Title: Lecture - Quantum Information, PHYS 635

Speakers: Alex May

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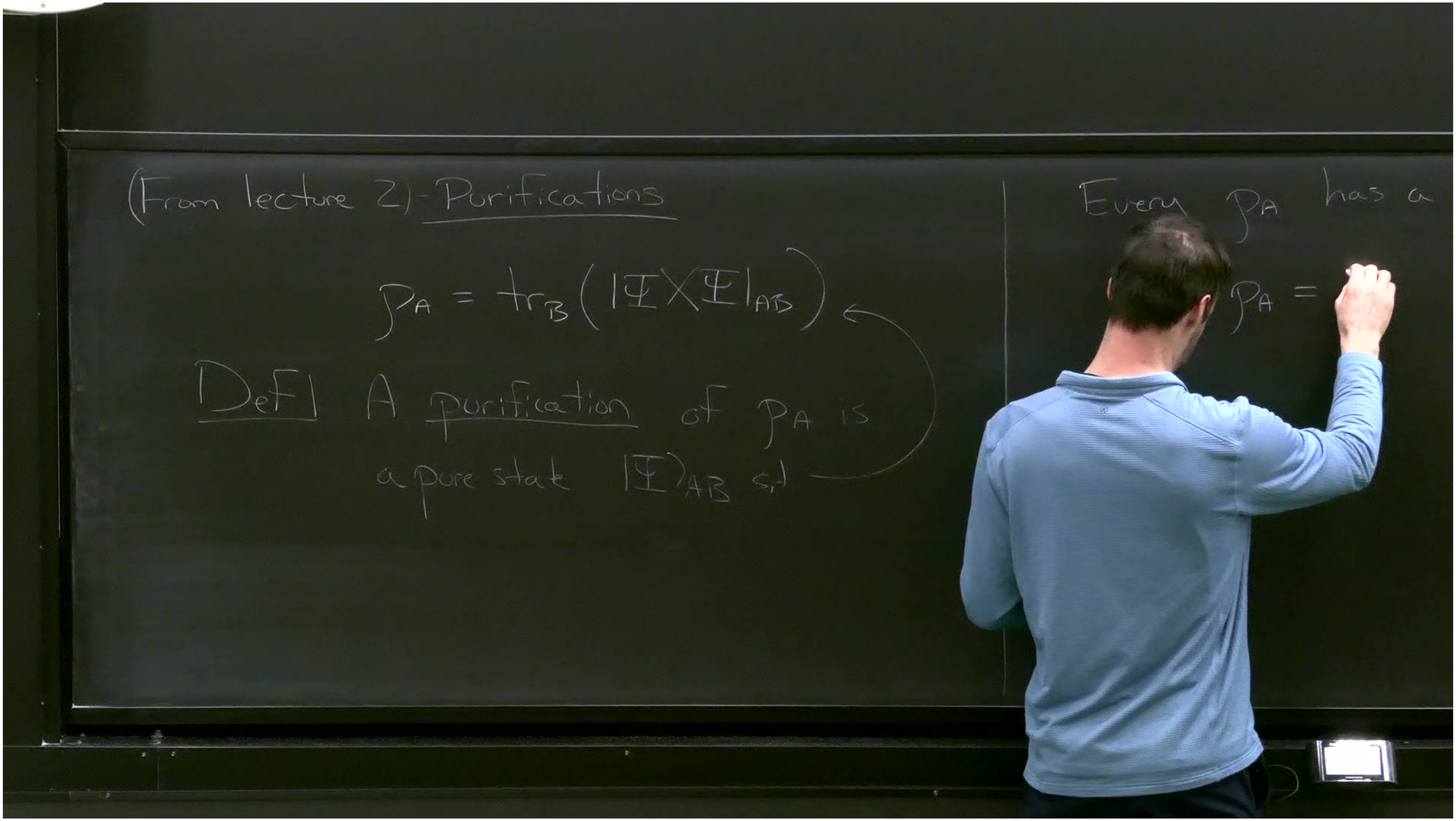
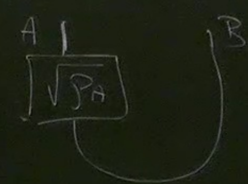

(From lecture 2) - Purifications
$\rho_A = \mathrm{tr}_B(|\Psi\rangle\langle\Psi|_{AB})$
Def| A purification of ρ_A is
a pure state $|\Psi\rangle_{AB}$ s.t
Every ρ_A has a
$\rho_A =$

Every ρ_A has a purification

$$\rho_A = \sum_i \lambda_i |\psi_i\rangle\langle\psi_i|_A$$

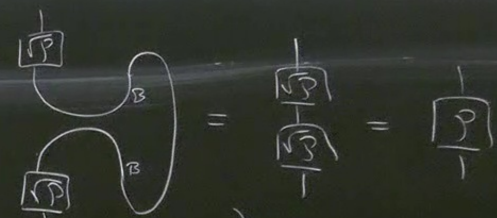
$$|\Psi\rangle_{AB} = \sum_i \sqrt{\lambda_i}\, |\psi_i\rangle_A |\varphi_i\rangle_B$$

$$|\Psi'\rangle_{AB} = \left(\sqrt{\rho_A} \otimes \mathbb{I}_B\right) \sum_i |\psi_i\rangle |i\rangle =$$



$$\ldots_{AB} = \left(\sqrt{\rho_A} \otimes \mathbb{I}_B\right) \sum \ldots$$

$$\mathrm{tr}_B\left(|\Psi'\rangle\langle\Psi'|_{AB}\right) =$$



$$\mathrm{tr}_B\left(\sum_i \sqrt{\lambda_i}\, |\psi_i\rangle|\varphi_i\rangle \sum_j \sqrt{\lambda_j}\, \langle\psi_j|\langle\varphi_j|\right)$$

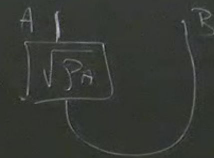
$$= \sum_{ij} \sqrt{\lambda_i \lambda_j}\, |\psi_i\rangle\langle\psi_j|_A \; \underbrace{\mathrm{tr}_B\left(|\varphi_i\rangle\langle\varphi_j|_B\right)}_{\delta_{ij}}$$

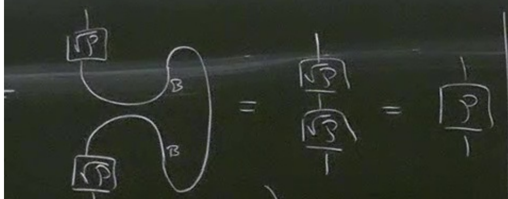
$|\Psi\rangle_{AB}$, $(\mathbb{I}_A \otimes V_{B\to C})|\Psi\rangle_{AB}$ also a purification

$V_{B\to C}$ isometric if $\quad V^{\dagger}_{B\to C} V_{B\to C} = \mathbb{I}_B$

$$\mathrm{tr}_C\left(V_{B\to C} |\Psi\rangle_{AB}\,{}_{AB}\langle\Psi| V^{\dagger}_{B\to C} \right)$$

$|\Psi\rangle_{AB} = \sum_i \sqrt{\lambda_i}\, |\psi_i\rangle_A |\varphi_i\rangle_B$

$|\Psi'\rangle_{AB} = \left(\sqrt{\rho_A} \otimes \mathbb{I}_B\right) \sum_i |\psi_i\rangle |i\rangle =$ 



$\sum_j \sqrt{\lambda_j}\, \langle\psi_j| \langle\varphi_j| \Big)$

$_A\ \mathrm{tr}_B\left(|\varphi_i\rangle\langle\varphi_j|_B\right)$

δ_{ij}

Thm If $|\Psi^0\rangle_{AB}$, $|\Psi'\rangle_{AC}$ both purify ρ_A, $d_B \leq d_C$

then $\exists\ V_{B\to C}$ s.t. $V_{B\to C} |\Psi^0\rangle_{AB} = |\Psi'\rangle_{AC}$

3- Quantum channels

Def. A quantum channel is a map $\mathcal{N}$ operators on $\mathcal{H}_A \to$ op. on $\mathcal{H}_A$ of the form:

$$\rho_A \to \mathcal{N}_A(\rho_A) = \mathrm{tr}_B\left(U_{AB} \left(\rho_A \otimes |0\rangle\langle 0|_B \right) U_{AB}^{\dagger} \right)$$

op. on $\mathcal{H}_A$.

$$\rho_A \longrightarrow \mathcal{N}(\rho_A) = |0\rangle\langle 0|_A$$

$$\mathrm{tr}_B\left(\mathrm{SWAP}_{AB}\left(\rho_A \otimes |0\rangle\langle 0|_B\right)\mathrm{SWAP}_{AB}^{\dagger}\right) = |0\rangle\langle 0|$$

$$\mathrm{tr}_B\left(|0\rangle\langle 0|_A \otimes \rho_B\right) = |0\rangle\langle 0|_A$$

$$U_{AB} \longrightarrow V_B U_{AB}$$

$$\mathrm{tr}_B\Big(\underbrace{V_B^\dagger V_B}_{\mathbb{I}} U_{AB} \left(\rho_A \otimes |0\rangle\langle 0|_B\right) U_{AB}^\dagger\Big) = \mathcal{N}(\rho)$$

Thm Every quantum operation is a map of the form:

$$\rho \to \mathcal{N}(\rho) = \sum_k E_A^k \, \rho_A \, (E_A^k)^\dagger$$

with E_A^k linear, $\sum_k (E^k)^\dagger E^k = \mathbb{1}$

with E^k_A linear, $\sum_k (E^k)^\dagger E^k = \mathbb{1}$

"Kraus decomposition"

q. operation → Kraus

$$\mathcal{N}(\rho_A) = \mathrm{tr}_B\left(U_{AB}\left(\rho_A \otimes |0\rangle\langle 0|_B\right) U_{AB}^\dagger\right)$$

$$= \sum_k \underbrace{{}_B\langle k|U_{AB}|0\rangle_B}_{E^k_A}\, \rho_A\, \underbrace{\langle 0|U^\dagger_{AB}|k\rangle_B}_{(E^k_A)^\dagger}$$

$$= \sum_k E^k_A\, \rho_A (E^k_A)^\dagger \quad ✓$$

$$|\Psi\rangle_{AB} = (\sqrt{\rho_A} \otimes \mathbb{I}_B) \sum_i |\psi_i\rangle|i\rangle = \boxed{\sqrt{\rho_A}}$$

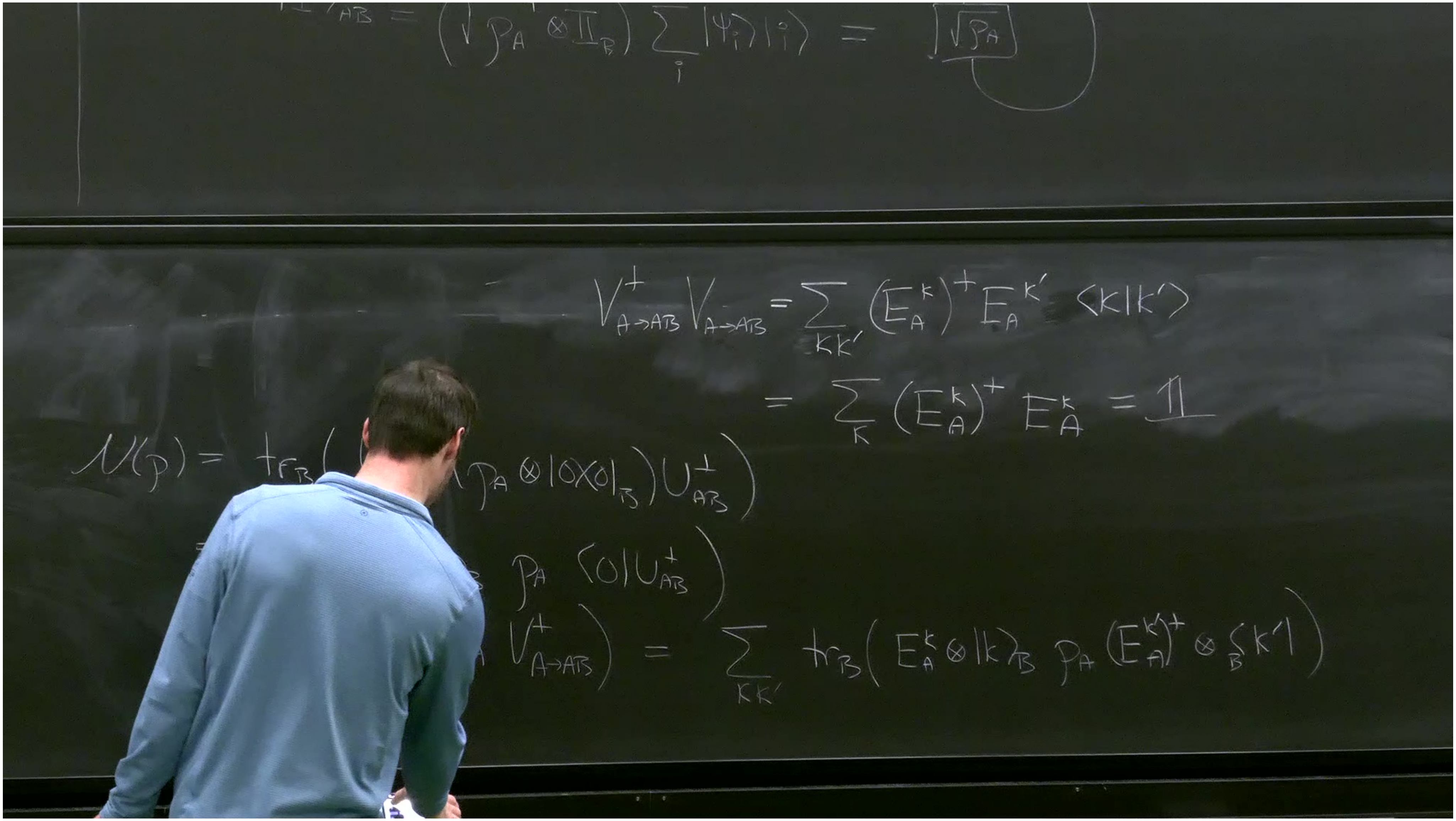
$$V^\dagger_{A\to AB} V_{A\to AB} = \sum_{kk'} (E_A^k)^\dagger E_A^{k'} \langle k|k'\rangle$$

$$= \sum_k (E_A^k)^\dagger E_A^k = \mathbb{1}$$

$$\mathcal{N}(\rho) = \mathrm{tr}_B\left(U_{AB} (\rho_A \otimes |0\rangle\langle 0|_B) U^\dagger_{AB} \right)$$

$$= \mathrm{tr}_B\left(U_{AB} |0\rangle_B \, \rho_A \, \langle 0| U^\dagger_{AB} \right)$$

$$= \mathrm{tr}_B\left(V_{A\to AB} \, \rho_A \, V^\dagger_{A\to AB} \right)$$


$V^{\dagger}_{A\to AB} V_{A\to AB} = \sum_{kk'} (E_A^k)^{\dagger} E_A^{k'} \langle k|k'\rangle$
$= \sum_k (E_A^k)^{\dagger} E_A^k = \mathbb{1}$
$\mathcal{N}(\rho) = \mathrm{tr}_B($
$\rho_A \otimes |0\rangle\langle 0|_B) U^{\dagger}_{AB})$
$\rho_A \langle 0| U^{\dagger}_{AB})$
$V^{\dagger}_{A\to AB}) = \sum_{kk'} \mathrm{tr}_B(E_A^k \otimes |k\rangle_B \rho_A (E_A^{k'})^{\dagger} \otimes {}_B\langle k'|)$

$$\mathrm{tr}_B\left(V_{A\to AB}\, \rho_A\, V^\dagger_{A\to AB} \right) = \sum_{kk'} \mathrm{tr}_B\left(E_A^k \otimes |k\rangle_B\, \rho_A\, (E_A^{k'})^\dagger \otimes {}_B\langle k'| \right)$$

$$= \sum_{kk'} E_A^k\, \rho_A (E_A^{k'})^\dagger\, \underbrace{\mathrm{tr}_B\left(|k\rangle\langle k'|_B \right)}_{\delta_{kk'}}$$

$$= \sum_k E_A^k\, \rho_A (E_A^k)^\dagger$$

$$ {}_B\left(V_{A\to AB}\,\rho_A\, V^{\dagger}_{A\to AB}\right) = \sum_{kk'} \mathrm{tr}_B\left(E_A^{k}\otimes |k\rangle_B\, \rho_A\, (E_A^{k'})^{\dagger} \otimes {}_B\langle k'| \right) $$

$$ = \sum_{kk'} E_A^{k}\,\rho_A\,(E_A^{k'})^{\dagger}\, \underbrace{\mathrm{tr}_B\left(|k\rangle\langle k'|_B \right)}_{\delta_{kk'}} $$

$$ = \sum_{k} E_A^{k}\,\rho_A\,(E_A^{k})^{\dagger} $$

$$ \rho = \sum \lambda_i\, |\psi_i\rangle\langle\psi_i|_A $$

$$ \to \sum \sqrt{\lambda_i}\, |\psi_i\rangle_A |\psi_i\rangle_B $$

$$\mathcal{N}(\rho) = p\rho + (1-p)X\rho X \qquad 0 \le p \le 1$$

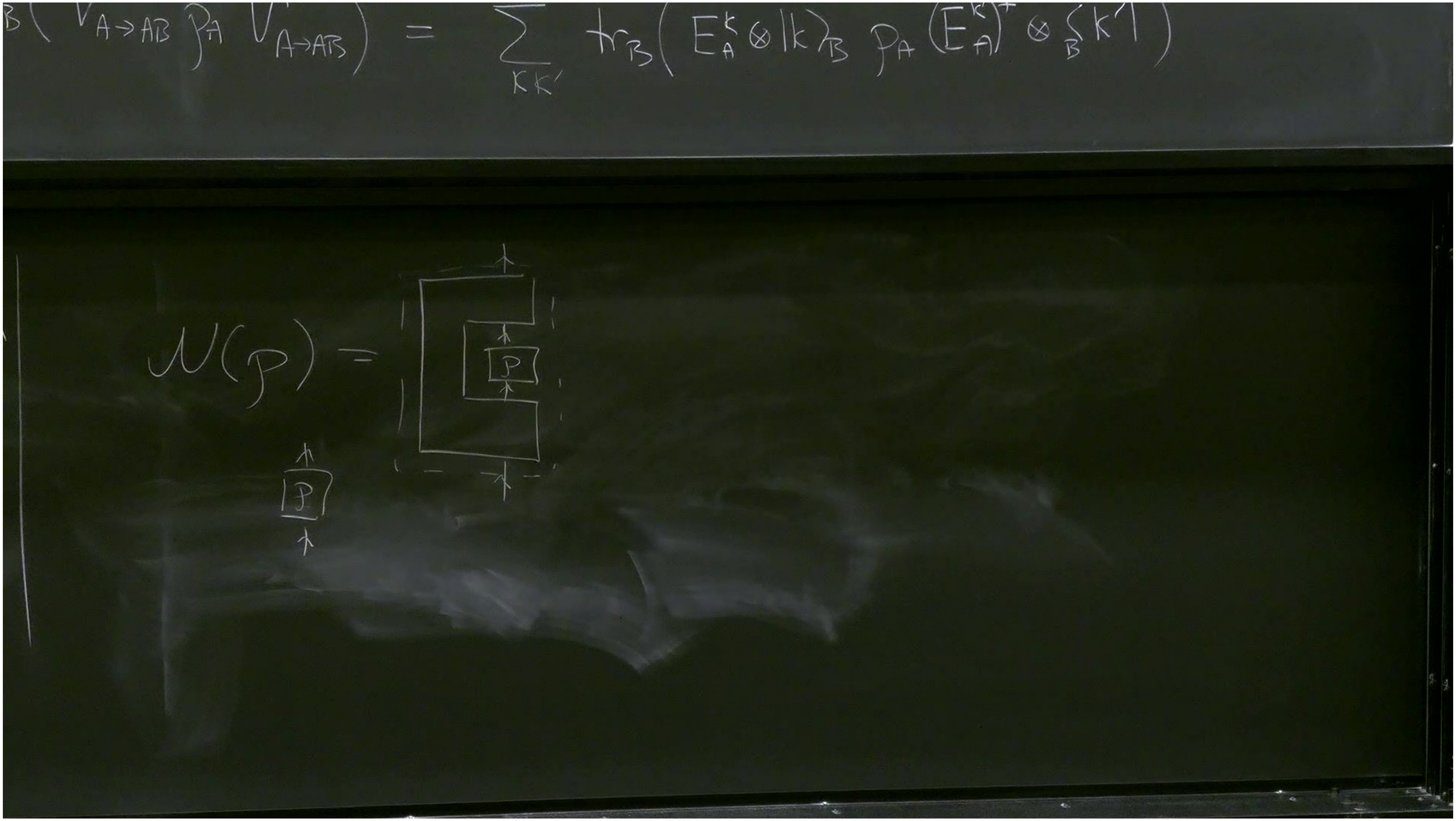
"bit flip"

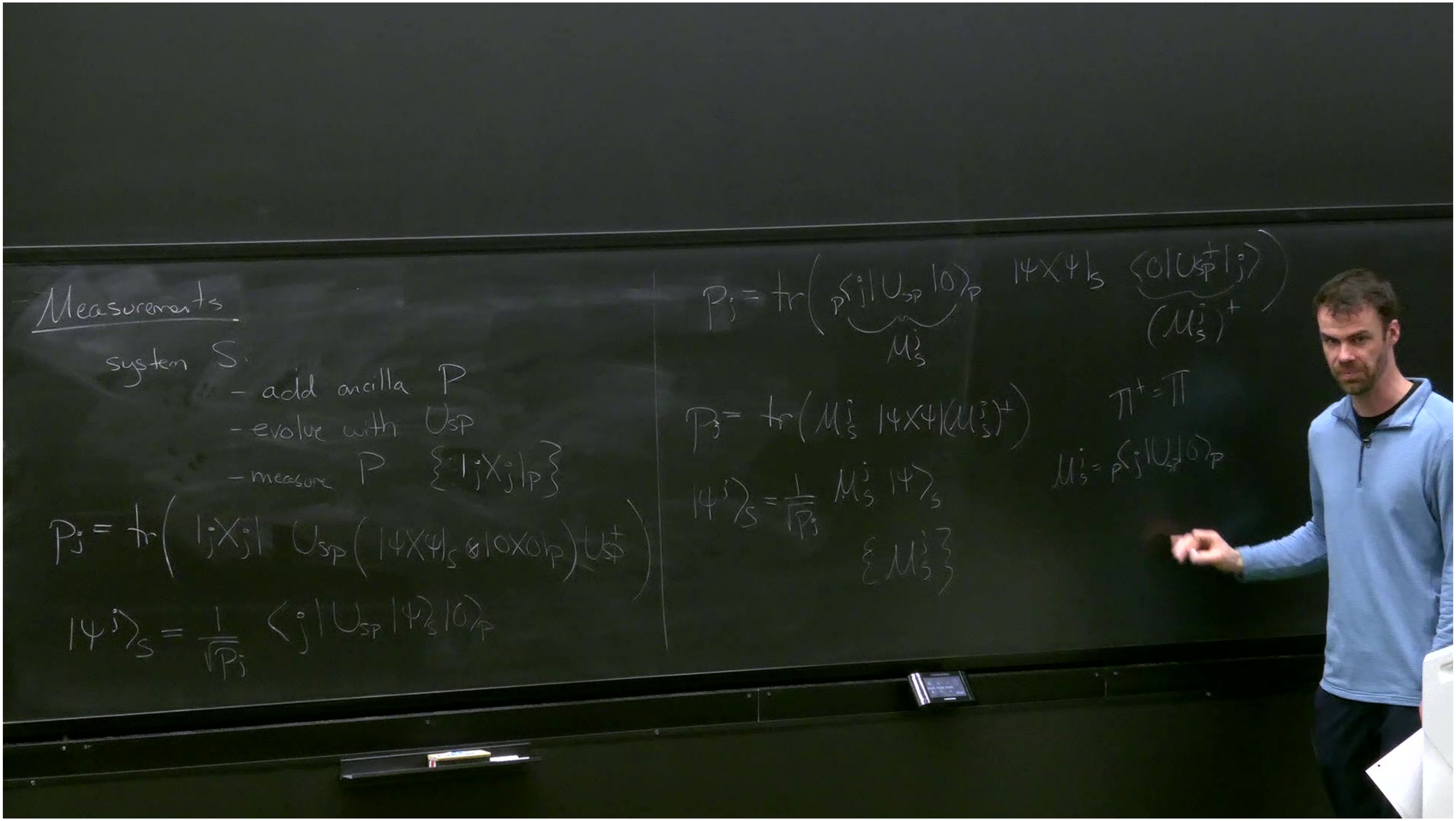
$$\{E_k\} = ?$$

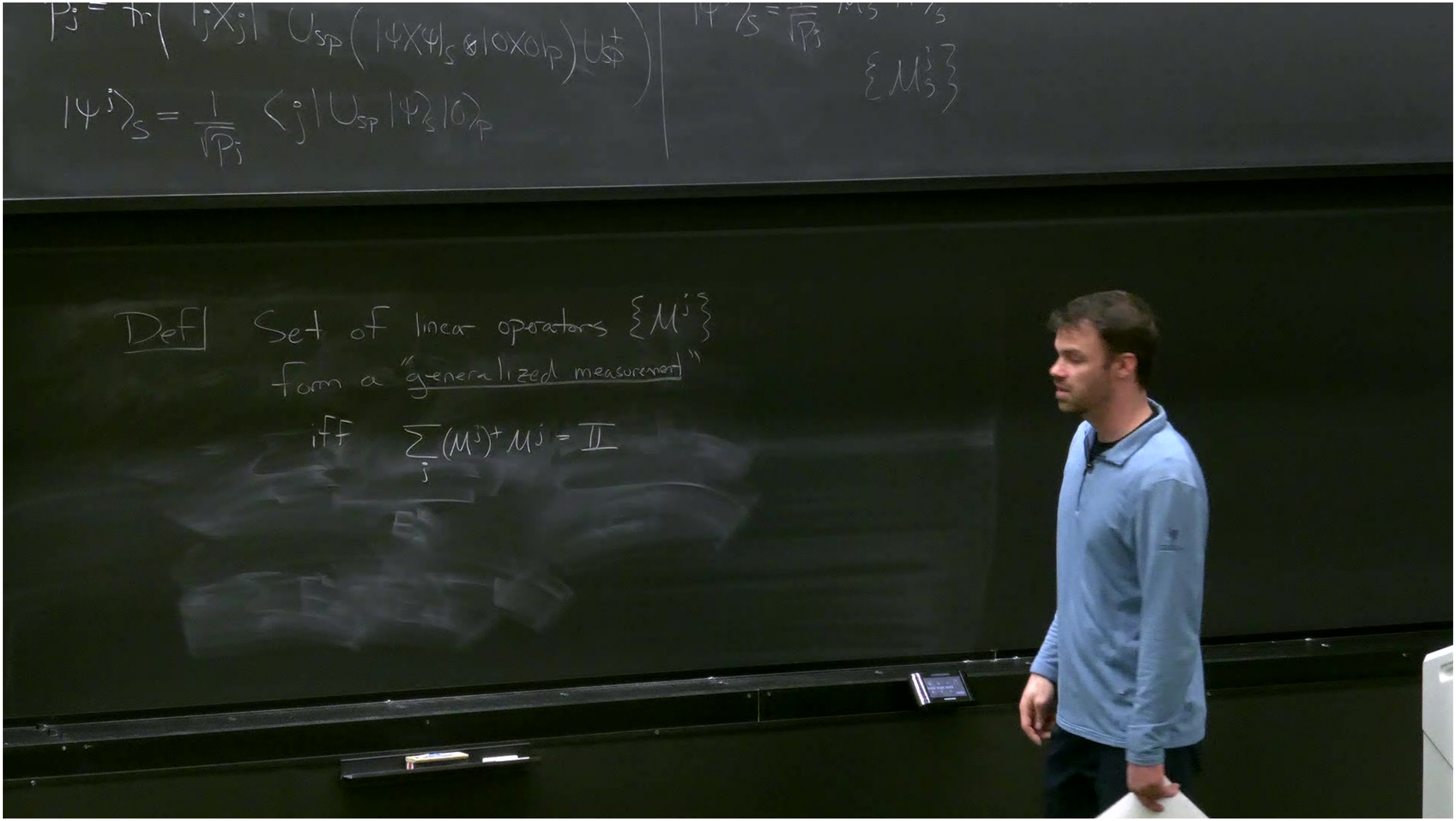
$$E_0 = \sqrt{p}\,\mathbb{I}$$

$$E_1 = \sqrt{1-p}\,X$$

$\rho =$


$= \sum_{kk'} tr_B(E^k_A \otimes |k\rangle_B \, \rho_A \, (E^{k'}_A)^\dagger \otimes {}_B\langle k'|)$
$\mathcal{N}(\rho) =$


Measurements
system S:
- add ancilla P
- evolve with U_{SP}
- measure P $\{|j\rangle\langle j|_P\}$
$p_j = \mathrm{tr}\left(|j\rangle\langle j| \, U_{SP} \left(|\psi\rangle\langle\psi|_S \otimes |0\rangle\langle 0|_P \right) U_{SP}^\dagger \right)$
$|\psi^j\rangle_S = \frac{1}{\sqrt{p_j}} \langle j| U_{SP} |\psi\rangle_S |0\rangle_P$
$p_j = \mathrm{tr}\left(\underbrace{{}_P\langle j| U_{SP} |0\rangle_P}_{M_S^j} |\psi\rangle\langle\psi|_S \underbrace{\langle 0| U_{SP}^\dagger |j\rangle}_{(M_S^j)^\dagger} \right)$
$p_j = \mathrm{tr}\left(M_S^j |\psi\rangle\langle\psi| (M_S^j)^\dagger \right)$
$|\psi^j\rangle_S = \frac{1}{\sqrt{p_j}} M_S^j |\psi\rangle_S$
$\{M_S^j\}$
$\Pi^\dagger = \Pi$
$M_S^j = {}_P\langle j| U_{SP} |0\rangle_P$


$|\psi^j\rangle_S = \frac{1}{\sqrt{p_j}} \langle j| U_{SP} |\psi\rangle_S |0\rangle_P$
$\{M^j_S\}$
Def
Set of linear operators $\{M^j\}$
form a "generalized measurement"
iff $\sum_j (M^j)^\dagger M^j = \mathbb{I}$