Title: Lecture - Quantum Gravity, PHYS 644

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Subject: Quantum Gravity

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Recap

- (P,ω) sympl if ω closed + nondeg , $\{\cdot,\cdot\} = \omega^{-1}$
- $X \in \mathfrak{X}^1(P)$ Hamilt. if $\exists f \in C^\infty(P)$ s.t. $X = \{f, \cdot\} \longleftrightarrow i_X\omega = -df$

- $X_\bullet : C^\infty(P) \longrightarrow \mathfrak{X}^1(P)$
  $f \longmapsto X_f := \{f, \cdot\}$

  homomorph of Lie alg. btw $(C^\infty(P), \{\cdot,\cdot\})$ and $(\mathfrak{X}^1(P), [\cdot,\cdot])$

  i.e $[X_f, X_g] = X_{\{f,g\}}$

- A
  f:
  i

- T

Ex

- A homomorph btw
$\rho : (\mathfrak{g}, [\,,\,]) \longrightarrow (\mathfrak{X}^{\prime}(P), [\,,\,])$
is called an <u>action</u>

- The action is Hamil'n $\forall \xi \in \mathfrak{g}$ iff
$\exists\ J : P \to \mathfrak{g}^*$ (<u>momentum map</u>)
s.t. $\rho(\xi) = \{ \langle J, \xi \rangle, \cdot \}$

i.e. $i_{\rho(\xi)} \omega = - d \langle J, \xi \rangle$

<u>Ex</u>: $\mathfrak{g} = so(3)$ $\omega = d\vec{p} \wedge d\vec{q}$ action by rot
$J = \vec{L}$ so that $\langle J, \xi \rangle = \vec{L} \cdot \vec{\xi}$, $\vec{L} = \vec{p} \times \vec{q}$

$\vec{L} = \vec{p} \times \vec{q}$

- J is said <u>equivariant</u> if

$$L_{\rho(\xi)} J = -\mathrm{ad}^*_{\xi} J$$

i.e. $\{\langle J, \xi\rangle, \langle J, \eta\rangle\} = \langle J, [\xi, \eta]\rangle$

iff

$$\begin{tikzcd}
(\mathfrak{g}, [\ ,\ ]) \arrow[rr, "\check{J}"] \arrow[dr, "\rho"'] & & (C^\infty(P), \{\ ,\ \}) \arrow[dl, "X_\cdot"] \\
& (\mathfrak{X}^1(P), [\ ,\ ]) &
\end{tikzcd}$$

- More generally, J is not necessarily equiv.

$$\{\check{J}(\xi), \check{J}(\eta)\} = \check{J}([\xi, \eta]) + \kappa(\xi, \eta)$$

① $d\kappa(\xi, \eta) = 0$
does not depend on p, q

② $\kappa(\xi, \eta) + \kappa(\eta, \xi) = 0$

③ cyclic sum $\kappa(\xi_1, [\xi_2, \xi_3]) = 0$

$\rightarrow \kappa \in \mathfrak{g}^* \wedge \mathfrak{g}^*$ is a CE 2-cocycle

κ is a trivial cocycle (co-boundary) if it is of the form $\kappa(\xi, \eta) = \langle \lambda, [\xi, \eta] \rangle$ for some $\lambda \in \mathfrak{g}^*$

b.c. then $J'(z) = J(z) + \lambda$ is equiv.

Physical Interpretation

* $\mathfrak{g}$ ~ abstract algebra of syms ($so(3)$, $su(2)$)
* $\rho(\xi) = (\delta_\xi \vec{q}) \frac{\partial}{\partial \vec{q}} + (\delta_\xi \vec{p}) \frac{\partial}{\partial \vec{p}}$

  ~ action/transformation of ph space variables/phys system under abstract sym (e.g. rotations, isospin)

* J ~ canonical generator of the transf $\rho = \{J, \cdot\}$

  ~ if the transf ρ is compatible w/ dynamics, $L_\rho H = 0$, the J is conserved charge.

$$\rho(\xi) = \{ \underset{\substack{\| \| \\ \langle J, \xi \rangle \\ \in C^\infty(P)}}{\check{J}(\xi)}, \cdot \} = \partial$$

$$\frac{d}{dt} J = \{J, H\} = L_\rho H = 0$$

$$* \quad \{J(\xi), J(\eta)\} = J([\xi,\eta]) + \underline{K(\xi,\eta)}$$

~ charge algebra

→ Quantization gives a rep. of this algebra

$$\rho(\xi) = \{\underset{\underset{\substack{\langle J, \xi\rangle \\ \in C^\infty(P)}}{\equiv}}{\check{J}(\xi)}, \cdot\} = \frac{\partial \check{J}(\xi)}{\partial q^i}\frac{\partial}{\partial p_i} - (q \leftrightarrow p)$$

$$\frac{d}{dt} J = \{J, H\} = L_\rho H = 0$$

~ if the transf ρ is com[...] w/ dynamics, $L_\rho H =$ [...] the J is conserved ch[...]

SYMPLECTIC REDUCTION

If $\rho(\xi)$ is a gauge sym,
then its momentum map must vanish
on all physical config.

ρ E&M gauge tr. $\Longrightarrow$ $J = \nabla_i E^i = 0$

Rmk. $J \overset{!}{=} 0$ $\quad 0 \overset{!}{=} \{J, J\} \sim J + \cancel{k}$

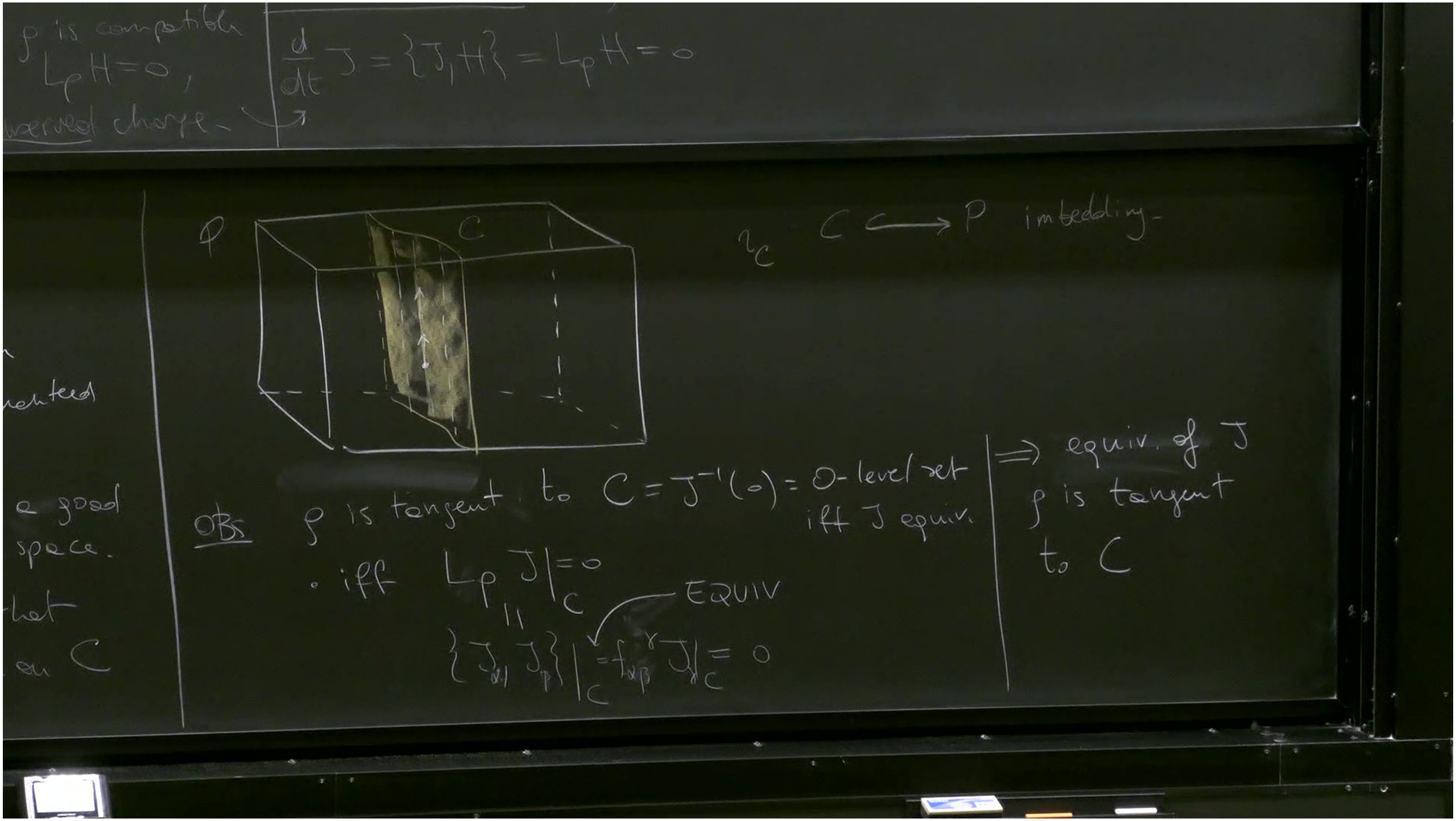
$\rightarrow$ J must be equivariant

$$C := \{ z \in P \mid J(z) = 0 \} = J^{-1}(0)$$ CONSTRAINT SURFACE

Note:
(P, ω) 2n-dim
$\mathfrak{g}$ & $\mathfrak{g}^*$ k-dim
$\hookrightarrow$ C is $2n-k$ dim
which is not guaranteed
to be even

$\rightarrow$ C cannot be a good
phys ph space.

IDEA: the problem is that
ω pulled back on C
is degenerate.



ρ is compatible
$L_\rho H = 0$,
charge
$\frac{d}{dt} J = \{J, H\} = L_\rho H = 0$
P
C
$\imath_C$ $C \hookrightarrow P$ imbedding.
good
space.
on C
OBS ρ is tangent to $C = J^{-1}(0)$ = 0-level set
iff J equiv.
iff $L_\rho J|_C = 0$
$\{J_\alpha, J_\beta\}|_C = f_{\alpha\beta}^{\gamma} J_\gamma|_C = 0$
EQUIV
⟹ equiv. of J
ρ is tangent
to C

Lemma

$$\ker(\iota_C^* \omega) = \mathrm{Im}(\rho)$$

makes sense bc $\rho \in TC \subset T_C P$

Pf:

$(\supseteq)$ $0 \overset{?}{=} i_\rho \iota_C^* \omega = \iota_C^* i_\rho \omega = -\iota_C^* dJ = -d\underbrace{\iota_C^* J}_{\equiv 0}$.

$(\subseteq)$ counting: ω is non deg

C is $2n-k$ dim

$\ker(\iota_C^* \omega)$ is at most k dim

$\mathrm{Im}(\rho)$ is k dim

~ if the transf ρ is com[illegible]
w/ dynamics, $L_\rho H = $[illegible]
the J is conserved ch[illegible]

SYMPLECTIC REDUCTION

If $\rho(\xi)$ is a gauge sym,
then its momentum map must vanish
on all physical config.

ρ E&M gauge tr. $\Longrightarrow$ $J = \nabla_i E^i = 0$

Rmk. " $J_\alpha \overset{!}{=} 0$, $0 \overset{!}{=} \{J_\alpha, J_\beta\} = f_{\alpha\beta}^{\ \ \gamma} J_\gamma + \cancel{k_{\alpha\beta}}$ "

$\longrightarrow$ J must be equivariant

$C := \{ z \in P \mid J_\alpha(z) = 0 \} = J^{-1}(0)$ CONSTRAINT SURFACE

Note:
(P, ω) 2n-dim
$\mathfrak{g}$ & $\mathfrak{g}^*$ k-dim
$\hookrightarrow$ C is 2n-k dim
which is not guaranteed
to be even

$\rightarrow$ C cannot be a good phys ph. space.

IDEA: the problem is that
ω pulled back on C
is degenerate.

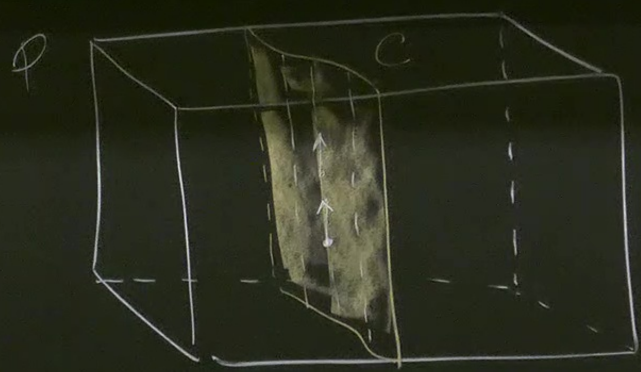
generating
$\rho = \{J, \cdot\}$
iff ρ is compatible
...ics, $L_\rho H = 0$,
is <u>conserved</u> charge.

$\in C^\infty(P)$

$\frac{d}{dt} J = \{J, H\} = L_\rho H = 0$

k dim
not guaranteed

not be a good
ph. space.
is that
back on C
...ete.



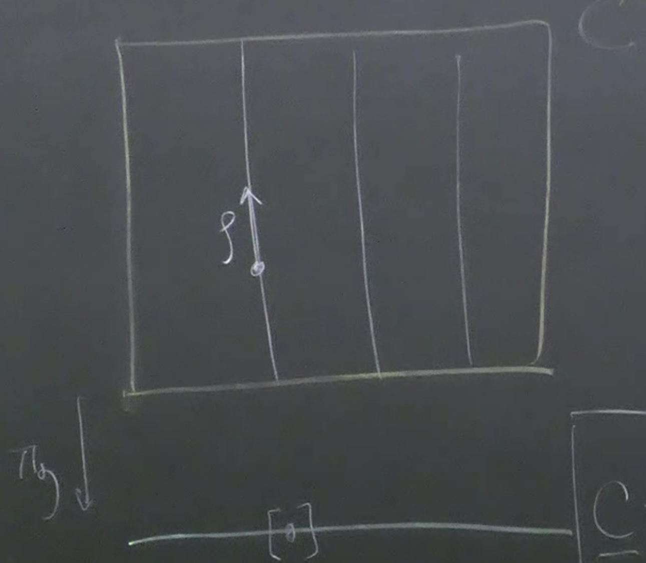
$i_C \quad C \hookrightarrow P$ imbedding.

OBs ρ is tangent to $C = J^{-1}(0)$ = 0-level set iff J equiv.

. iff $L_\rho J|_C = 0$

$\{J_\alpha, J_\beta\}|_C = f_{\alpha\beta}{}^\gamma J_\gamma|_C = 0$ EQUIV

$\Longrightarrow$ equiv. of J
ρ is tangent
to C



$$\underline{\mathcal{C}} = \mathcal{C}/\mathcal{G}$$

quotient out the gauge transf.

$$\boxed{i_{\mathcal{C}}^{*}\omega} = \pi_{\mathcal{G}}^{*}\,\underline{\omega}$$

$\rightarrow (\underline{\mathcal{C}}, \underline{\omega})$ Reduced ph. space.

More heuristic argument:

$$J \overset{!}{=} 0 \rightarrow 0 \overset{!}{=} \{J, \cdot\}$$

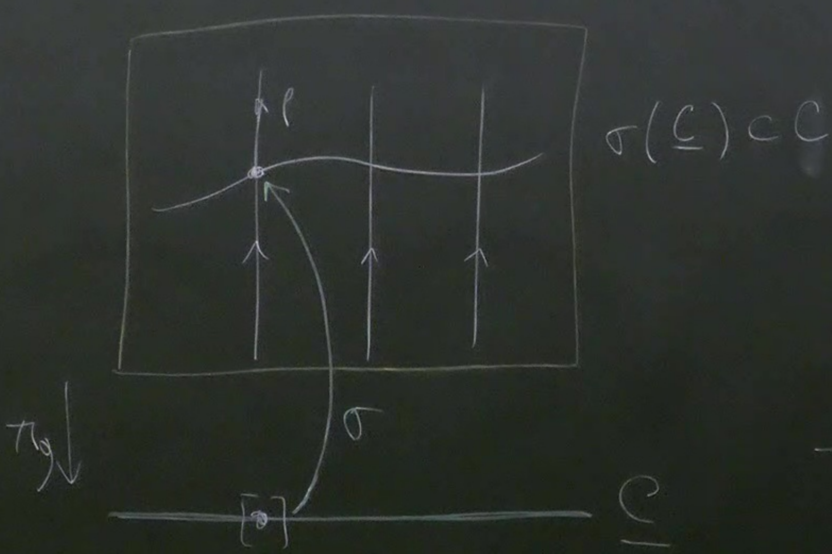
instead of tampering w/ bracket, just say that only observables $\mathcal{O}$ which are allowed are those s.t.

$$0 = \{J(\xi), \mathcal{O}\} = L_{\rho(\xi)}\mathcal{O} \equiv \delta_\xi \mathcal{O}$$

invariant under gauge transf.

<u>Rmk</u> such $\mathcal{O}$ are of the form

$$\mathcal{O} = \pi_{\mathcal{G}}^* \underline{\mathcal{O}} \quad \text{for some } \underline{\mathcal{O}} \in C^\infty(\underline{\mathcal{C}})$$



$\pi_g \circ \sigma = \mathrm{id}_{\underline{C}}$ "section" of π

$\sigma(\underline{C})$ in practice defined as $\underline{\Psi}^{-1}(0)$

$\underline{\Psi}$ = gauge fixing condition

for ex

$\underline{\Psi}_{\text{Coulomb}} = 0 \iff \underline{\Psi}(A, E) = D_i A^i = 0$

gauge fixing is meaningful iff $\Psi^{-1}(0)$ is transverse to $\mathrm{Im}(\rho)$

iff $\det(M^\alpha_\beta) \neq 0$ Faddeev Popov.

$M^\alpha_\beta = \{\Psi^\alpha, J_\beta\}$