Title: Lecture - Quantum Field Theory III - PHYS 777

Speakers: Jaume Gomis

Collection/Series: Quantum Field Theory III, PHYS 777-, February 24 - March 28, 2025

Subject: Quantum Fields and Strings

Date: February 26, 2025 - 10:15 AM

URL: https://pirsa.org/25020007

CFT's ≡ QFT's w/ additional <u>spacetime</u> symmetries

1. kinematics (geometry, algebra ...)
   - local operators (represent symmetry on local operators)
2. Dynamics. (consistency conditions)

Kinematics

$\mathbb{Z}_2: \phi \to -\phi$

- Geometry of conformal transformations
- Symmetries form a group, in particular conformal transf. form a group G

Look at transformations connected to identity,
captured by the Lie algebra of G $\quad [t_a, t_b] = i f_{ab}^{\ \ c}\, t_c$

Jacobi identity

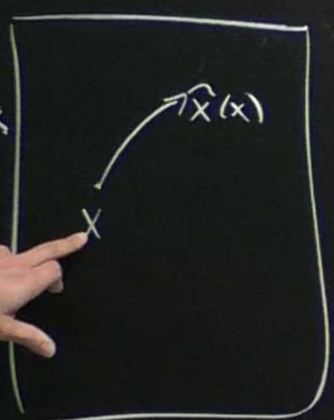
Conformal Transformations

1. Preserve angles between vectors (but not distances)

2. Preserves the lightcones.

$\mathbb{R}^{1,D-1}$

Minkowski



$$ds^2 \longrightarrow e^{2\sigma(x)} ds^2$$

$$\eta_{\rho\sigma}\, d\hat{x}^\rho d\hat{x}^\sigma = e^{2\sigma(x)}\, \eta_{\mu\nu}\, dx^\mu dx^\nu$$

$$\eta_{\rho\sigma} \frac{\partial \tilde{x}^\rho}{\partial x^\mu} \frac{\partial \tilde{x}^\sigma}{\partial x^\nu} = \left| \frac{\partial \tilde{x}}{\partial x} \right|^{2/D} \eta_{\mu\nu} \qquad R^\rho_\mu(x) = \frac{\partial \tilde{x}^\rho}{\partial x^\mu} e^{-\sigma(x)}$$

- Look at ∞'al transformations

$$\tilde{x}^\mu = x^\mu + \xi^\mu(x) \qquad R^T(x)\,\eta\, R(x) = \eta$$

$$\eta_{\rho\sigma} \left(\delta^\rho_\mu + \partial_\mu \xi^\rho \right)\left(\delta^\sigma_\nu + \partial_\nu \xi^\sigma \right) = \left(1 + \frac{2}{D} \partial\cdot\xi \right) \eta_{\mu\nu} \quad O(\xi)$$

$O(\xi^0)$ ✓

$O(\xi)$: $\boxed{\partial_\mu \xi_\nu + \partial_\nu \xi_\mu = \frac{2}{D}(\partial\cdot\xi)\, \eta_{\mu\nu}}$

Weyl transformation that can be undone by a diffeomorphism

$$\begin{cases} \text{Weyl} & \delta_\sigma g_{\mu\nu} = 2\sigma(x)\, g_{\mu\nu} \\ \text{Diffeo:} & \delta_\xi \eta_{\mu\nu} = \mathcal{L}_\xi \eta_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \end{cases}$$

CKV must obey

$$\left(\partial_\rho \partial_\sigma \partial_\mu \xi_\nu = 0\right)$$

Killing vector: $\partial_\rho \partial_\sigma \xi_\nu = 0$

$D = 2$

$\mu = \nu = 1$

$\mu = 1, \nu = 2$

$$\boxed{\begin{aligned} \partial_1 \xi_1 &= \partial_2 \xi_2 \\ \partial_1 \xi_2 &= -\partial_2 \xi_1 \end{aligned}}$$

Cauchy-Riemann

$$\Downarrow$$

$z = x^1 + i x^2$

$$\partial_{\bar z} \xi = 0$$

$\xi = z^{n+1}$

Virasoro algebra

#	D	$\frac{D(D-1)}{2}$	1	D	total #
	P_μ	$M_{\mu\nu}$	$\hat{D}$	K_μ	$\frac{(D+2)(D+1)}{2}$
			(dilatation operator)	special conformal	dim of conformal algebra

captured by the Lie algebra of G $\quad [t_a, t_b] = i f_{ab}^{\ \ c}\, t_c$

Jacobi identity

$M_{\mu\nu}$ D K_μ

(dilatation operator) special conformal dim of conformal algebra

$\xi^\mu_{(1)}$, $\xi^\mu_{(2)}$

$[\xi_{(1)}, \xi_{(2)}] = \xi_{(3)}$

Poincaré
$$\begin{cases} [M^{\mu\nu}, M^{\rho\sigma}] = i\eta^{\mu\rho} M^{\nu\sigma} + \text{antisymmetize in } \mu \leftrightarrow \nu,\ \rho \leftrightarrow \sigma \\ [M^{\mu\nu}, P^\rho] = i(\eta^{\mu\rho} P^\nu - \eta^{\nu\rho} P^\mu) \end{cases}$$

$[M^{\mu\nu}, K^\rho] = i(\eta^{\mu\rho} K^\nu - \eta^{\nu\rho} K^\mu)$

$[M^{\mu\nu}, \hat{D}]$

$$[\hat{D}, P_\mu] = -i P_\mu \qquad [\hat{D}, K_\mu] = i K_\mu$$

$$[P^\mu, K^\nu] = -2i\left(M^{\mu\nu} + \eta_{\mu\nu}\hat{D}\right).$$

$$\hat{D} = x^\mu \frac{\partial}{\partial x^\mu}$$

$$P_\mu = \frac{\partial}{\partial x^\mu}$$

$$\left[x^\mu \frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu} \right] f = x^\mu \cancel{\partial_{\mu\nu}} f - \left(\frac{\partial}{\partial x^\nu} x^\mu\right) \frac{\partial}{\partial x^\mu} \propto \frac{\partial}{\partial x^\nu}$$

$$[\hat{D}, P_\mu] = -i P_\mu \qquad [\hat{D}, K_\mu] = i K_\mu$$

$$[P^\mu, K^\nu] = -2i\left(M^{\mu\nu} + \eta_{\mu\nu}\hat{D}\right)$$

$$M = (\mu, D, D+1)$$

$$[L_{MN}, L_{PQ}] = i\,\hat{\eta}_{MP} L_{NQ} + \begin{array}{c}\text{anti}\\ M \leftrightarrow N\\ P \leftrightarrow Q\end{array}$$

$$\hat{\eta} = (-1, \underbrace{1, \ldots, 1}_{D}, 1, -1)$$

$$L_{MN} = \begin{cases} L_{\mu\nu} = M_{\mu\nu} \\ L_{D\,D+1} = \hat{D} \\ L_{\mu D} = \frac{1}{2}(P_\mu + K_\mu) \\ L_{\mu\,D+1} = \frac{1}{2}(P_\mu - K_\mu) \end{cases}$$

$$[\hat{D}, P_\mu] = -i P_\mu \qquad [\hat{D}, K_\mu] = i K_\mu$$

$$[P^\mu, K^\nu] = -2i\left(M^{\mu\nu} + \eta_{\mu\nu}\hat{D}\right)$$

$$M = (\mu, D, D+1)$$

$$[L_{MN}, L_{PQ}] = i\,\hat{\eta}_{MP} L_{NQ} + \begin{smallmatrix}\text{anti}\\ M \leftrightarrow N \\ P \leftrightarrow Q\end{smallmatrix}$$

$$\hat{\eta} = (\underbrace{-1, 1, \dots, 1}_{D}, 1, -1)$$

$$L_{MN} = \begin{cases} L_{\mu\nu} = M_{\mu\nu} \\ L_{D\,D+1} = \hat{D} \\ L_{\mu D} = \frac{1}{2}(P_\mu + K_\mu) \\ L_{\mu\,D+1} = \frac{1}{2}(P_\mu - K_\mu) \end{cases}$$

Lie algebra $SO(2, D)$

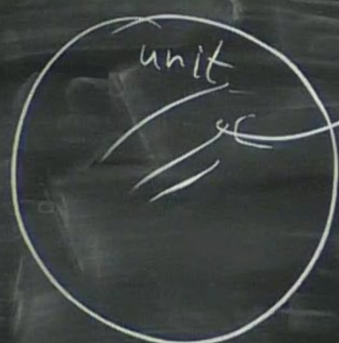
$\Updownarrow$

isometries of AdS_{D+1}

Inversion: $\mathbb{Z}_2$ transformation

$I: \; x^\mu \to x^\mu / x^2 \qquad I^2 = 1$

1. Show it is conformal



2. Reverses orientation of Spacetime

$dx^1 \wedge \cdots \wedge dx^D \to -(dx^1 \wedge \cdots \wedge dx^D)$

conformal group: combining Poincaré w/ I.

Inversion

$I: x^\mu \to x^\mu / x^2$ $\quad I^2 = 1$ $\quad dx^1 \wedge \cdots \wedge dx^D \to -(dx^1 \wedge \cdots \wedge dx^D)$

1. Show it is conformal

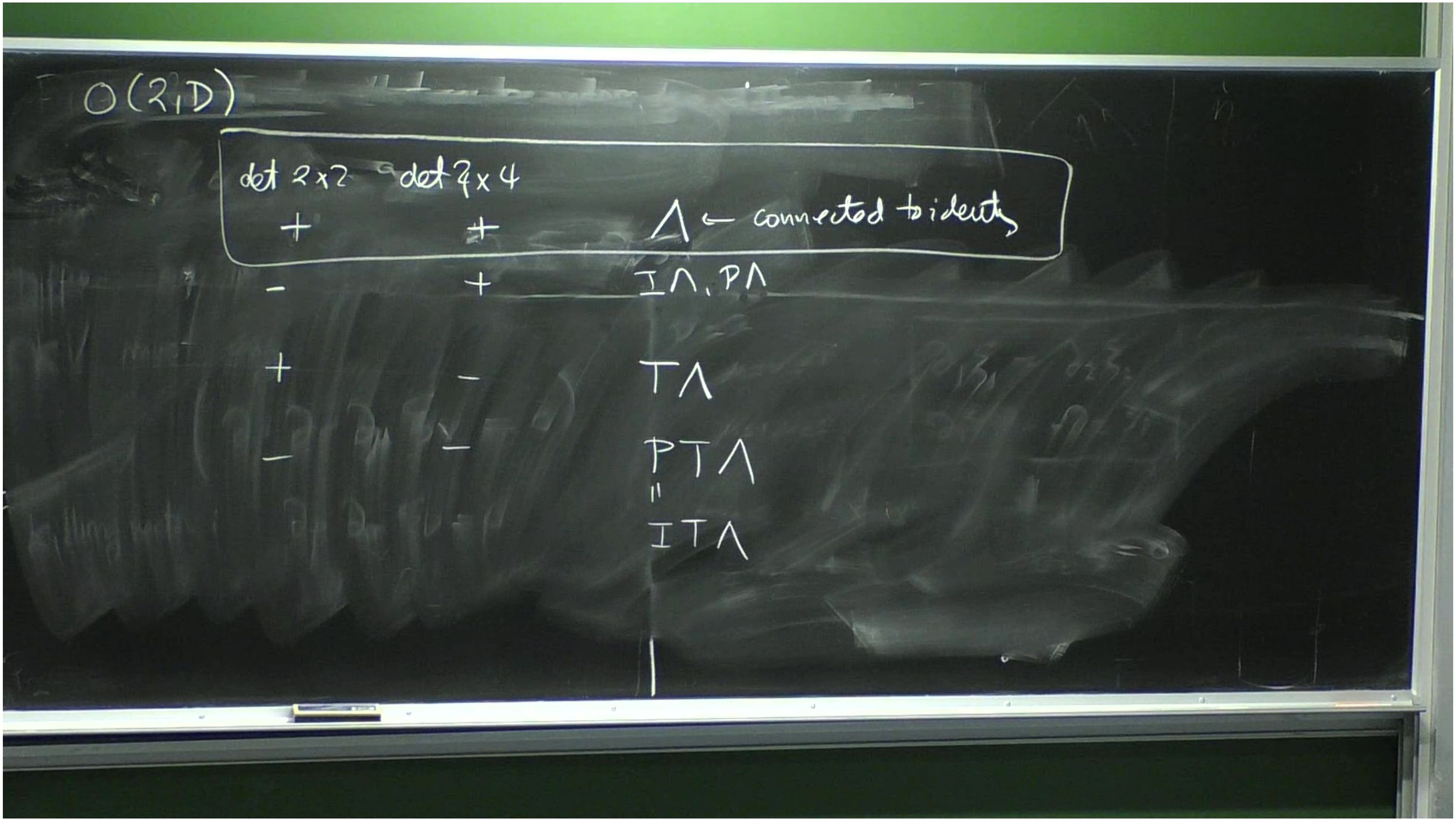
conformal group: combining Poincaré w/ I.

$I P_\mu I = K_\mu \Longrightarrow$ Derive the form of a finite conformal transformation

$I K_\mu I = P_\mu$

$I M_{\mu\nu} I = M_{\mu\nu}$

$I \hat{D} I = -\hat{D}$


O(2,D)
det 2x2
det 4x4
+
+
Λ ← connected to identity
−
+
IΛ, PΛ
+
−
TΛ
−
−
PTΛ
=
ITΛ