**Title:** Lecture - Strong Gravity, PHYS 777

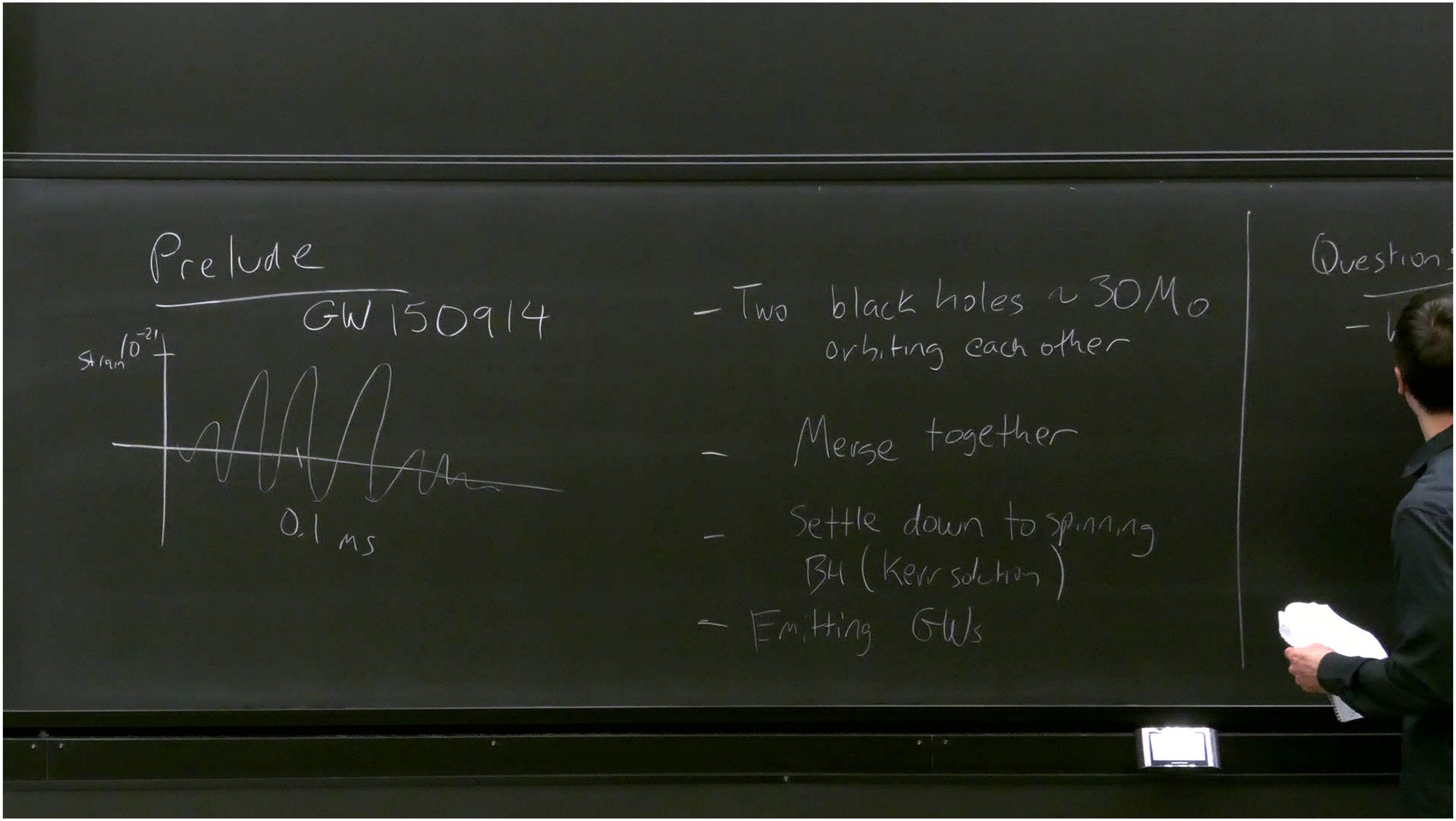
**Speakers:** William East

**Collection/Series:** Strong Gravity (Elective), PHYS 777, February 24 - March 28, 2025

**Subject:** Strong Gravity

**Date:** February 24, 2025 - 2:00 PM

**URL:** https://pirsa.org/25020018

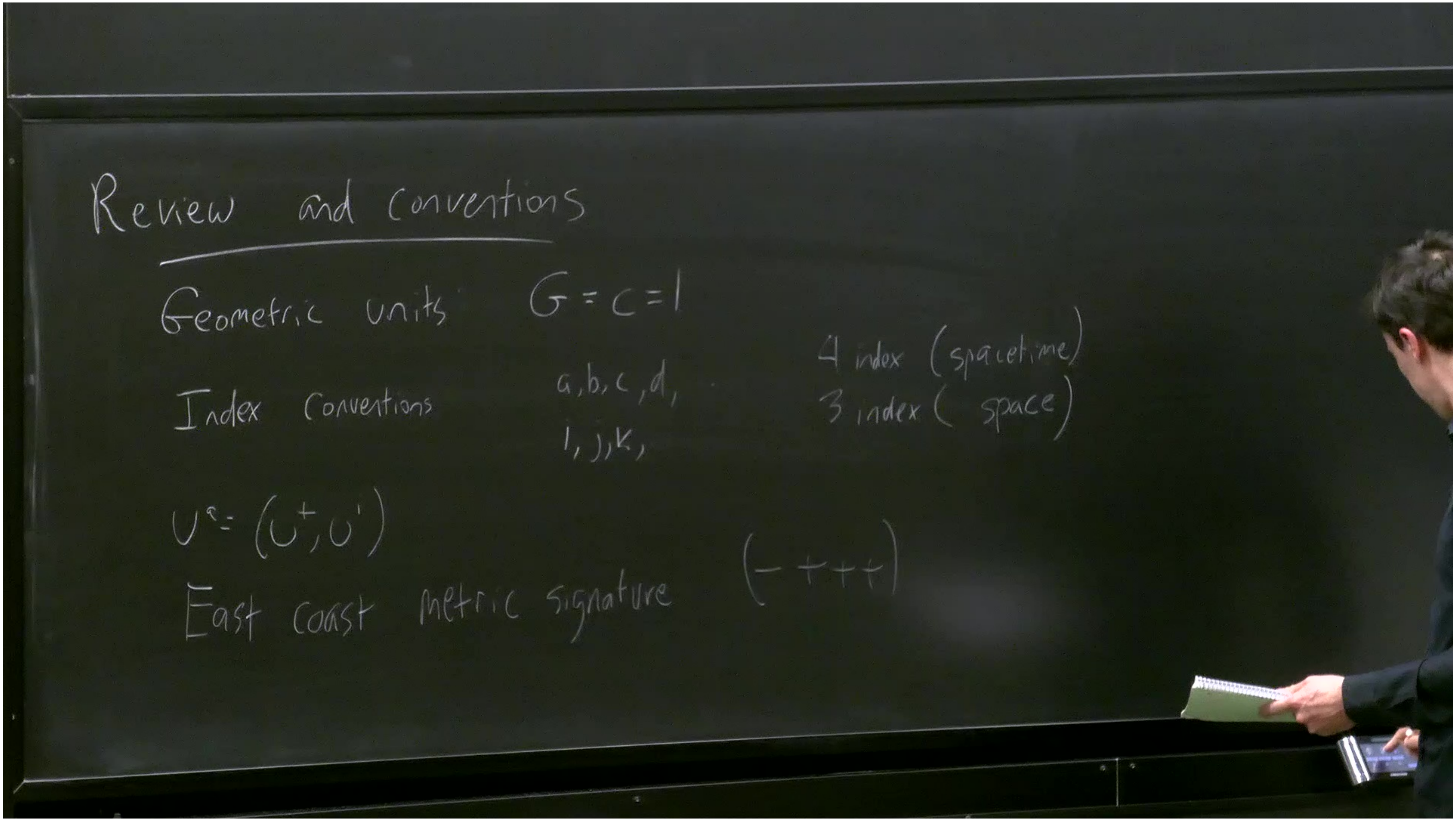

Prelude
GW 150914
Strain
$10^{-21}$
0.1 ms
– Two black holes ~ 30 $M_\odot$
orbiting each other
– Merge together
– Settle down to spinning
BH (Kerr solution)
– Emitting GWs
Questions

$0 M_\odot$

ng

Questions:

- What are the properties of spinning black holes? How gain (or lose) mass? Orbits around them?
- How do we use GR to describe dynamical spacetimes?
- Hows are gravitational waves sourced? How do they propagate?


Review and conventions
Geometric units G = c = 1
Index conventions
a, b, c, d, 4 index (spacetime)
i, j, k, 3 index ( space)
$U^a = (U^t, U^i)$
East coast metric signature
(− + + +)

Metric $ds^2 = g_{ab}\, dx^a dx^b$

Christoffel symbols

$$\Gamma^c{}_{ab} = \frac{1}{2} g^{cd}\left(\partial_a g_{bd} + \partial_b g_{ad} - \partial_d g_{ab}\right)$$

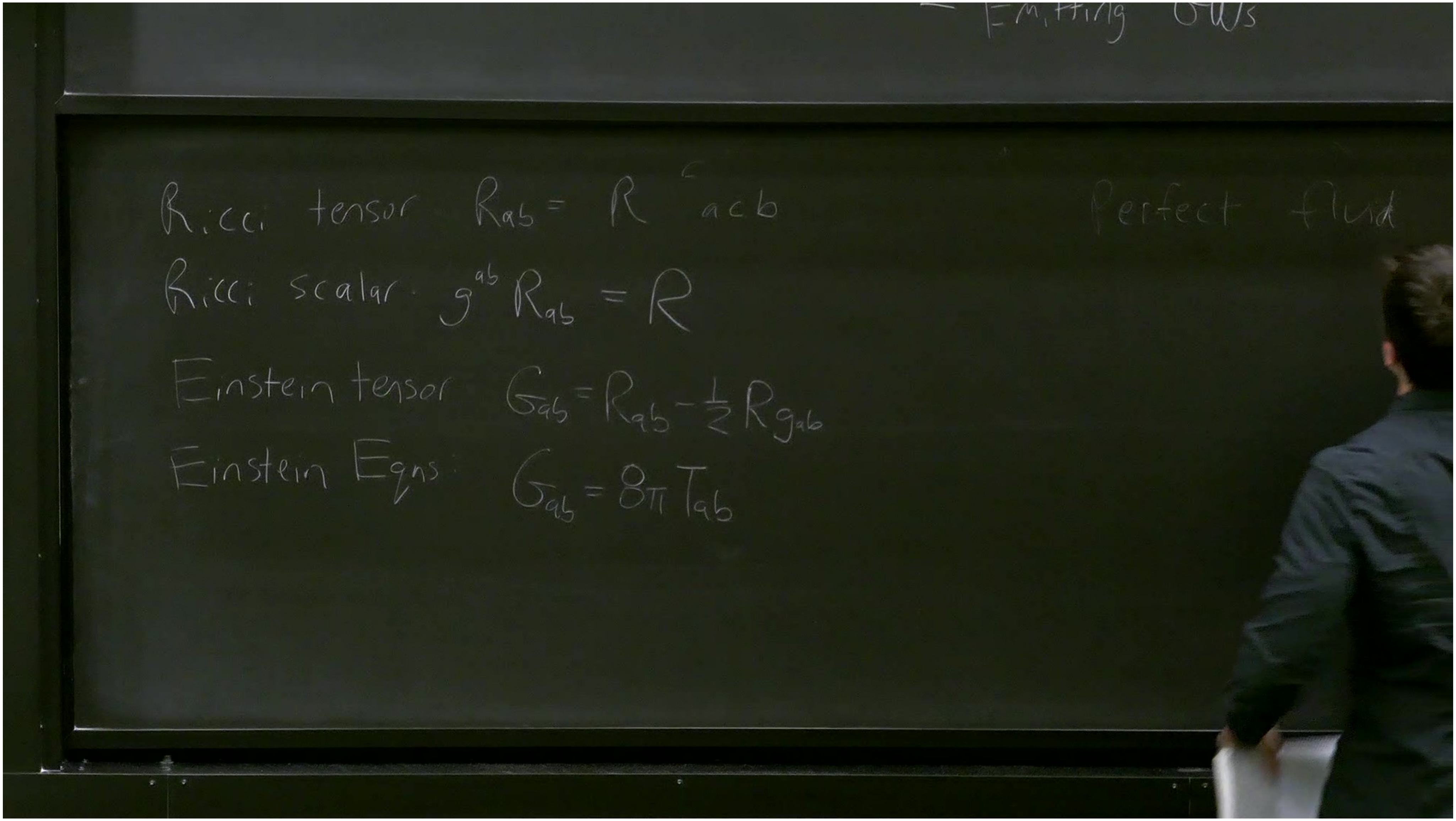
$$\nabla_a T^{b_1 \dots b_n}{}_{c_1 \dots c_n} = \partial_a T^{b_1 \dots b_n}{}_{c_1 \dots c_n} + \Gamma^{b_1}{}_{ad} T^{d b_2 \dots b_n}{}_{c_1 \dots c_n} + \Gamma^{b_2}{}_{ad} T^{b_1 d b_3 \dots b_n}{}_{c_1 \dots c_n} - \Gamma^{d}{}_{c_1 a} T^{b_1 \dots b_n}{}_{d c_2 \dots c_n} - \Gamma^{d}{}_{c_2 a} T^{b_1 \dots b_n}{}_{c_1 d c_3 \dots}$$

Metric compatible

$$\nabla_a g_{bc} = 0$$

$$(\nabla_a \nabla_b - \nabla_b \nabla_a) V^c = R^c{}_{dab} V^d$$

$$R^c{}_{dab} = \partial_a \Gamma^c{}_{bd} - \partial_b \Gamma^c{}_{ad} + \Gamma^c{}_{ae} \Gamma^e{}_{bd} - \Gamma^c{}_{be} \Gamma^e{}_{ad}$$


Emitting GWs
Ricci tensor $R_{ab} = R^{c}{}_{acb}$
Perfect fluid
Ricci scalar: $g^{ab} R_{ab} = R$
Einstein tensor $G_{ab} = R_{ab} - \frac{1}{2} R g_{ab}$
Einstein Eqns: $G_{ab} = 8\pi T_{ab}$

Perfect fluid S-E tensor

$$T_{ab} = (\rho + P) u_a u_b + P g_{ab}$$

S-E conservation

$$\nabla_a T^{ab} = 0$$

S-E tensor

$$\ldots U_a U_b + P g_{ab}$$

...on

$$\ldots = 0$$

Null energy condition

$k^a$ future pointing, null

$$T_{ab} k^a k^b \geq 0$$

$$(\rho + P)(U_a k^a)^2 \geq 0 \text{ for perfect fluid}$$

## Kerr Black Holes

- Only possible stationary, vaccum solution
  (Israel, Carter, Hawking, Robinson 1967-1971)
- Perturbations decay rapidly ( QNM lecture)
- Final state of generic collapse : BH $M + J$ + gravitational waves
  "No hair"

Metric in Boyer-Lindquist Coördinates

$$ds^2 = -\left(\frac{\Delta - a^2\sin^2\theta}{\Sigma}\right)dt^2 - \frac{2a\sin^2\theta(r^2+a^2-\Delta)}{\Sigma}\,dt\,d\phi$$

$$+\left[\frac{(r^2+a^2)^2-\Delta a^2\sin^2\theta}{\Sigma}\right]\sin^2\theta\, d\phi^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma\, d\theta^2$$

$$\Sigma = r^2 + a^2\cos^2\theta$$

$$\Delta = r^2 + a^2 - 2Mr$$

$$a = \frac{J}{M} \quad (\text{total ang mom / mass})$$

$$\bar{a} = \frac{J}{M^2}$$

$a = 0$ , Schwarzschild solution

Fix $\bar{a}$ , $M \to 0$

$$ds^2 = -dt^2 + \left(\frac{r^2 + a^2\cos^2\theta}{r^2 + a^2}\right) dr^2 + \left(r^2 + a^2\cos^2\theta\right) d\theta^2 + \left(r^2 + a^2\right)\sin^2\theta \, d\phi^2$$

$$x = \left(r^2 + a^2\right)^{1/2} \sin\theta \cos\phi$$

$$y = \left(r^2 + a^2\right)^{1/2} \sin\theta \sin\phi$$

$$z = r\cos\theta$$

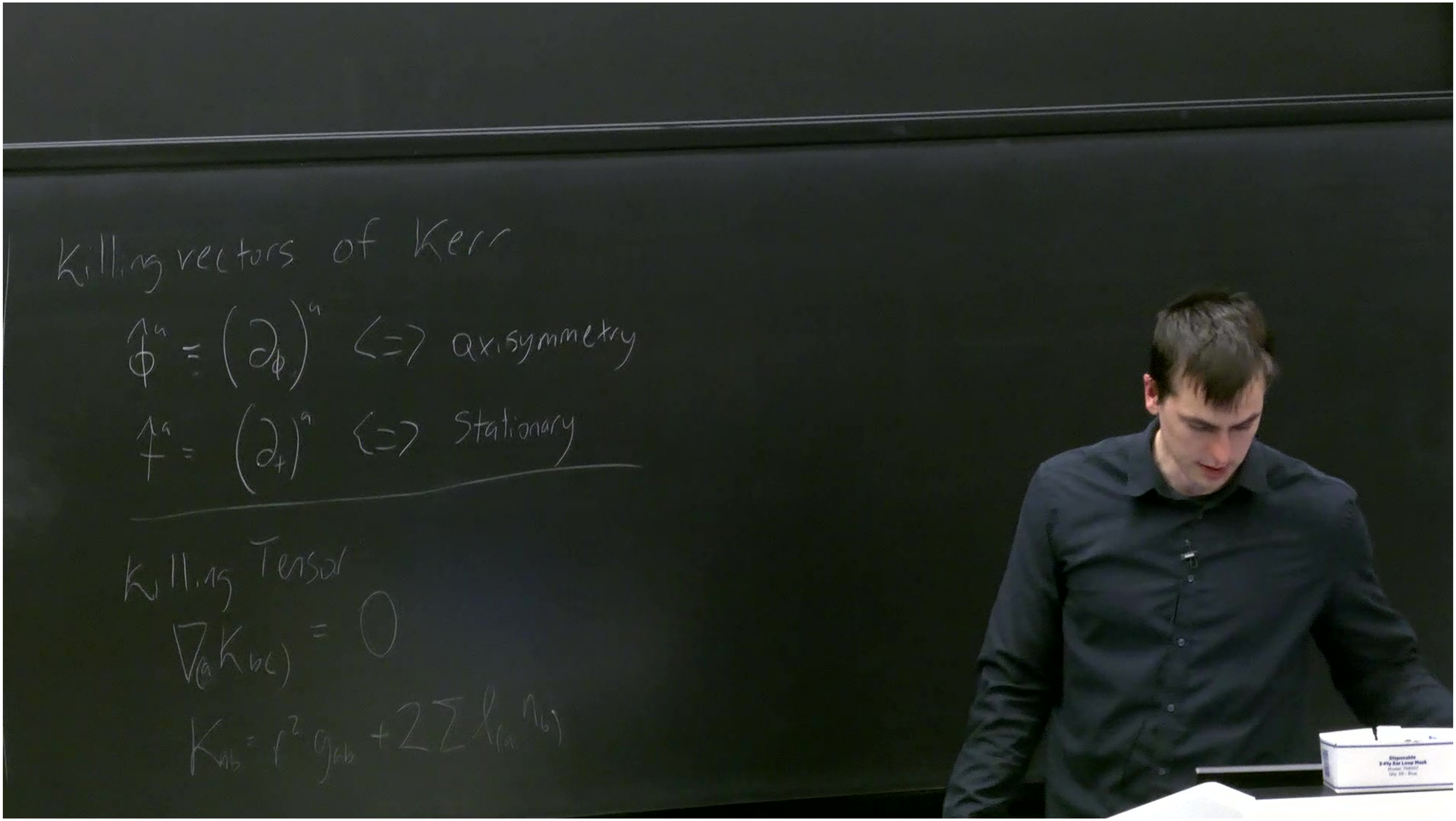
Symmetries of Kerr

Killing Vector $\mathcal{L}_K g_{ab} = 0$

$$\mathcal{L}_K T_{ab} = K^c \partial_c T_{ab} + (\partial_a K^c) T_{cb} + (\partial_b K^c) T_{ac}$$

$$\partial_a \to \nabla_a$$

$$\mathcal{L}_K g_{ab} = 0 + \nabla_a K_b + \nabla_b K_a = 2 \nabla_{(a} K_{b)}$$



Killing vectors of Kerr
$\phi^a = (\partial_\phi)^a$ <=> axisymmetry
$t^a = (\partial_t)^a$ <=> Stationary
Killing Tensor
$\nabla_{(a}K_{bc)} = 0$
$K_{ab} = r^2 g_{ab} + 2\Sigma l_{(a} n_{b)}$

Coordinates breakdown: $\Sigma = 0,\ \Delta = 0$

$R_{abcd}R^{abcd}$ blows up $\Sigma = 0$

$$r^2 + a^2\cos^2\theta = 0,\ r = 0,\ \theta = \frac{\pi}{2}$$

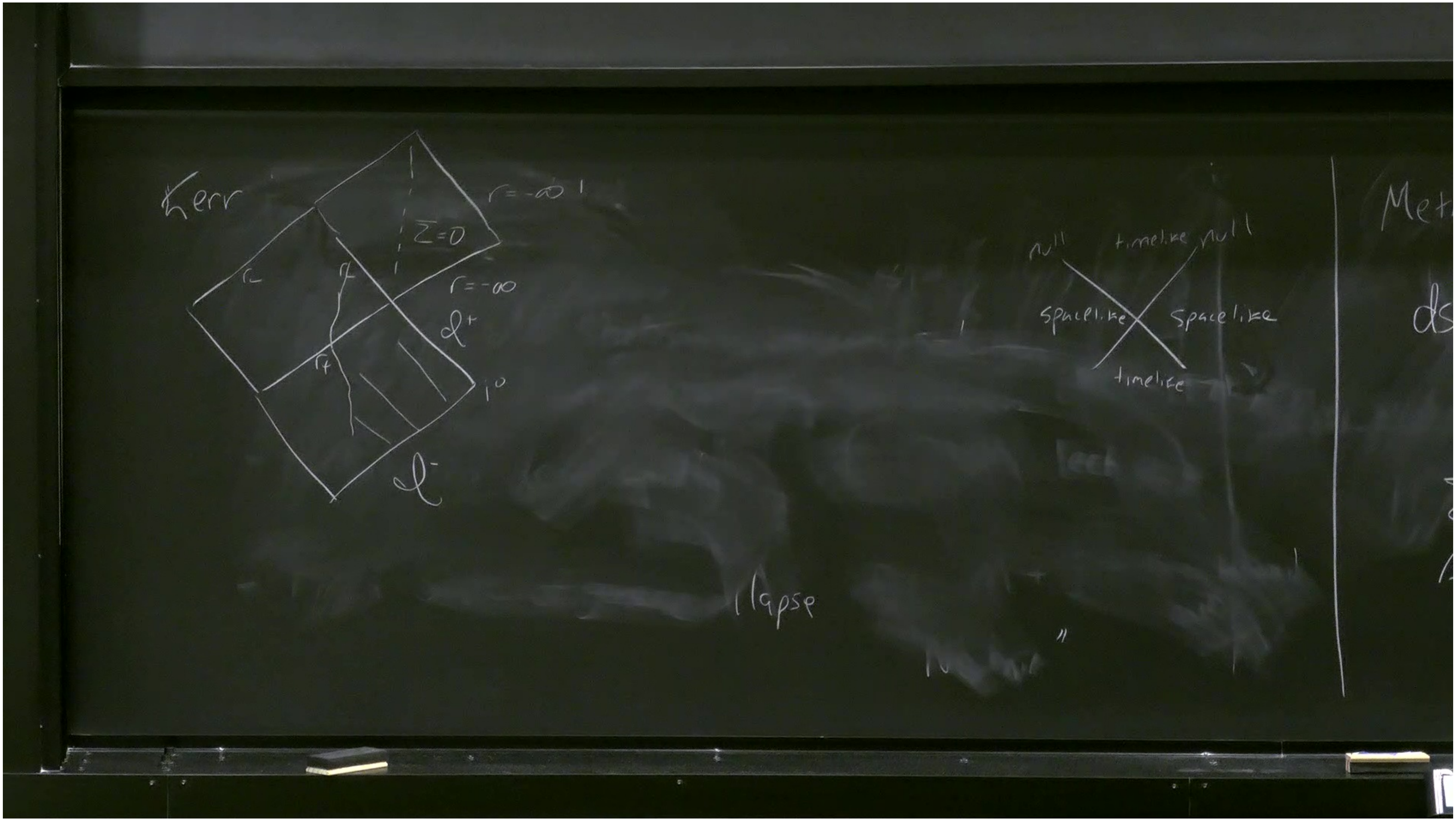
Horizon:

$r = \text{constant}$

$$g^{ab}\partial_a r\, \partial_b r \Rightarrow g^{rr} = \frac{\Delta}{\Sigma}$$

$$\Delta = (r - r_+)(r - r_-)$$

$$r_\pm = M \pm \sqrt{M^2 - a^2}$$


Kerr
r = -∞
Σ = 0
r = -∞
null
timelike
null
spacelike
spacelike
timelike
lapse