Title: Lecture - Quantum Gravity, PHYS 644

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Subject: Quantum Gravity

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Recap

sympl. mfd ("phase space")

(P, ω) $\quad P$ 2n-dim $\ni z^I$

$\omega \in \Omega^2(P)$

↳ non deg $\rightsquigarrow \{\cdot,\cdot\} := (\omega_{IJ})^{-1} \frac{\partial}{\partial z^I} \wedge \frac{\partial}{\partial z^J}$ $\searrow$ $\{,\}$ Poisson

↳ closed $d\omega = 0 \Rightarrow \{\cdot,\cdot\}$ is Jacobi $\nearrow$

Darboux thm: locally (in a chart!), $\exists$ coords $z^I = (p_i, q^i)$ such that $\omega = \sum_{i=1}^{n} dp_i \wedge dq^i$

"canonical coordinates" $\Rightarrow \{\cdot,\cdot\} = \frac{\partial}{\partial q} \otimes \frac{\partial}{\partial p} - \frac{\partial}{\partial p} \otimes \frac{\partial}{\partial q}$

$\{\cdot,\cdot\}$

$\to$ Poisson

$q^j)$

$\frac{\partial}{\partial p} - \frac{\partial}{\partial p} \otimes \frac{\partial}{\partial q}$

<u>Ex</u> particle in an ext EM field

$$L(q,\dot{q}) = \frac{1}{2}\dot{\vec{q}}^{\,2} - \vec{A}(q)\cdot\dot{\vec{q}} - V(q)$$

$\downarrow$ Legendre tr.

$$\vec{p} = \frac{\partial L}{\partial \dot{\vec{q}}} = \dot{\vec{q}} - \vec{A}(q)$$

$$\mathcal{H}(p,q) = \frac{1}{2}\left(\vec{p} + \vec{A}(q)\right)^2 + V(\vec{q})$$

$$\omega = d\vec{p} \wedge d\vec{q}$$

$$\frac{d}{dt} = X_{\mathcal{H}} \quad : \quad i_{X_{\mathcal{H}}}\omega = -d\mathcal{H}$$

$$\text{pf} : X_{\mathcal{H}} = \begin{pmatrix} X_{\mathcal{H}}^q \\ X_{\mathcal{H}}^p \end{pmatrix} \longrightarrow \vec{X}_{\mathcal{H}}^p\, d\vec{q} - \vec{X}_{\mathcal{H}}^q\, d\vec{p} = -\frac{\partial \mathcal{H}}{\partial \vec{q}} d\vec{q} - \frac{\partial \mathcal{H}}{\partial \vec{p}} d\vec{p}$$

$\Rightarrow$

$^2 + V(\vec{q})$

$$\boxed{\omega = -dH}$$

$d\vec{p} = -\frac{\partial H}{\partial \vec{q}} d\vec{q} - \frac{\partial H}{\partial \vec{p}} d\vec{p}$

$$\Rightarrow \frac{d}{dt}\begin{pmatrix} \vec{q} \\ \vec{p} \end{pmatrix} = \begin{pmatrix} X_H^q \\ X_H^p \end{pmatrix} = \begin{pmatrix} -\partial H/\partial p \\ \partial H/\partial q \end{pmatrix}$$

What about a coord change over P?

$$\vec{v} := \vec{p} + \vec{A}(q)$$

$$(q, p) \longmapsto (q, v)$$

$$\rightsquigarrow \mathcal{H}(q, v) = \frac{1}{2}\vec{v}^2 + V(\vec{q})$$

$$\omega(q, v) = d\vec{v} \wedge d\vec{q} - d\vec{A}(q) \wedge d\vec{q}$$

$$= d\vec{v} \wedge d\vec{q} - \frac{1}{2} B_i \epsilon^{ijk} dq^j \wedge dq^k$$

Sanity check, recover Lagrange eom $(q, \vec{v} \sim \dot{q})$ from $i_{X_H}\omega = -dH$

Def $X \in \mathfrak{X}^1(P)$

- symplectic $L_X \omega = 0$
- Hamiltonian if $\exists f_X \in C^\infty(P)$: $i_X \omega = -df_X$

Lemma Assume θ, X : $\omega = d\theta$, $L_X \theta = 0$

then X is Hamiltonian w/ $f_X = i_X \theta$

Pf: $0 = L_X \theta = i_X d\theta + d i_X \theta = i_X \omega + df_X$ $\square$

Rmk: $\theta = p\,dq$, $f_X = i_X \theta = p\,\delta_X q$

Lemma

$$[X_f, X_g] = X_{\{f,g\}}$$

where $X_f := \{f, \cdot\} \longleftrightarrow i_{X_f}\omega = -df$

Pf

$$L_{X_f} i_{X_g}\omega = L_{X_f}(-dg) = -d\, L_{X_f} g = -d\, X_f(g) = -d\{f,g\}$$

$$\parallel$$

$$i_{[X_f,X_g]}\omega + i_{X_g} \underbrace{L_{X_f}\omega} \qquad = i_{[X_f,X_g]}\omega$$

$$\begin{aligned} &= d\, i_{X_f}\omega + i_{X_f}\underbrace{d\omega}_{=0} \\ &= -d^2 f \\ &= 0 \end{aligned}$$

$\boxed{\checkmark}$

$-X_f\, g = -d\, X_f(g) = -d\{f,g\}$

$\boxed{\checkmark}$

$= i_{[X_f, X_g]}\omega$

$i_f d\omega$
$= 0$

This means that:

$$\left(C^\infty(P), \{\cdot,\cdot\}\right) \xrightarrow{X_\bullet} \left(\mathfrak{X}^1(P), [\cdot,\cdot]_{TP}\right)$$

$X_\bullet$ homomorphism of Lie algebras over $\mathbb{R}$

<u>Def</u> [Action] $\mathfrak{g}$ is a Lie alg.

$$\rho : \mathfrak{g} \longrightarrow \mathfrak{X}^1(P)$$

is called an action if it is a homomorph. of Lie algebras (over $\mathbb{R}$)

$$= d\, i_{X_f}\omega + i_{X_f}\underbrace{d\omega}_{=0} = -d^2 f = 0$$

Rmk: $\theta = p\,dq$, $f_X = i_X\theta = p\,\delta_X q$

Unpacked: $\xi \in \mathfrak{g}$

$$\begin{cases} \rho([\xi,\eta]) = [\rho(\xi), \rho(\eta)] \\ \rho(a\xi) = a\,\rho(\xi), \quad a \in \mathbb{R} \end{cases}$$

Examples: • translations

$$(P,\omega) = (T^*\mathbb{R}^3,\ d\vec{p}\wedge d\vec{q})$$

$$\mathfrak{g} \simeq (\mathbb{R}^3, +)$$

$$\rho(\xi) = \vec{\xi}\cdot\frac{\partial}{\partial\vec{q}} + \text{zero}\,\frac{\partial}{\partial\vec{p}}$$

• rotations

$$\mathfrak{g} = \mathfrak{so}(3) \simeq (\mathbb{R}^3, \times)$$

$$\rho(\xi) = \underbrace{(\vec{\xi}\times\vec{q})}_{\delta_\xi\vec{q}}\cdot\frac{\partial}{\partial\vec{q}} + \underbrace{(\vec{\xi}\times\vec{p})}_{\delta_\xi\vec{p}}\frac{\partial}{\partial\vec{p}}$$

Idea: close the triangle:

$$\begin{tikzcd} (\mathfrak{g}, [\cdot,\cdot]) \arrow[dr, "\rho"'] & & (C^\infty(P), \{\cdot,\cdot\}) \arrow[dl, "X_\cdot"] \\ & (\mathfrak{X}^1(P), [\cdot,\cdot]_{TP}) & \end{tikzcd}$$

Rmk: $\theta = p\,dq,\ f_X = i_X\theta = p\,\delta_X q$.

$= d i_{X_f}\omega + i_{X_f}\underbrace{d\omega}_{=0}$
$= -d^2 f$
$= 0$

Unpacked:

$$\left[\begin{array}{l} \rho([\xi,\eta]) = [\rho(\xi),\rho(\eta)] \\ \rho(a\xi) = a\,\rho(\xi)\,, \quad a\in\mathbb{R} \end{array}\right.$$

$\xi \in \mathfrak{g}$

Examples: • translations

$(P,\omega) = (T^*\mathbb{R}^3,\ d\vec{p}\wedge d\vec{q})$

$\mathfrak{g} \simeq (\mathbb{R}^3, +)$

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Idea: close the triangle.

$$\begin{tikzcd}
(\mathfrak{g},[\,,\,]) \arrow[rr, dashed, "\tilde{J}"] \arrow[dr, "\rho"'] & & (C^\infty(P),\{\cdot,\cdot\}) \arrow[dl, "X_\bullet"] \\
& (\mathfrak{X}^1(P),[\,,\,]_{TP})\big|_{HVF} \arrow[ur, bend right, "f_\bullet"'] &
\end{tikzcd}$$

$= d i_{X_f}\omega + i_{X_f} \underbrace{d\omega}_{=0}$

$= -d^2 f$

$= 0$

is called an action if it is a homomorph. of Lie algebras (over $\mathbb{R}$)

$(C^\infty(P), \{\cdot,\cdot\})$

$X_\circ \qquad f_\circ$

HVF

$\boxed{\text{Assumption}}$

$\rho(\xi)$ is Hamiltonian for all $\xi \in \mathfrak{g}$.

that is

$\exists\ \check{J} : \mathfrak{g} \to C^\infty(P)$ s.t. $\quad i_{\rho(\xi)}\omega = -d\check{J}(\xi) \Leftrightarrow \rho(\xi) = \{\check{J}(\xi), \cdot\}$

(Co)momentum map

<u>Rmk</u>: $\rho(\xi)$ is $\mathbb{R}$-linear in $\xi \Rightarrow \check{J}(\xi)$ must be $\mathbb{R}$-linear in ξ.

this means that $\check{J}$ defines a map $J : P \to \mathrm{Lin}(\mathfrak{g}) = \mathfrak{g}^*$

momentum map.

$\check{J}(\xi)|_z = J(z)|_\xi \equiv \langle J(z), \xi \rangle$ pairing btw $\mathfrak{g}^*$ & $\mathfrak{g}$

$$\rightsquigarrow \quad i_{\rho(\xi)}\omega = -d\langle J, \xi \rangle$$

Examples

- rotations $\quad \mathfrak{g} = \mathfrak{so}(3) \simeq \mathbb{R}^3, \quad \mathfrak{g}^* \simeq \mathbb{R}^3$

$$\begin{aligned} i_{\rho(\xi)}\omega &= \delta_\xi \vec{p}\cdot d\vec{q} - \delta_\xi \vec{q}\cdot d\vec{p} \\ &= (\vec{\xi}\times\vec{p})\cdot d\vec{q} - (\vec{\xi}\times\vec{q})\cdot d\vec{p} \\ &= d\left[(\vec{\xi}\times\vec{p})\cdot\vec{q}\right] \\ &= -d\,\vec{L}\cdot\vec{\xi} \end{aligned}$$

Angular momentum

$\vec{L}$ is the mom. map J for rotations

Q: is $\mathfrak{g} \xrightarrow{\check{J}} C^\infty(P)$ a homomorph of Lie alg?

(assumption ρ is Ham.)

Desiderata: $\quad \check{J}([\xi,\eta]) \overset{?}{=} \{\check{J}(\xi), \check{J}(\eta)\}$

Lemma: $\quad d\left(\{\check{J}(\xi), \check{J}(\eta)\} - \check{J}([\xi,\eta])\right) = 0$

Pf: $\quad d\{\check{J}(\xi), \check{J}(\eta)\} = d\, L_{\rho(\xi)}\check{J}(\eta)$

$\underline{Q}$: is $\mathfrak{g} \xrightarrow{\check{J}} C^\infty(P)$ a homomorph of Lie alg?

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Desiderata: $\check{J}([\xi,\eta]) \overset{?}{=} \{\check{J}(\xi), \check{J}(\eta)\}$

$\underline{\text{Lemma}}$: $d\left(\{\check{J}(\xi), \check{J}(\eta)\} - \check{J}([\xi,\eta])\right) = 0$

$\underline{\text{Pf}}$: $d\{\check{J}(\xi), \check{J}(\eta)\} = d\, L_{\rho(\xi)} \check{J}(\eta) = L_{\rho(\xi)}\, d\check{J}(\eta) = -L_{\rho(\xi)}\, i_{\rho(\eta)}\, \omega$

$= -i_{\rho(\eta)} \underbrace{L_{\rho(\xi)}\, \omega}_{=0} - i_{[\rho(\xi),\rho(\eta)]}\, \omega = -i_{\rho([\xi,\eta])}\, \omega = -d\check{J}([\xi,\eta])$ $\square$

[ ] is the mom. map ... $= 0$
for rotations

Rmk a priori one cannot do any better!

For ex. let $\lambda \in \mathfrak{g}^*$, redefine $J(z) \mapsto J'(z) = J(z) + \lambda$

note that $J'(z)$ is also a mom map for ρ!

But: if $\{\check{J}(\xi), \check{J}(\eta)\} = \check{J}([\xi,\eta])$

then $\{\check{J}'(\xi), \check{J}'(\eta)\} = \check{J}([\xi,\eta]) = \check{J}'([\xi,\eta]) - \langle \lambda, [\xi,\eta] \rangle$

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Note: $d\langle \lambda, [\xi,\eta] \rangle = 0 \rightarrow$ Lemma is ok.

Rmk: not all obstructions are of this form.
they are classified by the 1st CE cohomology of $\mathfrak{g}$.

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Def $\check{J}$ is said equivariant if

$$\{\check{J}(\xi), \check{J}(\eta)\} = \check{J}([\xi,\eta])$$

iff

$$L_{\rho(\xi)} J = -\mathrm{ad}^*_\xi J$$

note: $\{\check{J}(\xi), \check{J}(\eta)\} = L_{\rho(\xi)} \check{J}(\eta) = \langle L_{\rho(\xi)} J, \eta \rangle$

$$\check{J}([\xi,\eta]) = \langle J, [\xi,\eta] \rangle = \langle J, \mathrm{ad}_\xi \eta \rangle =: -\langle \mathrm{ad}^*_\xi J, \eta \rangle$$

(1)

(2)

$\ldots \xi, \eta \rangle$

$\ldots \mathrm{ad}_\xi \eta \rangle =: -\langle \mathrm{ad}^*_\xi J, \eta \rangle$

If J $\exists$ & equivariant

then

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\check{J}"] \arrow[dr, "\rho"'] & C^\infty(P) \arrow[d, "X_\cdot"] \\ & \mathfrak{X}'(P) \end{tikzcd}$$

are all homomorph of Lie alg.

Ex $\xi_i \eta_j \{L^i, L^j\} = \epsilon^{ij}{}_k L^k$ rotations (1)

$\delta_\xi \vec{L} = \rho(\xi) \vec{L} = \vec{\xi} \times \vec{L}$ (2)