Title: Lecture - Mathematical Physics, PHYS 777-

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Recap The principal bundle $G \xrightarrow{\pi} P \rightarrow B$

- G - Lie grp
- P is a manifold with smooth right G-action $R_g \cdot p \mapsto pg \ \forall \ p \in P, g \in G$.
- $B = P/G$ is a quotient space, with a smoot proj $\pi : P \rightarrow P/G = B$.
  $p \mapsto [p] = \{q \in P \mid \exists g \in G. \ q = pg\}$

  Since the action is free $\Rightarrow \pi^{-1}(x) \cong G$ (as manifolds).
- The bundle is locally trivial: $\forall \ x \in B \ \exists \ U \ni x, \ \varphi : \pi^{-1}(U) \rightarrow U \times G$ s.t.

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\varphi"] \arrow[d, "\pi"'] & U \times G \arrow[dl, "proj_1"] \\ U & \end{tikzcd}$$

& $\varphi(pg) = \varphi(p) \cdot g \quad \forall \ p \in P, g \in G$

$R_g(x, h) = (x, h \cdot g)$

φ "respects" G-action

φ is equivariant map

- The local trivialization (U_i, φ_i), $\varphi_i^{-1} \circ \varphi_j : U_i \cap U_j \times G \to U_i \cap U_j \times G$,
$(x, h) \mapsto (x, g_{ij}(x) \cdot h)$

$g_{ij} : U_i \cap U_j \to G$ satisfies all the gluing cocycle axioms.

<u>Remark</u> Principal bundles are like vector bundles but without vector spaces.

- Sections $s \in \Gamma(B, E)$ are smooth $s : B \to P$ s.t. $\pi \circ s = id$.
- <u>Theorem</u> $G \to P \to B$ is trivial $\Leftrightarrow$ $\exists$ smooth global section $s \in \Gamma(B, P)$
- <u>Example</u> homogeneous space $SO(2) \to SO(3) \to S^2$.
  - action $SO(3) \circlearrowleft SO(2)$ is free. $g h = g \Leftrightarrow h = id$. $\left(\begin{array}{c|c} * & \begin{matrix} 0 \\ 0 \end{matrix} \\ \hline \begin{matrix} 0 & 0 \end{matrix} & 1 \end{array}\right) \subset \left(* \right)$

  $SO(2)$ - rotation angle

  S^2 - rotation axis.

Associated bundles - how to construct V.b out of p.b?

- Representation ρ of Lie G. $\rho: G \to \mathrm{End}(V) \cong GL(V) = GL(\dim V)$

  that satisfies $\rho(e) = id$, $\rho(g_1 g_2) = \rho(g_1)\cdot\rho(g_2)$

Examples $\rightsquigarrow GL(V) \curvearrowright V$ $\rightsquigarrow GL(V) \curvearrowright V^*$: $\rho^*(g) = (g^{-1})^T$

$$(\rho^*(g)\cdot w)^T \cdot \rho(g) v = w^T v$$

$\rightsquigarrow$ If V_1, V_2 - two representations $\Rightarrow$ $\rho(g)\cdot v_1 \otimes v_2 = \rho_1(g) v_1 \otimes \rho_2(g)\cdot v_2$

$\Rightarrow$ This makes $T^p_q(V)$ to be $GL(V)$-representation

• The associated bundle. $G \to P \xrightarrow{\pi} B$ - principal bundle

\+ vector space F & left representations of G on F.

$\rho : G \longrightarrow \mathrm{End}(F)$

$$\Rightarrow \quad E = (P \times F)/G, \text{ where } \quad R_g(p, v) = (p \cdot g, \ \rho(g^{-1}) \cdot v)$$

$$\rho((g_1 g_2)^{-1}) = \rho(g_2^{-1})\, \rho(g_1^{-1})$$

The projection map is $\tilde{\pi} : E \longrightarrow B$.

$$\tilde{\pi}(p, v) = \pi(p),$$

• This defines a structure of a vector bundle $F \longrightarrow E \xrightarrow{\tilde{\pi}} B$

and trivialization $\varphi_i \circ \varphi_j^{-1} : (x, v) \longmapsto (x, \rho(g_{ij}) v)$

the cocycle

Example Frame bundle LM, M – n-dim manifold

$$LM = \bigcup_{p\in M} L_pM, \quad L_pM = \left\{ \begin{array}{l} \text{sets of } n \text{ linearly independent} \\ \text{vectors } u=(X_1,\dots,X_n),\ X_i \in T_pM \end{array} \right\}$$

The right action by $GL(n)$ $R_g\,(X_1,\dots,X_n) = X_\beta\, g^\beta{}_\alpha$

The action is free and transitive $\Rightarrow LM \to LM/GL(n) = M$ defines principal bundle

$\rightsquigarrow$ The local trivialization: $X_\alpha = X^\mu{}_\alpha \frac{\partial}{\partial x^\mu}$ $X^\mu_\alpha \in GL(n)$

Under coordinate change, on $U\cap V$ (x on U, y on V) $X_\alpha = X^\mu{}_\alpha \left.\frac{\partial}{\partial x^\mu}\right|_p = \tilde{X}^\nu{}_\beta \left.\frac{\partial}{\partial y^\nu}\right|_p$

$$\Rightarrow g_{UV}(x) = \left.\left(\frac{\partial x^\mu}{\partial y^\nu}\right)\right|_p \in GL(V), \quad g_{UV}: U\cap V \to GL(n).$$

Claim All the vector bundles $T^p_q(TM)$ are associated with LM.

Connections on vector bundles

"how to identify neighbouring fibers", "how to define derivation on sections"?

Let $F \to E \xrightarrow{\pi} B$, $\Gamma(B, E)$ - global sections $\ni s_i$

The covariant derivative (connection) $\nabla : \Gamma(E) \to \Gamma(T^*B \otimes E)$

bundle

- $\mathbb{R}$-linear : $\alpha, \beta \in \mathbb{R}$ $\nabla(\alpha s_1 + \beta s_2) = \alpha \nabla(s_1) + \beta \nabla(s_2)$.
- $f \in C^\infty(B, \mathbb{R})$: $\nabla(f \cdot s) = df \otimes s + f \cdot \nabla(s)$

Remark ∇ can also act on $\Omega^k(E) = \Gamma(\Lambda^k T^* B \otimes E)$

$$\nabla(\omega \otimes s) = d\omega \otimes s + (-1)^{\deg \omega} \omega \wedge \nabla(s)$$

$$\nabla : \Omega^k(E) \to \Omega^{k+1}(E)$$

The parallel transport along $\gamma : [0,1] \to B$, $\gamma(0) = x_0$ $\gamma(1) = x_1$

$$\tau_\gamma : F_{x_0} \to F_{x_1}$$

The section s is parallel to ∇ iff $\nabla_{\dot\gamma} s \equiv i_{\dot\gamma(t)} \gamma^*(\nabla s) = 0$

linear differential equation on s

$\tau_\gamma(v) = s|_{x_1}$, where s is parallel to ∇, and $s|_{x_0} = v$.

Since the differential equation is linear, corresp. initial condition → →solution at given point is also linear.

Remark Result τ_γ depends on γ.

- the curvature – how much two infinitesimal parallel transports do not commute

$$F_\nabla \in \Omega^2(\mathrm{End}\, E) : F_\nabla(X,Y)(s) = \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s \quad \forall X,Y \in \mathrm{Vect}(B),\ s \in \Gamma(B,E)$$

$$F_\nabla(X,Y)(f s) = f\, F_\nabla(X,Y)(s)$$

Now let's coordinatize (U,φ) - local trivialization, $\varphi: \pi^{-1}(U) \to U \times F$

Let's pick a frame of local sections $e_i \in \Gamma(U,E)$, $i=1,\dots,\mathrm{rk}\,E = k$

$\Rightarrow s = \sum_i s^i e_i \quad \forall s \in \Gamma(U,E)$.

$$\Rightarrow \nabla(e_i) = \sum_{j=1}^{k} A_i^j \otimes e_j, \quad A_i^j \in \Omega^1(U)$$

$\Rightarrow$ the matrix (A_i^j) can be interpreted as an element of $\Omega^1(\mathrm{End}\,E)$

local connection form.

$$\nabla(s) = \nabla\Big(\sum s^i e_i\Big) = \sum_{j=1}^{k}\Big(ds^j + \sum_{i=1}^{k} A_i^j s^i\Big) \otimes e_j, \quad \nabla(s) = ds + As, \quad s = \begin{pmatrix} s^1 \\ \vdots \\ s^k \end{pmatrix}$$

Remark: It is easy to check that under change of the chart:

$x \rightsquigarrow A_i^j = A_{i\mu}^j dx^\mu$

$$A' = gAg^{-1} - dg \cdot g^{-1}$$

(for change of trivialization the formulas would be similar)

$$\nabla(s') = g \cdot \nabla(s), \quad F' = gFg^{-1}.$$