Title: Lecture - Quantum Gravity, PHYS 644

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Collection/Series: Quantum Gravity (Elective), PHYS 644, February 24 - March 28, 2025

Subject: Quantum Gravity

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PLAN OF THE COURSE

[Read outline on portal → GenAI policy]

1) Covariant Phase Space
   - → Noether I & II (and more)
     - "universal" tool for study of holography, and relationship btw GR & thermodyn.
   - → Wald's entropy & 1st law of BH mech.

2) From Cov. Ph Sp. to Canonical Ph Sp.
   - → constraint algebra + general covariance as phys principle
     - used in studies of boundaries etc

Review of Cl. Mech & Sympl. geom

Action principle

- Space of histories $\Gamma \ni C^\infty(\underset{\text{time}}{\mathbb{R}}, \underset{\text{config. space}}{Q}) \ni \gamma$
- Action $S : \Gamma \to \mathbb{R}$

$$S = \int_{t_1}^{t_2} dt \, \mathcal{L}(\gamma, \dot\gamma, t) \qquad \text{quadratic in } \dot\gamma$$

action principle: $\delta S \big|_{\delta\gamma|_{t=t_1,t_2}=0} = 0$ @ phys. history.

...ys principle

• used in studies of boundaries etc

$$\delta S = \int_{t_1}^{t_2} dt \Big(\underbrace{\frac{\partial L}{\partial \gamma} - \frac{d}{dt}\frac{\partial L}{\partial \dot\gamma}}_{=0} \Big)\delta\gamma + \Big[\frac{\partial L}{\partial \dot\gamma}\delta\gamma\Big]_{t_1}^{t_2}$$

$\delta\dot\gamma \equiv \frac{d}{dt}\delta\gamma$

boundary cond in action principle.

Action principle $\Leftrightarrow$ Euler-Lagrange eqs (2nd order)

To go to first order: introduce momenta $p = \frac{\partial L}{\partial \dot\gamma} \leftrightarrow \boxed{\dot\gamma = \dot\gamma(p,\gamma)}$

Legendre transf. $\mathcal{H}(p,\gamma) = p\dot\gamma - L(\gamma,\dot\gamma,t)$

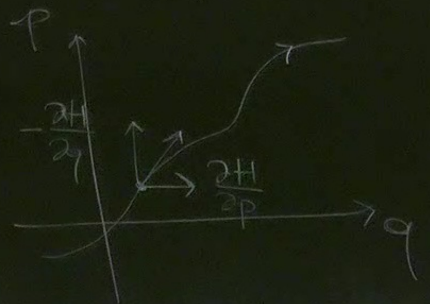
$$\begin{cases} \dot{p} = -\dfrac{\partial H}{\partial q} \\ \dot{q} = \dfrac{\partial H}{\partial p} \end{cases} \rightsquigarrow \begin{cases} \dot{p}_i = -\dfrac{\partial H}{\partial q} \\ \dot{q}_i = \dfrac{\partial H}{\partial p} \end{cases}$$

Phase space: $P = T^*Q \ni (q, p)$

$\omega = \sum_i dp_i \wedge dq^i$ symplectic

$d\omega = 0$

ω is nondegenerate.



$X_H \in \mathfrak{X}^1(P)$

$$X_H = \begin{pmatrix} \partial H / \partial p \\ -\partial H / \partial q \end{pmatrix}$$

$$\boxed{i_{X_H} \omega = -dH}$$

X_H = Ham. Vect. field of $H \in C^\infty(P)$

Note

Q mfd, $P = T^*Q$ is sympl.

Pf in coords. (q^i) over $U \subset Q$

$$T_q^*Q \text{ is a vector sp with basis } dq^i$$

$$\rotatebox{90}{$\in$}$$

that is $$\alpha = \sum_i \underbrace{p_i(\alpha)}_{\text{the components of } \alpha \text{ in basis } (dq^i)} dq^i$$

$\Rightarrow (q^i, p_i)$ are coords on $T^*U \subset T^*Q$

$\rightsquigarrow$ define

- sympl. pot. $\theta(q,p) = \sum_i p_i \, dq^i \in \Omega^1(T^*U)$
- sympl. 2-form $\omega = d\theta = \sum_i dp_i \wedge dq^i$

(dq^i)

$V \ni \underline{v} = \sum_i v^i \underline{e}_i = \sum_i \boxed{\langle \underline{f}^i, \underline{v} \rangle} \underline{e}_i$

$(\underline{e}_i)$ $\qquad$ $p_i(\alpha)$

$(\underline{f}^i)$ $\quad \langle \underline{f}^i, \underline{e}_j \rangle = \delta^i_j$

dq^i

basis (dq^i)

T^*Q

$\rightsquigarrow$ define

- sympl pot. $\theta(q,p) = \sum_i p_i \, dq^i \in \Omega^1(T^*U)$
- sympl. 2-form $\omega = d\theta = \sum_i dp_i \wedge dq^i$

θ: tautological 1-form

$$\alpha \in \Omega^1(Q) \leftrightarrow \tilde{\alpha} : Q \to T^*Q$$
$$q \mapsto (q, \alpha(q))$$

$$\Rightarrow \boxed{\tilde{\alpha}^* \theta = \alpha} \quad \forall \alpha \qquad \square$$

Thm (Dar

Locally (

$\left(-\partial H / \partial q \right)$ $\quad X_H =$ Ham. Vect. field of $H \in C^\infty(P)$

Thm (Darboux) (P, ω) sympl.

Locally ($\forall z \in P$, $\exists U_z \subset P$ open such that)

there exists coords $z^I = (q^i, p_i)$

$$\omega = \sum_i dp_i \wedge dq^i .$$

Def. A symplectomorphism is a vector field $X \in \mathfrak{X}^1(P)$ s.t.

$$L_X \omega = 0 .$$

□

Rmk $0 = L_X \omega = i_X d\omega + d i_X \omega$

$= d(i_X \omega)$

$\Rightarrow$ if $H^1(P) = 0$ (i.e. all closed 1-forms are exact)

then all symplectomorphisms are Hamiltonian -

that is $\exists f_X \in C^\infty(P) : i_X \omega = -df_X$

Relation to Poisson brackets

$$\omega = \frac{1}{2}\omega_{IJ}(z)\, dz^I \wedge dz^J \qquad z \in P$$

non deg $\Longleftrightarrow$ $\omega_{IJ}(z)$ invertible

$$\Pi^{IJ}(z) = -\left(\omega_{IJ}(z)\right)^{-1}$$

$$\Pi = \frac{1}{2}\Pi^{IJ}\frac{\partial}{\partial z^I}\wedge\frac{\partial}{\partial z^J}$$ Poisson bivector

$$\{f,g\} = \Pi(df \wedge dg) = \Pi^{IJ}\frac{\partial f}{\partial z^I}\frac{\partial g}{\partial z^J}$$ is Poisson

Jacobi $\Longleftrightarrow$ $d\omega = 0$

$\omega = -df_X$

Hamiltonia

$i_{X_H}\omega = -$

$\in P$

Poisson bivector

$^{I}\frac{\partial f}{\partial z^I}\frac{\partial g}{\partial z^J}$ is Poisson

Hamiltonian vf

$i_{X_H}\omega = -dH \iff X_H = \{H,\cdot\} = \Pi^{IJ}\frac{\partial H}{\partial z^I}\frac{\partial}{\partial z^J}$

Def ω presympl if closed & const rank.

Rmk ω presympl $\neq$ Poisson geom w/ non invertible Π^{IJ}