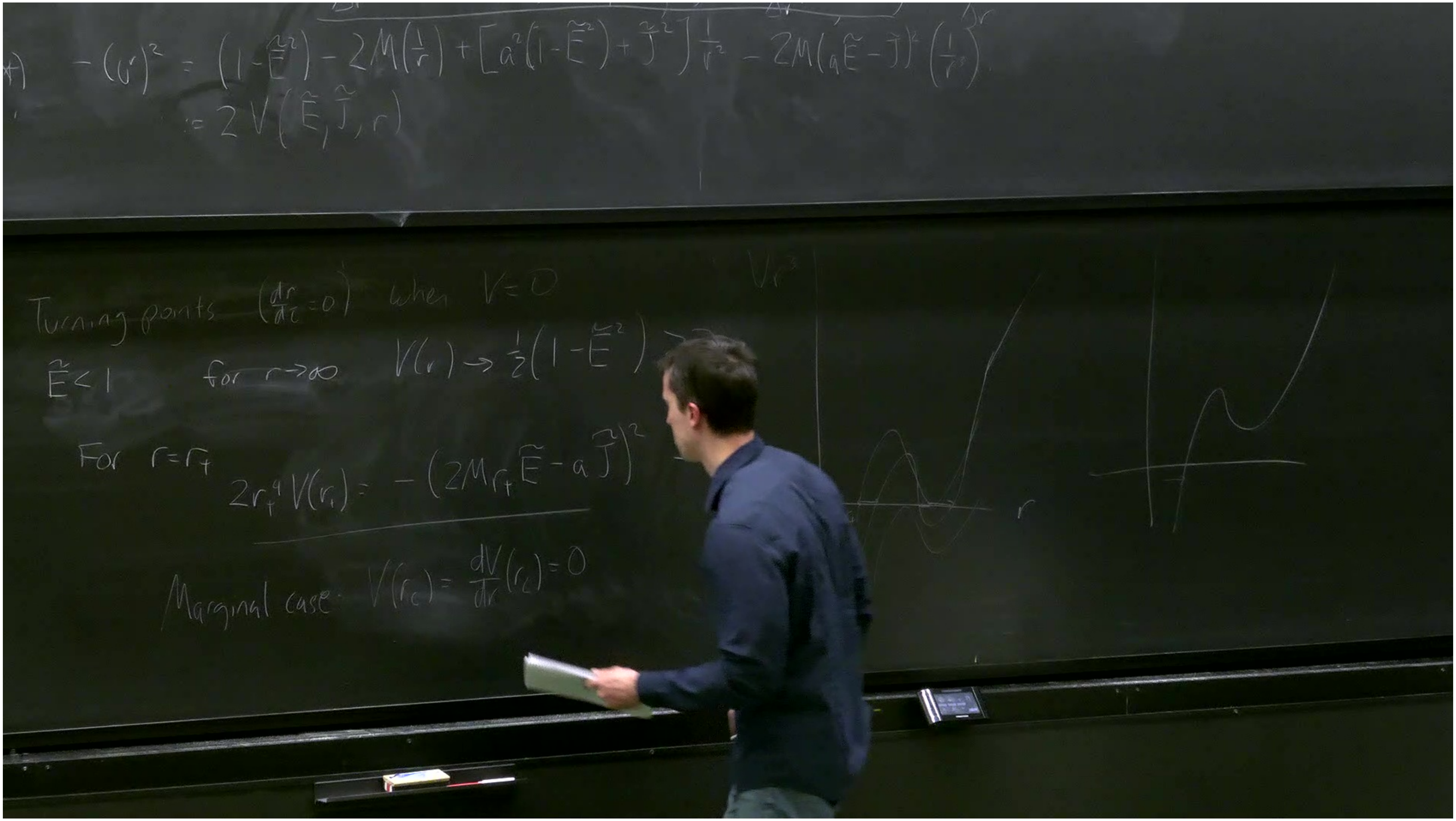
$$C=K_{ab}u^a u^b$$

$$u^a\nabla_a(C)=0$$

Restrict to equatorial geodesics $\theta=\frac{\pi}{2}$, $u^\theta=u_\theta=0$


(*) $-1 = g^{ab} u_a u_b = g^{tt}\tilde{E}^2 + g^{rr}(u_r)^2 + g^{\phi\phi}\tilde{J}^2 - 2g^{t\phi}\tilde{E}\tilde{J}$
$u_r = g_{rr} u^r = \frac{r^2}{\Delta}\frac{dr}{d\tau}$
$(\theta = \frac{\pi}{2})$
$g^{tt} = \frac{-((r^2+a^2)^2 - a^2\Delta)}{\Delta r^2}$, $g^{rr} = \frac{\Delta}{r^2}$, $g^{\phi\phi} = \frac{\Delta - a^2}{\Delta r^2}$, $g^{t\phi} = \frac{-2Ma}{\Delta r}$
(**) $-(u^r)^2 = (1-\tilde{E}^2) - 2M\left(\frac{1}{r}\right) + \left[a^2(1-\tilde{E}^2)+\tilde{J}^2\right]\frac{1}{r^2} - 2M(a\tilde{E}-\tilde{J})^2\left(\frac{1}{r^3}\right)$
$:= 2V(\tilde{E},\tilde{J},r)$


(**)
$:= 2V(\tilde{E}, \tilde{J}, r)$
Turning points $\left(\frac{dr}{d\tau} = 0\right)$ when $V = 0$
$\tilde{E} < 1$ for $r \to \infty$ $V(r) \to \frac{1}{2}(1 - \tilde{E}^2) > 0$
For $r = r_+$
$2r_+^4 V(r_+) = -(2Mr_+\tilde{E} - a\tilde{J})^2 \leq 0$
Vr^3
r_+
r


−(u^r)² = (1−Ẽ²) − 2M(1/r) + [a²(1−Ẽ²) + J̃²] 1/r² − 2M(aẼ − J̃)² (1/r³)
= 2V(Ẽ, J̃, r)
Turning points (dr/dτ = 0) when V = 0
Ẽ < 1 for r → ∞ V(r) → ½(1 − Ẽ²)
For r = r+
2r+⁴ V(r+) = −(2Mr+Ẽ − aJ̃)²
Marginal case: V(r_c) = dV/dr (r_c) = 0
Vr³
r

$2V_+ V(r_+) = \ldots$

Marginal case: $V(r_c) = \frac{dV}{dr}(r_c) = 0 \qquad r_c > r_+$

$$\tilde{E}_c = \frac{1 - \frac{2M}{r_c} + \frac{a}{r_c}\sqrt{\frac{M}{r_c}}}{\sqrt{1 - \frac{3M}{r_c} + \frac{2a}{r_c}\sqrt{\frac{M}{r_c}}}}$$

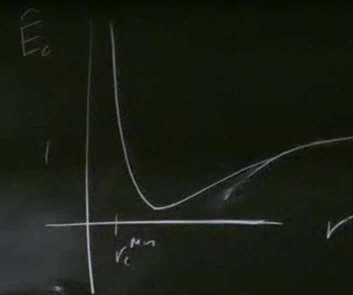
$$\tilde{J}_c = \frac{\sqrt{Mr_c} - 2a\frac{M}{r_c} + \frac{a^2}{r_c}\sqrt{\frac{M}{r_c}}}{\sqrt{1 - \frac{3M}{r_c} + \frac{2a}{r_c}\sqrt{\frac{M}{r_c}}}}$$

$r_c >> M$

$\tilde{E}_c \sim 1 - \frac{M}{2r_c}$

$\tilde{J}_c \sim \sqrt{Mr_c}$

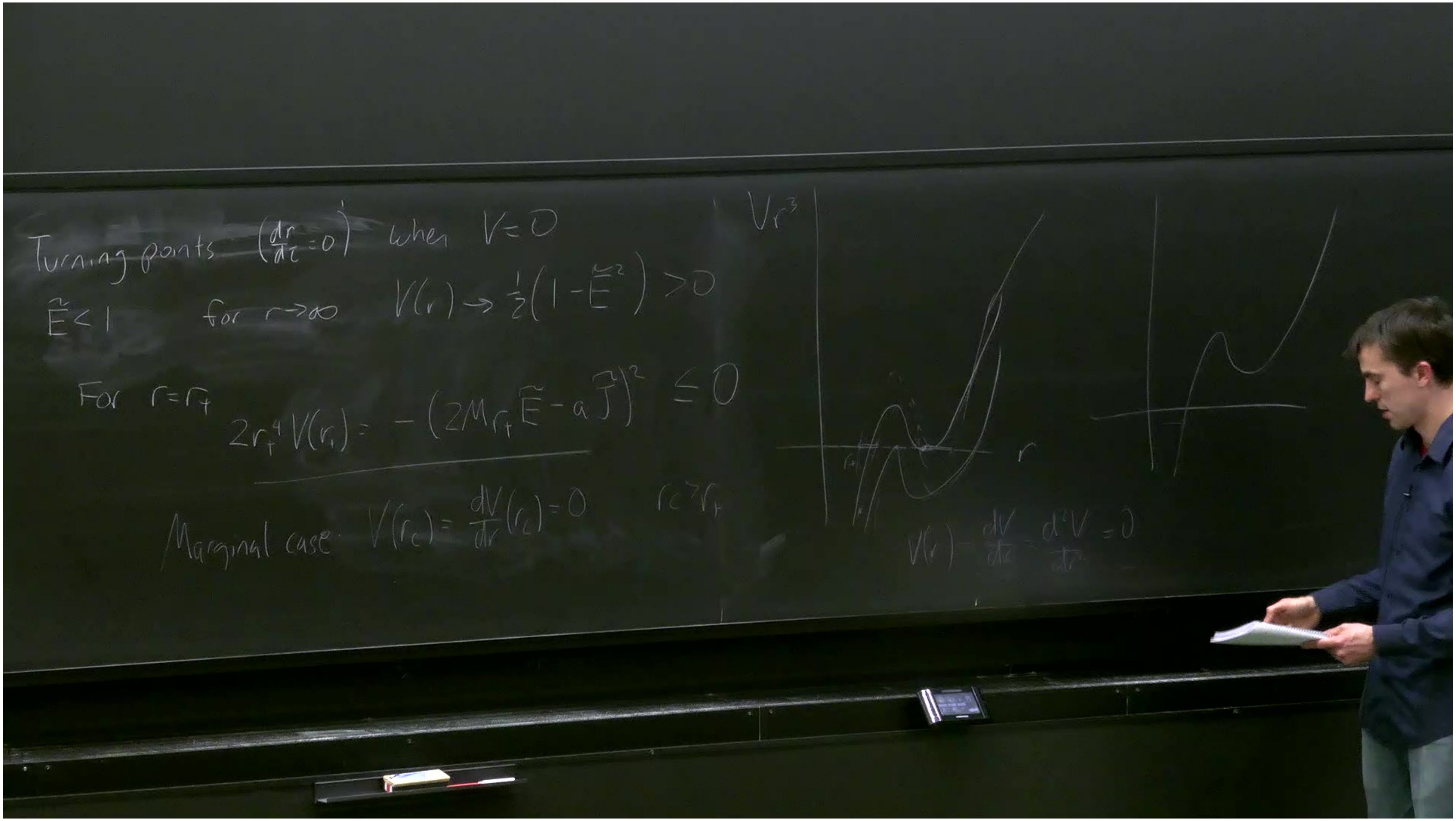
$\tilde{J}_c, \tilde{E}_c \to \infty \qquad r_c^{3/2} - 3Mr_c^{1/2} + 2a\sqrt{M} \to 0.$

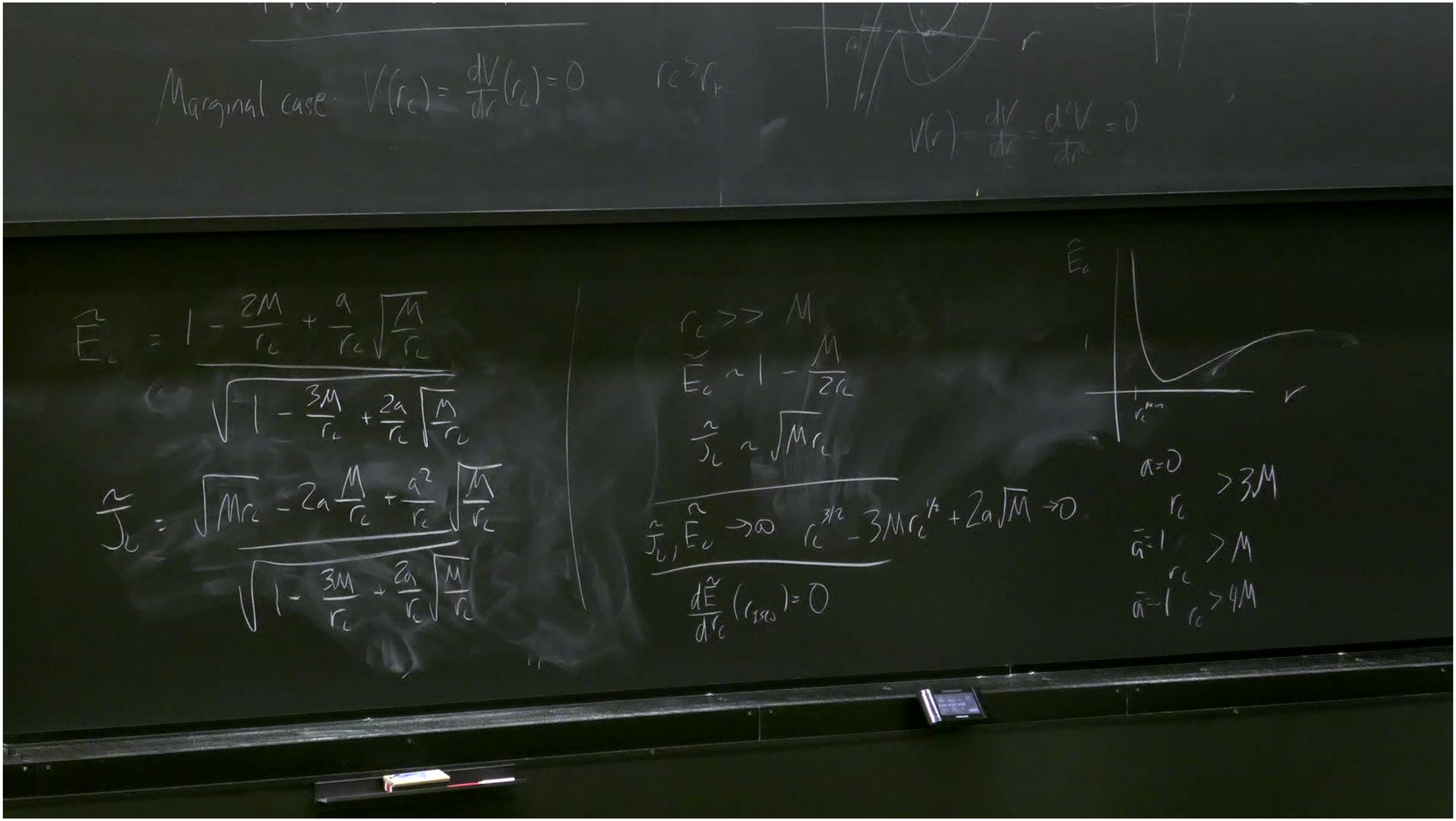


$a = 0 \quad r_c > 3M$

$\tilde{a} = 1 \quad r_c > M$

$\tilde{a} = -1 \quad r_c > 4M$


Turning points $\left(\frac{dr}{d\tau}=0\right)$ when $V=0$
$\tilde{E}<1$ for $r\to\infty$ $V(r)\to\frac{1}{2}(1-\tilde{E}^2)>0$
For $r=r_+$ $2r_+^4 V(r_+) = -(2Mr_+\tilde{E}-a\tilde{J})^2 \le 0$
Marginal case $V(r_c)=\frac{dV}{dr}(r_c)=0$ $r_c>r_+$
Vr^3
r
$V(r)=\frac{dV}{dr}=\frac{d^2V}{dr^2}=0$


Marginal case: $V(r_c) = \frac{dV}{dr}(r_c) = 0$
$r_c > r_+$
$V(r) = \frac{dV}{dr} = \frac{d^2V}{dr^2} = 0$
$\tilde{E}_c = \dfrac{1 - \frac{2M}{r_c} + \frac{a}{r_c}\sqrt{\frac{M}{r_c}}}{\sqrt{1 - \frac{3M}{r_c} + \frac{2a}{r_c}\sqrt{\frac{M}{r_c}}}}$
$\tilde{J}_c = \dfrac{\sqrt{Mr_c} - 2a\frac{M}{r_c} + \frac{a^2}{r_c}\sqrt{\frac{M}{r_c}}}{\sqrt{1 - \frac{3M}{r_c} + \frac{2a}{r_c}\sqrt{\frac{M}{r_c}}}}$
$r_c >> M$
$\tilde{E}_c \sim 1 - \frac{M}{2r_c}$
$\tilde{J}_c \sim \sqrt{Mr_c}$
$\tilde{J}_c, \hat{E}_c \to \infty \quad r_c^{3/2} - 3Mr_c^{1/2} + 2a\sqrt{M} \to 0$
$\frac{d\tilde{E}}{dr_c}(r_{isco}) = 0$
$\hat{E}_c$
1
r
r_c^{min}
$a=0 \quad r_c > 3M$
$\bar{a}=1 \quad r_c > M$
$\bar{a}=-1 \quad r_c > 4M$

$\phi_{KS} = \phi_{BL} + a\int \frac{dr}{\Delta}$

$z = r\cos\theta$

$x^2+y^2+z^2 = r^2+a^2[1-$

$$r_{ISCO}^2 - 6M r_{ISCO} + 8a\sqrt{M}\, r_{ISCO}^{1/2} - 3a^2 = 0$$

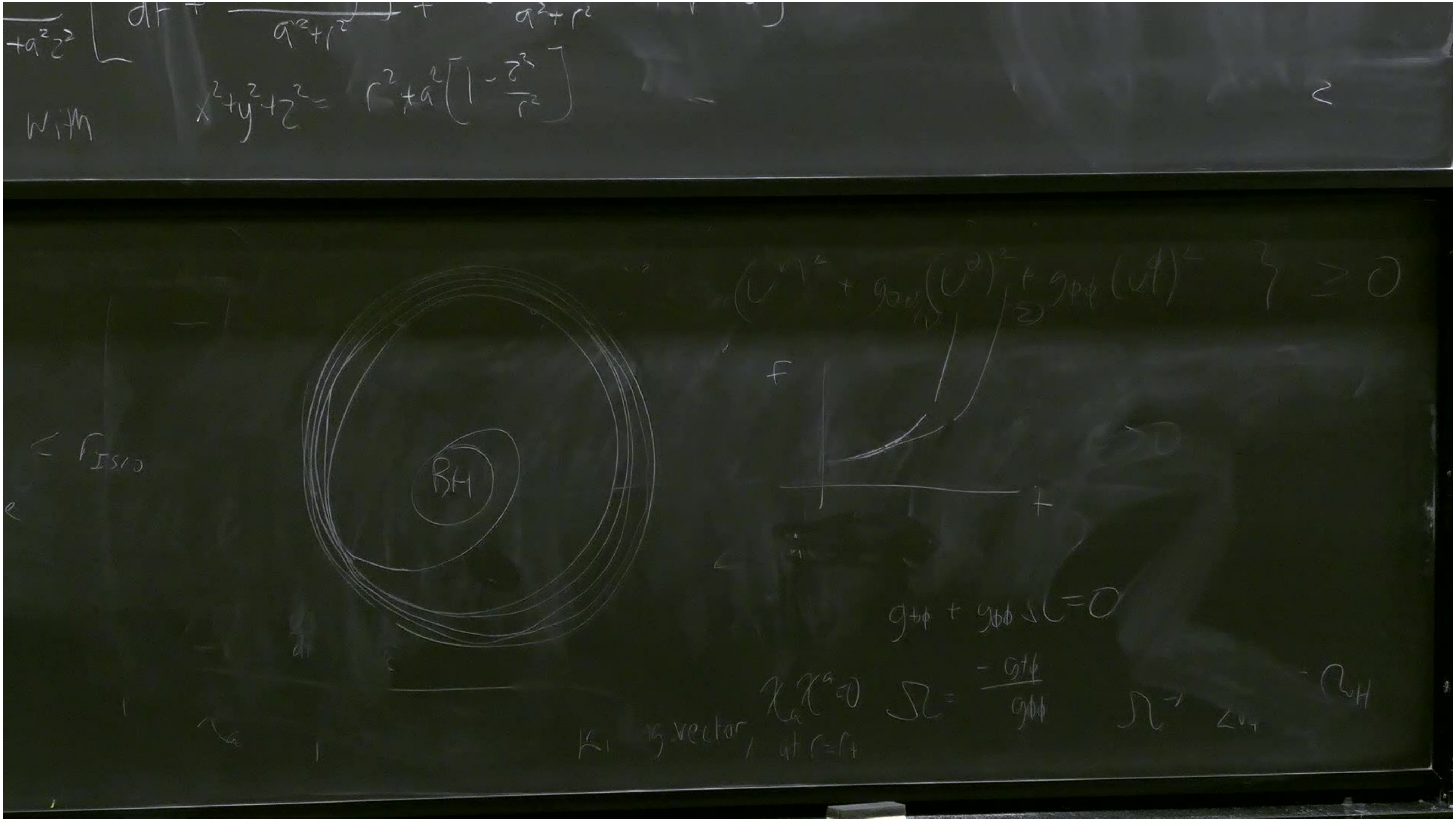
$a=0$, $r_{ISCO} = 6M$

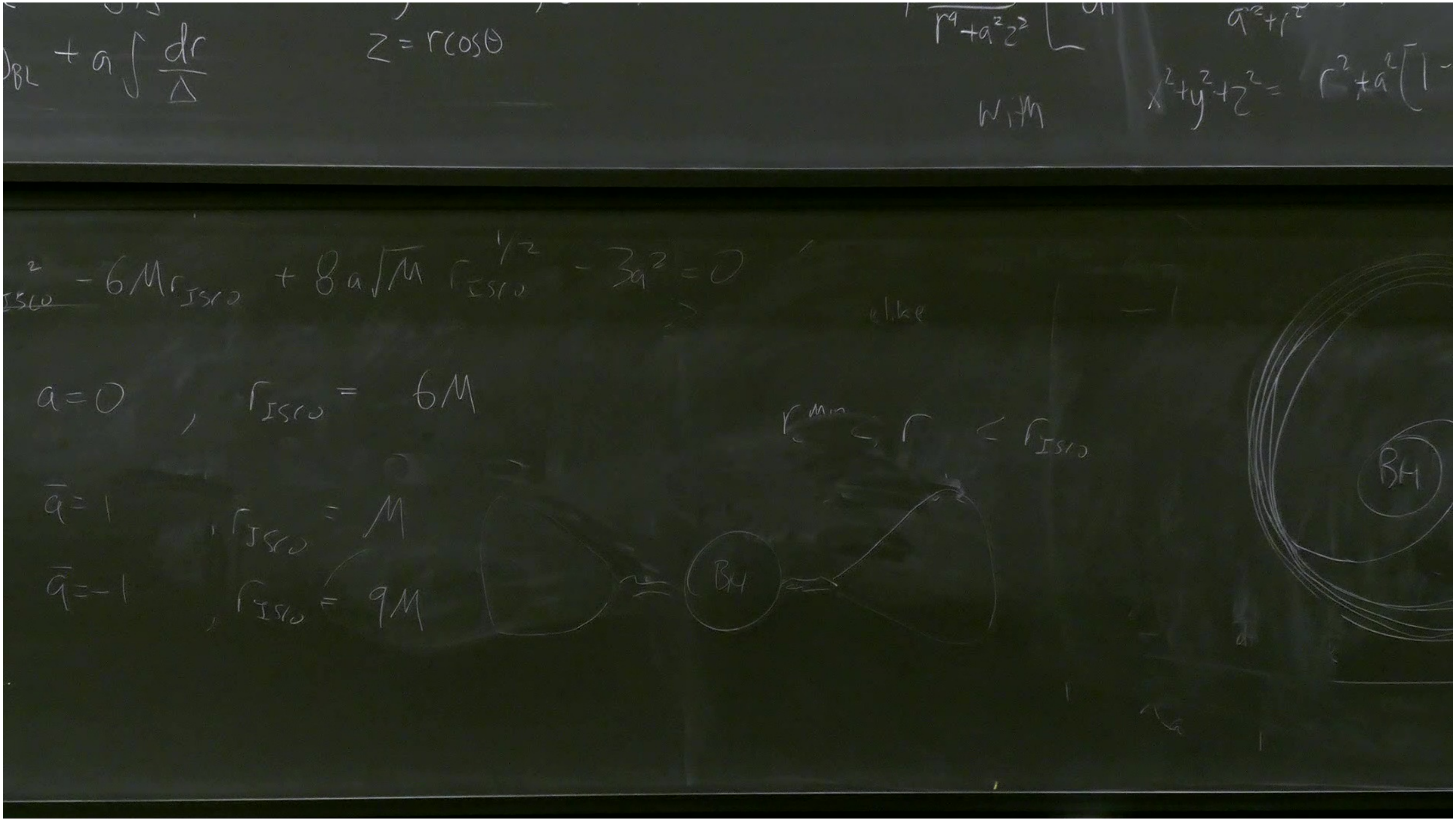
$\bar{a}=1$, $r_{ISCO} = M$

$\bar{a}=-1$, $r_{ISCO} = 9M$

$r_c^{min} < r < r_{ISCO}$

Unstable


with
x^2+y^2+z^2 = r^2+a^2[1 - z^2/r^2]
< r_ISCO
BH
≥ 0
g_tφ + g_φφ Ω = 0
χ_a χ^a = 0
Ω = - g_tφ / g_φφ
vector
at r=r_+


+ a ∫ dr/Δ
z = r cos θ
r^4 + a^2 z^2
with
x^2 + y^2 + z^2 = r^2 + a^2 [1 −
− 6M r_ISCO + 8a √M r_ISCO^1/2 − 3a^2 = 0
a = 0 , r_ISCO = 6M
ā = 1 , r_ISCO = M
ā = −1 , r_ISCO = 9M
r_min ≤ r ≤ r_ISCO
BH
BH