

Title: Lecture - Quantum Gravity, PHYS 644

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Subject: Quantum Gravity

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Recap

sympl. mfd ("phase space")

(P, ω)

P $2n$ -dim $\ni \mathbb{R}^I$

$\omega \in \Omega^2(P)$

\rightarrow non deg $\leadsto \{ \cdot, \cdot \} := (\omega_{IJ})^{-1} \frac{\partial}{\partial z^I} \wedge \frac{\partial}{\partial z^J} \rightarrow \{ \cdot, \cdot \}$ Poisson

\rightarrow closed $d\omega = 0 \rightarrow \{ \cdot, \cdot \}$ is Jacobi \rightarrow

Darboux thm: locally (in a chart!) \exists coords $z^I = (p_i, q^i)$
such that $\omega = \sum_{i=1}^n dp_i \wedge dq^i$

"canonical coordinates" $\rightarrow \{ \cdot, \cdot \} = \frac{\partial}{\partial q^i} \otimes \frac{\partial}{\partial p_i} - \frac{\partial}{\partial p_i} \otimes \frac{\partial}{\partial q^i}$

Ex particle in an ext EM field

$$L(q, \dot{q}) = \frac{1}{2} \dot{q}^2 - \vec{A}(q) \cdot \dot{q} - V(q)$$



Legendre tra.

$$\vec{p} = \frac{\partial L}{\partial \dot{q}} = \dot{q} - \vec{A}(q)$$

$$\mathcal{H}(p, q) = \frac{1}{2} (\vec{p} + \vec{A}(q))^2 + V(q)$$

$$\omega = d\vec{p} \wedge d\vec{q}$$

$$\frac{d}{dt} = X_{\mathcal{H}} \quad ; \quad i_{X_{\mathcal{H}}} \omega = -d\mathcal{H}$$

pf: $X_{\mathcal{H}} = \begin{pmatrix} X_{\mathcal{H}}^q \\ X_{\mathcal{H}}^p \end{pmatrix} \rightarrow \vec{X}_{\mathcal{H}}^p d\vec{q} - \vec{X}_{\mathcal{H}}^q d\vec{p} = -\frac{\partial \mathcal{H}}{\partial \vec{q}} d\vec{q} - \frac{\partial \mathcal{H}}{\partial \vec{p}} d\vec{p}$

→ } , ?
→ Poisson

q')

$\frac{e}{c} \vec{A}$
 $\frac{e}{c} \vec{A}$
 $\frac{e}{c} \vec{A}$

⇒

$$^2 + V(\vec{q})$$

$$\omega = -dH$$

$$d\vec{p} = -\frac{\partial H}{\partial \vec{q}} d\vec{q} - \frac{\partial H}{\partial \vec{p}} d\vec{p}$$

$$\Rightarrow \frac{d}{dt} \begin{pmatrix} \vec{q} \\ \vec{p} \end{pmatrix} = \begin{pmatrix} X_H^q \\ X_H^p \end{pmatrix} = \begin{pmatrix} -\partial H / \partial \vec{p} \\ \partial H / \partial \vec{q} \end{pmatrix}$$

What about a coord change over P?

$$\vec{v} := \vec{p} + \vec{A}(q)$$

$$(q, p) \mapsto (q, v)$$

$$\leadsto \mathcal{H}(q, v) = \frac{1}{2} \vec{v}^2 + V(\vec{q})$$

$$\omega(q, v) = d\vec{v} \wedge d\vec{q} - d\vec{A}(q) \wedge d\vec{q}$$

$$= d\vec{v} \wedge d\vec{q} - \frac{1}{2} B_i \epsilon^{ijk} dq^j \wedge dq^k$$

Sanity check, recover Lagrange eqn $(q, \vec{v} = \dot{q})$
 from $X_H \omega = -dH$

Def $X \in \mathcal{X}^1(P)$

- symplectic $L_X \omega = 0$
- Hamiltonian if $\exists f_X \in C^\infty(P) : i_X \omega = -df_X$

Lemma Assume $\exists X : \omega = d\theta, L_X \theta = 0$

then X is Hamiltonian w/ $f_X = i_X \theta$

Pf: $0 = L_X \theta = i_X d\theta + d i_X \theta = i_X \omega + df_X \quad \square$

Rmk: $\theta = p dq, f_X = i_X \theta = p f_X q$

Lemma

$$[X_f, X_g] = X_{\{f, g\}}$$

where $X_f = -\{f, \cdot\} \leftrightarrow i_{X_f} \omega = -df$

Pf $L_{X_f} i_{X_g} \omega = L_{X_f} (-dg) = -d L_{X_f} g = -d X_f(g) = -d\{f, g\}$

$$i_{[X_f, X_g]} \omega + i_{X_g} L_{X_f} \omega = i_{[X_f, X_g]} \omega$$

$$= di_{X_f} \omega + i_{X_f} d\omega$$

$$= -d^2 f$$

$$= 0$$



$$X_f g = -dX_f(g) = -d\{f, g\}$$

$$= i_{[X_f, X_g]} \omega$$

$$i_{L} d\omega = 0$$

This means that:

$$(\mathcal{C}^\infty(P), \{ \cdot, \cdot \}) \xrightarrow{X_\bullet} (\mathfrak{X}'(P), [\cdot, \cdot]_{TP})$$

X_\bullet homomorphism of Lie algebras over \mathbb{R}

Def [Action] \mathfrak{g} is a Lie alg.

$$\rho: \mathfrak{g} \rightarrow \mathfrak{X}'(P)$$

is called an action if it is a homomorph. of Lie algebras (over \mathbb{R})

Rmk: $\Theta = p dq, f_x = i_x \Theta = p \delta_x q$

$$= d i_x \omega + i_x d\omega = 0$$

$$= -d^2 f = 0$$

Unpacked:

$$p([\xi, \eta]) = [e(\xi), p(\eta)]$$

$$p(a\xi) = a p(\xi), a \in \mathbb{R}$$

$\xi \in \mathfrak{g}$ • rotations

$$\mathfrak{g} = \mathfrak{so}(3) \simeq (\mathbb{R}^3, \times)$$

$$p(\xi) = \underbrace{(\xi \times \bar{q})}_{\delta_\xi \bar{q}} \cdot \frac{\partial}{\partial \bar{q}} + \underbrace{(\xi \times \bar{p})}_{\delta_\xi \bar{p}} \cdot \frac{\partial}{\partial \bar{p}}$$

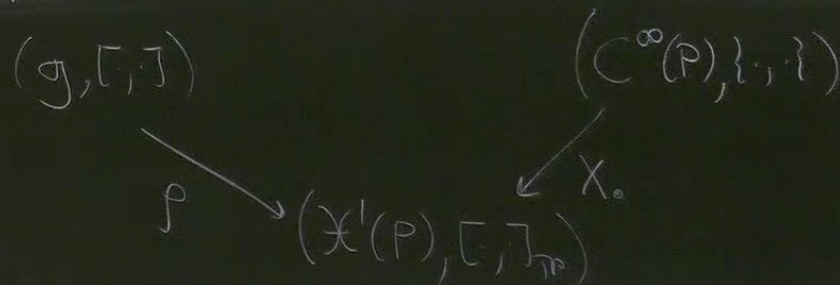
Examples: • translations

$$(P, \omega) = (T^*\mathbb{R}^3, d\bar{p} + d\bar{q})$$

$$\mathfrak{g} \simeq (\mathbb{R}^3, +)$$

$$p(\xi) = \xi \cdot \frac{\partial}{\partial \bar{q}} + \text{zero} \cdot \frac{\partial}{\partial \bar{p}}$$

Idea: close the triangle:



Rmk: $\Theta = p dq, f_x = 1_x \Theta = p \delta_x q$

$$= dx_x \omega + 1_x \omega$$

$$= -d^2 f$$

$$= 0$$

Unpacked:

$$p([\xi, \eta]) = [e(\xi), p(\eta)]$$

$$p(a\xi) = a p(\xi), a \in \mathbb{R}$$

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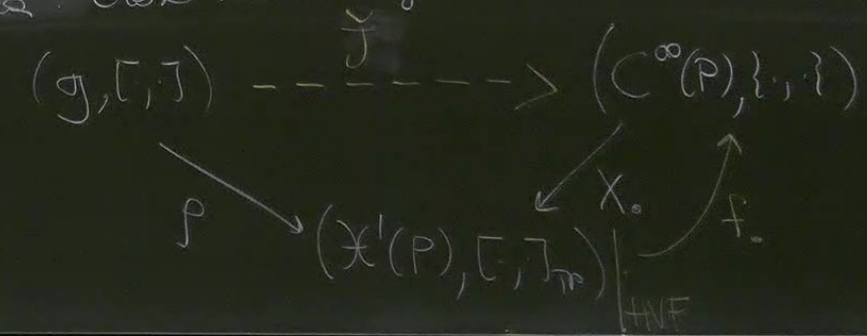
Examples: • translations

$$(P, \omega) = (T^*\mathbb{R}^3, d\vec{p} \wedge d\vec{q})$$

$$\mathfrak{g} \simeq (\mathbb{R}^3, +)$$

$$p(\xi) = \vec{\xi} \cdot \frac{\partial}{\partial \vec{q}} + \text{zero} \frac{\partial}{\partial \vec{p}}$$

Idea: close the triangle



$$\begin{aligned}
 &= dx_{\mu} \omega + i x_{\mu} d\omega \\
 &= -d^2 f \\
 &= 0
 \end{aligned}$$

is called an action if it is a homomorph of Lie algebras (over \mathbb{R})

Assumption

$\rho(\xi)$ is Hamiltonian for all $\xi \in \mathfrak{g}$.

that is

$$\exists \check{J} : \mathfrak{g} \rightarrow C^{\infty}(P) \text{ s.t. } \rho(z)\omega = -d\check{J}(z) \Leftrightarrow \rho(z) = \{ \check{J}(z), - \}$$

↑ (Co)momentum map

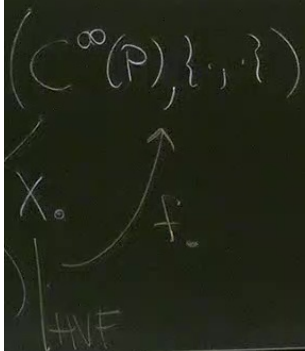
Remark: $\rho(\xi)$ is \mathbb{R} -linear in $\xi \Rightarrow \check{J}(\xi)$ must be \mathbb{R} -linear in ξ .

this means that \check{J} defines a map $\check{J} : P \rightarrow \text{Lin}(\mathfrak{g}) = \mathfrak{g}^*$

$$\check{J}(\xi)|_z = \check{J}(z)|_{\xi} \equiv \langle \check{J}(z), \xi \rangle$$

↖ pairing btw \mathfrak{g}^* & \mathfrak{g}

↑ momentum map.



$$\leadsto i_{\rho(\xi)} \omega = -d\langle J, \xi \rangle$$

Examples

rotations $\mathfrak{g} = \mathfrak{so}(3) \cong \mathbb{R}^3$, $\mathfrak{g}^* = \mathbb{R}^3$

$$\begin{aligned} i_{\rho(\xi)} \omega &= \delta_{\vec{q}} \vec{p} \cdot d\vec{q} - \delta_{\vec{p}} \vec{q} \cdot d\vec{p} \\ &= (\vec{\xi} \times \vec{p}) \cdot d\vec{q} - (\vec{\xi} \times \vec{q}) \cdot d\vec{p} \\ &= d[(\vec{\xi} \times \vec{p}) \cdot \vec{q}] \end{aligned}$$

$= -d \vec{L} \cdot \vec{\xi}$
 \vec{L} is the mom. map for rotations
 Angular momentum

Q: is $\mathfrak{g} \xrightarrow{J} C^\infty(P)$ a homomorph of Lie alg?
 (assumption ρ is Ham.)

Desiderate: $\check{J}([\xi, \eta]) \stackrel{?}{=} \{\check{J}(\xi), \check{J}(\eta)\}$

Lemma: $d(\{\check{J}(\xi), \check{J}(\eta)\} - \check{J}([\xi, \eta])) = 0$

Pf: $d\{\check{J}(\xi), \check{J}(\eta)\} = dL_{\rho(\xi)} \check{J}(\eta)$

Q: Is $\check{J} \rightarrow C^\infty(P)$ a homomorph of Lie alg?
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Desiderate: $\check{J}([\xi, \eta]) \stackrel{?}{=} \{\check{J}(\xi), \check{J}(\eta)\}$

Lemma: $d(\{\check{J}(\xi), \check{J}(\eta)\} - \check{J}([\xi, \eta])) = 0$

Pf: $d\{\check{J}(\xi), \check{J}(\eta)\} = d L_{\rho(\xi)} \check{J}(\eta) = L_{\rho(\xi)} d\check{J}(\eta) = -L_{\rho(\xi)} i_{\rho(\eta)} \omega$
 $= -i_{\rho(\eta)} \underbrace{L_{\rho(\xi)} \omega}_{=0} - i_{[\rho(\xi), \rho(\eta)]} \omega = -i_{\rho([\xi, \eta])} \omega = -d\check{J}([\xi, \eta]) \quad \square$

L is the mom. map for rotations

=0

Rmk a priori one cannot do any better!

For ex. let $\lambda \in \mathfrak{g}^*$, redefine $J(z) \mapsto J'(z) = J(z) + \lambda$

note that $J'(z)$ is also a mom map for ρ !

But: if $\{\check{J}(\xi), \check{J}(\eta)\} = \check{J}([\xi, \eta])$

then $\{\check{J}'(\xi), \check{J}'(\eta)\} = \check{J}([\xi, \eta]) = \check{J}'([\xi, \eta]) - \langle \lambda, [\xi, \eta] \rangle$

Rmk a priori, one cannot do any better!

For ex. let $\lambda \in \mathfrak{g}^*$, redefine $J(z) \mapsto \check{J}'(z) = J(z) + \lambda$

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Note: $d\langle \lambda, [\xi, \eta] \rangle = 0 \rightarrow$ Lemma is ok.

Rmk: not all obstructions are of this form.
they are classified by the 1^{st} CE cohomology of \mathfrak{g} .

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Def \check{J} is said equivariant if
 $\{\check{J}(\xi), \check{J}(\eta)\} = \check{J}([\xi, \eta])$

iff

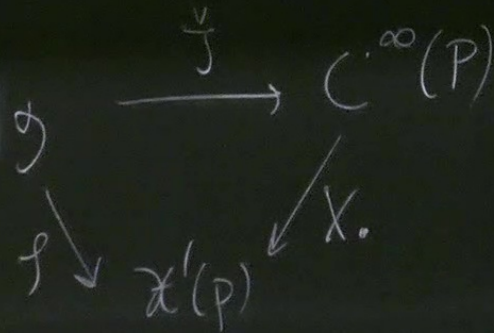
$$L_{\rho(\xi)} J = -\text{ad}_{\xi}^* J$$

note: $\{\check{J}(\xi), \check{J}(\eta)\} = L_{\rho(\xi)} \check{J}(\eta) = \langle L_{\rho(\xi)} J, \eta \rangle$

$\check{J}([\xi, \eta]) = \langle J, [\xi, \eta] \rangle = \langle J, \text{ad}_{\xi} \eta \rangle = -\langle \text{ad}_{\xi}^* J, \eta \rangle$

If J is equivariant

then



are all homomorph of Lie alg.

Ex $\{L^i, L^j\} = \epsilon^{ij}_k L^k$ rotations (1)

$\vec{\zeta} \vec{L} = \rho(\vec{\zeta}) \vec{L} = \vec{\zeta} \times \vec{L}$ (2)

$\langle \xi, \eta \rangle$

$\langle \text{ad}_\xi^* J, \eta \rangle = - \langle \text{ad}_\xi^* J, \eta \rangle$