Title: TBA - Quantum Foundations Seminar

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WHY BOSONS AND FERMIONS?

A COMBINATORIAL APPROACH

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SKETCH OF A PROGRAM

The problem of identical particles

An operational perspective

Combinatorics and representations

A theorem on partition functions

New statistics = New physics

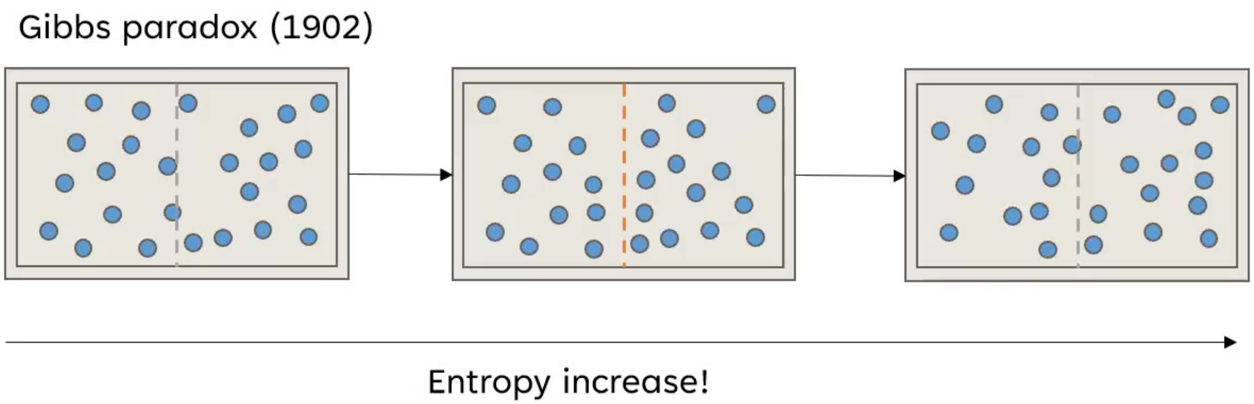
Back to algebras and symmetries

A theorem on canonical relations

Introduction to combinatorial operationalism

Ideas to take home

WHY IDENTICAL PARTICLES?



Problem: entropy fails to be extensive (additive on subsystems)

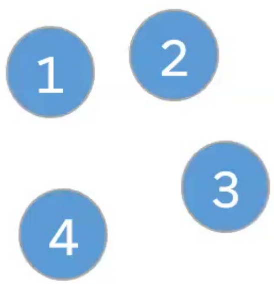
Solution: many-particle states differing only by particle interchange are numerically the same

IDENTICAL PARTICLES IN QUANTUM MECHANICS

Symmetrization postulate

a label is attached to each particle, and it is necessary to introduce permutation symmetry to express indistinguishability.

Problem: artificial labels for indistinguishable particles should not persist in time, hence symmetrization is not feasible operationally.



Canonical relations

identical particles are generated by a single field and algebraic (bilinear) relations between field operators (e.g. CCR, CAR).

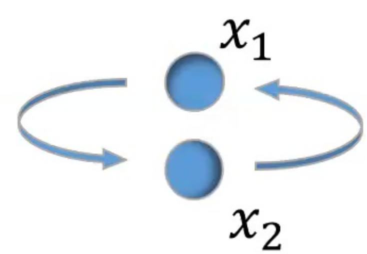
Problem: these relations are *ad hoc*, introduced as a postulate rather than derived from any physical principle.

$$[a(f), a^\dagger(g)] = \langle f \mid g \rangle$$
$$\{a(f), a^\dagger(g)\} = \langle f \mid g \rangle$$

Topology

particles are embedded in the configuration space of identical particles and their statistical behavior is derived from the topology of such space.

Problem: relies on the non-quantum abstract assumption of the configuration space for identical particles and only applies to structureless particles.



THE QUESTION IS…

"Other more complicated kinds of symmetry are possible mathematically, but do not apply to any known particles"
Dirac, (1930) Principles of Quantum Mechanics

So far, we have not observed generalized statistics of particles in Nature. Hence:

1. We need more precise and sophisticated experiments, or
2. These generalizations collide with basic laws of physics

Operational approach: define a typical quantum experiment and address physical questions

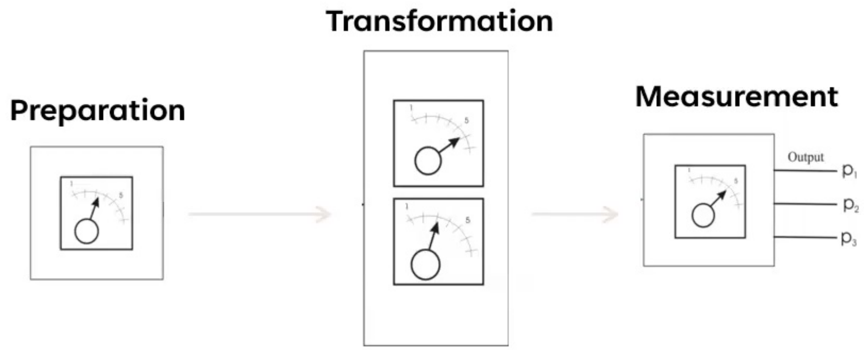
1. How to define identical particles experimentally?
2. How to establish a operational differentiation between types of particles?

Property of a system ⟷ Feasible protocol to verify it

"IT SEEMS TO US, HOWEVER, THAT NO COMPLETELY SATISFACTORY DISCUSSION ON THE CONSEQUENCES OF INDISTINGUISHABILITY, IN THE CONTEXT OF NONRELATIVISTIC QUANTUM MECHANICS, HAS EMERGED SO FAR."

J. M. Leinaas, J. Myrheim, *On theory of identical particles* (1976)

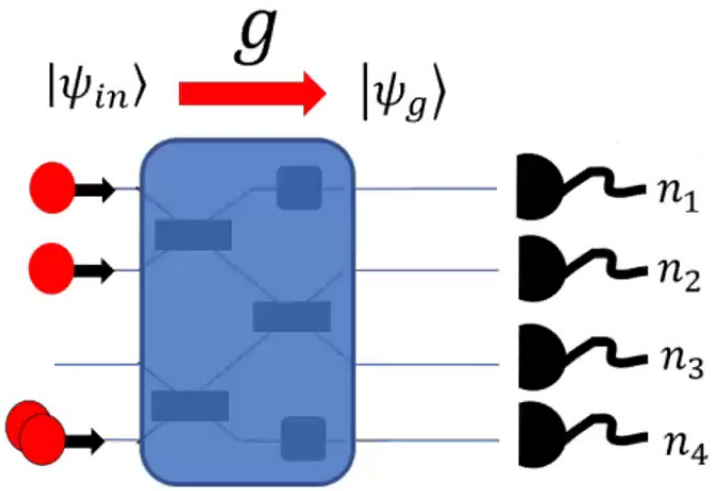
A TYPICAL QUANTUM EXPERIMENT



Operationally well-motivated assumptions → Classification of quantum particle statistics

MY EXPERIMENTAL SETUP

This is an interferometer:



One (quantum) particle:

1. A state is an element of a complex Hilbert space of dimension d (e.g. d=4)
2. A transformation is defined by an element (e.g. g) of the unitary group U(d)
3. Probability of a detection is calculated using the Born rule.

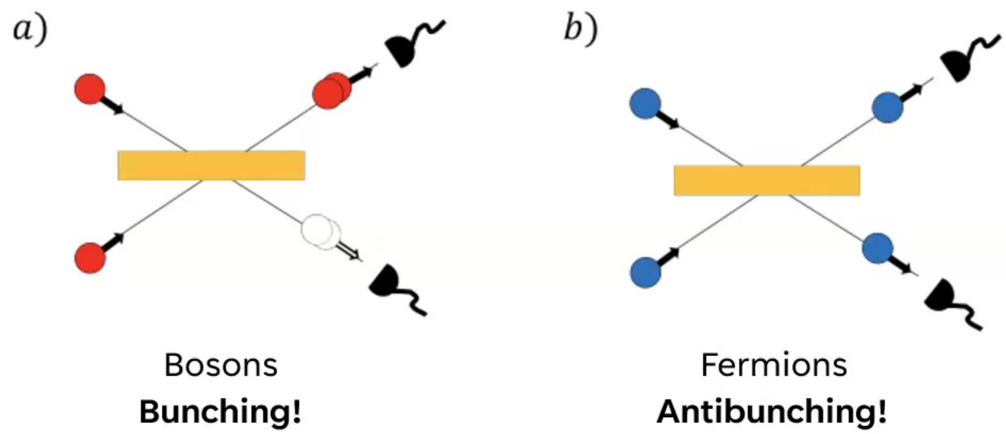
What happens if many particles are injected in this setup?

INSPIRED ON HONG-OU-MANDEL

Outcomes	Distinguishable	Bosons	Fermions
(2,0)	1/4	1/2	0
(1,1)	1/2	0	1
(0,2)	1/4	1/2	0

(R,L)

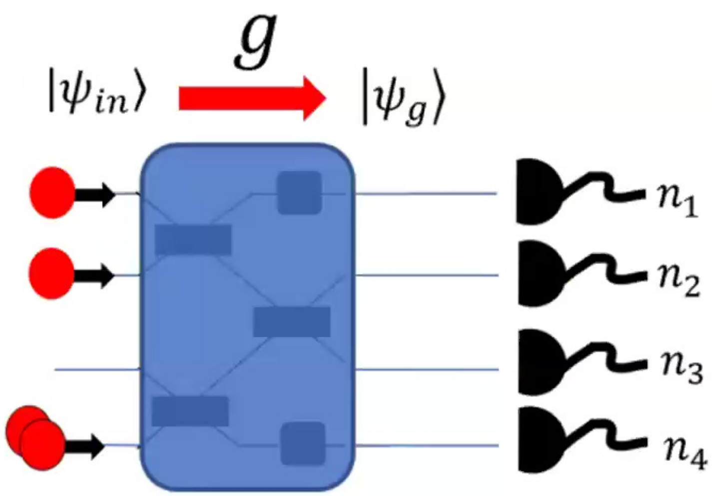
Probabilities



Simple interference experiments distinguish between different types of particles!

IN MATHEMATICAL TERMS?

1. We have a representation $\Delta: U(d) \to GL(F)$, with F the state space of the system.
2. We are claiming that each state can be characterized by a string of numbers $(n_1, \ldots, n_d)$ corresponding to the number of particles detected in each mode. In other words, each mode carries a representation of $U(1)$ generated by the „number of particles" observable.



Main problem: which representations of the unitary group are compatible with a factorization in d isomorphic reducible representations of $U(1)$

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A PRIMER ON CHARACTER THEORY OF $U(d)$

Remark 1: A representation Δ of $U(d)$ is in 1-to-1 correspondence with a function χ_Δ called the **character** of the representation

$$g \in U(d) \to \Delta(g) \to \chi_\Delta(g) = Tr(\Delta(g))$$

Remark 2: Character is a class function. Given $s \in U(d)$.

$$\chi(s^{-1}gs) = \chi(g)$$

Remark 3: A Unitary matrix can be diagonalized.

$$g = sts^{-1}, \qquad t = diag\left[e^{i\theta_1}, \dots, e^{i\theta_d}\right] \in T_d$$

Remark 4: Character is a function on the maximal torus

$$\chi = \chi(e^{i\theta_1}, \dots, e^{i\theta_d})$$

Remark 5: For the unitary group $U(d)$ an **irreducible representation** characterized by a Young diagram λ has as character a symmetric polynomial known as a **Schur polynomial** on the variables $x_i = e^{i\theta_i}$

$$\chi_\lambda(x_1, \dots, x_d) = s_\lambda(x_1, \dots, x_d)$$

LOCALITY ASSUMPTION

Main problem: which representations of the unitary group are compatible with a factorization in d isomorphic reducible representations of $U(1)$

Translating to character language

$$\underbrace{\chi_F(x_1,\dots,x_d)}_{\text{Character of the } U(d) \text{ representation on the state space } F} = \sum_\lambda \underbrace{c_\lambda s_\lambda(x_1,\dots,x_d)}_{\text{An integer non-negative combination of irreducible characters}} = \prod_{i=1}^{d} \underbrace{\chi(x_i)}_{\text{Character of the } U(1) \text{ representation on each mode}}$$

This is the **locality assumption**

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EXAMPLE: BOSONS AND FERMIONS

$$\chi_{fermions}(x_1, \dots, x_d) = \prod_{i=1}^{d} (1 + x_i)$$

$$\chi_{bosons}(x_1, \dots, x_d) = \prod_{i=1}^{d} \frac{1}{(1 - x_i)}$$

MAIN PROBLEM REVISITED

$$\sum_{\lambda} \underbrace{c_\lambda s_\lambda(x_1, \ldots, x_d)} = \prod_{i=1}^{d} \underbrace{\chi(x_i)}$$

An integer non-negative combination of irreducible characters

Character of the $U(1)$ representation on each mode

SOLVE THIS EQUATION: Find for which $\chi(x_i)$ the left-hand side is **always** a valid representation of $U(d)$, i.e., such that c_λ is **always** a non-negative integer.

What are we doing? Recall, this comes from the interferometric setup. Finding all $\chi(x_i)$ that solve the previous equation is the same as **finding all valid state spaces** for a multiparticle system described by our setup!

A valid state space is then a vector space with a specific decomposition into irreducibles of the unitary group

EXAMPLE: BOSONS AND FERMIONS

$$\chi_{fermions}(x_1, \dots, x_d) = \prod_{i=1}^{d} (1 + x_i) \longrightarrow F_{fermions} = \bigoplus_{vertical\ \lambda} F_\lambda$$

$$\chi_{bosons}(x_1, \dots, x_d) = \prod_{i=1}^{d} \frac{1}{(1 - x_i)} \longrightarrow F_{bosons} = \bigoplus_{horizontal\ \lambda} F_\lambda$$

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THERMODYNAMICS TO THE RESCUE

Recall the definition of character:

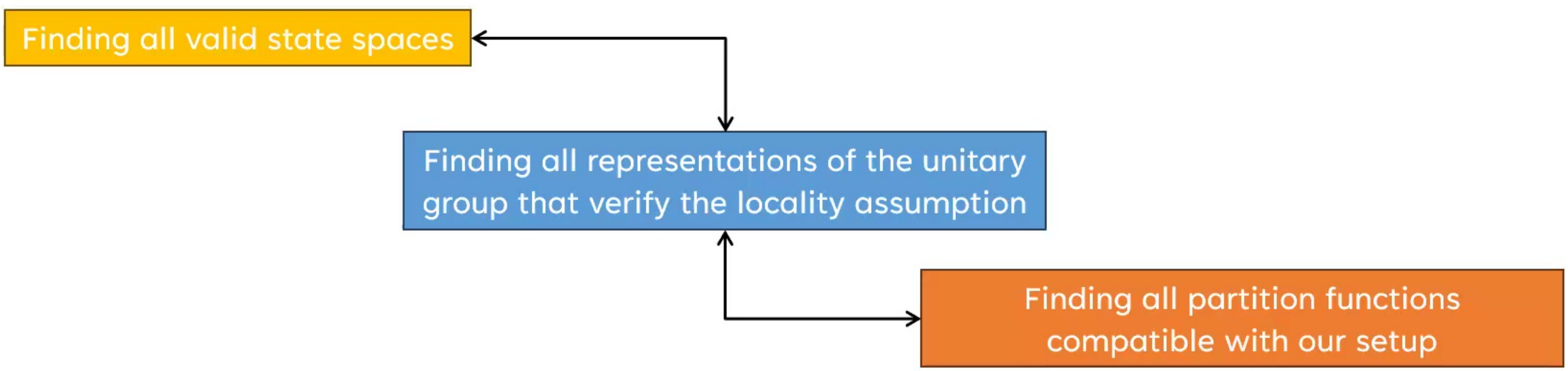
$$\chi_\Delta(g) = Tr(\Delta(g))$$

Given that the unitary group is connected, we can write this in terms of its Lie algebra w.l.o.g

$$\chi_\Delta\left(e^{iH}\right) = Tr(e^{i\Delta(H)})$$

If we perform a rotation on the complex plane, what we get is the **partition function** of the system

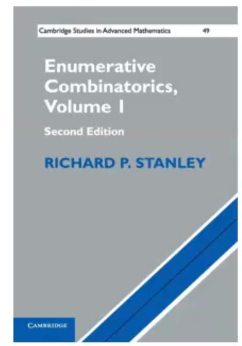
$$\chi_\Delta\left(e^{\beta H}\right) = Tr\left(e^{\beta\Delta(H)}\right) = Z(\beta)$$



EUREKA!

1. Solve this equation $\sum_{\lambda} c_{\lambda} s_{\lambda}(x_1, \dots, x_d) = \prod_{i=1}^{d} \chi(x_i)$
2. Using Lie theory it can be proven that c_{λ} is non-negative if and only if for $\chi(x) = \sum a_i x^i$ all the minors of the upper triangular Töplitz matrix $a_{ij} = a_{i-j}$ are non-negative
3. A deep theorem in algebraic combinatorics (Edrei-Thoma) classifies all upper triangular Töplitz matrices with non-negative minors! (Thanks to Richard Stanley for showing us this!)

$$A = \begin{bmatrix} a_0 & a_{-1} & a_{-2} & \cdots & \cdots & a_{-(n-1)} \\ a_1 & a_0 & a_{-1} & \ddots & & \vdots \\ a_2 & a_1 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & a_{-1} & a_{-2} \\ \vdots & & \ddots & a_1 & a_0 & a_{-1} \\ a_{n-1} & \cdots & \cdots & a_2 & a_1 & a_0 \end{bmatrix}$$



Edrei-Thoma theorem (adapted): An upper triangular Töplitz matrix has non-negative minors if and only if the function $f(x) = \sum a_i x^i$ has the form

$$f(x) = \frac{\prod(1+\alpha_i x)}{\prod(1-\beta_j x)}$$

With α_i, β_j non-negative real numbers.

MAIN THEOREM OF PARTITION FUNCTIONS

A partition function for an interferometric system that verifies the locality assumption can be written as

$$Z(\beta;\alpha,\beta) = \prod_{i=1}^{d} Z\left(e^{\beta\epsilon_i}\right) = \prod_{i=1}^{d} \left(\frac{\prod(1+\alpha_j e^{\beta\epsilon_i})}{\prod(1-\beta_k e^{\beta\epsilon_i})}\right)$$

Where the products are finite and the parameters such that the single-mode functions are rational functions

Contrast!

$$Z_{fermions}(\beta) = \prod_{i=1}^{d}(1+e^{\beta\epsilon_i}) \qquad Z_{bosons}(\beta) = \prod_{i=1}^{d}\frac{1}{(1-e^{\beta\epsilon_i})}$$

Medina Sanchez & Dakic, arXiv:2306.05919, 2023

EUREKA!

1. Solve this equation $\sum_{\lambda} c_{\lambda} s_{\lambda}(x_1, \dots, x_d) = \prod_{i=1}^{d} \chi(x_i)$
2. Using Lie theory it can be proven that c_{λ} is non-negative if and only if for $\chi(x) = \sum a_i x^i$ all the minors of the upper triangular Töplitz matrix $a_{ij} = a_{i-j}$ are non-negative
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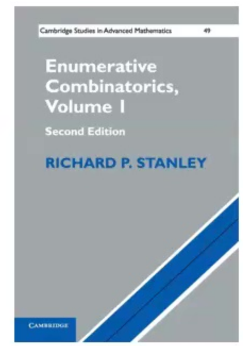

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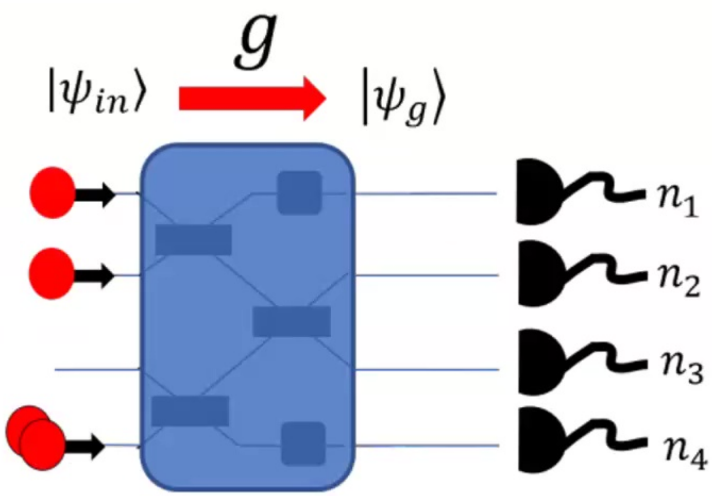
With α_i, β_j non-negative real numbers.

CLOSE ENOUGH!



This device is almost enough to discriminate between bosons and fermions. Other types of statistics **can be detected**, but they are similar to bosons and fermions up to degeneracies!

BACK TO STATE SPACES: TRANSTATISTICS

Recall the starting point: to find the valid state spaces. Which state spaces do we have?

The most general state space for a multiparticle system that can be described by our interferometric setup is a tensor product of the following spaces

$$F_f = \bigoplus_\lambda s_{\lambda^t}(\alpha_1, \ldots, \alpha_k) F_\lambda$$

Transfermionic

$$F_b = \bigoplus_\lambda s_{\lambda^t}(\beta_1, \ldots, \beta_l) F_\lambda$$

Transbosonic

The crucial difference with standard fermionic and bosonic statistics is the degeneracy of the irreducible sectors, i.e., **existence of unknown degrees of freedom.** Hence if we assume total knowledge about the internal structure of the particles **we recover bosonic and fermionic statistics**

NEW STATISTICS

=

NEW PHYSICS

NEW STATISTICS

=

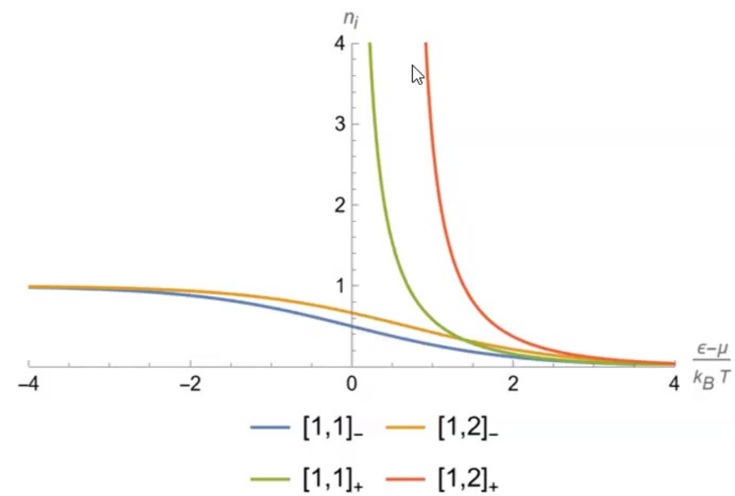
NEW PHYSICS

IDEAL GASES

Given the partition function we can calculate standard thermodynamic quantities

$$N = \sum_i n_i = \sum_i \frac{1}{\frac{1}{q} e^{\beta(\epsilon_i - \mu)} \pm 1}$$

Only 1 transfermionic or transbosonic parameter



Mean particle number for ordinary (blue and green) and generalized (orange and red) statistics

This additional parameter generates a residual entropy not present in ordinary statistics. This entropy does not vanish at T=0, hence the ground state is highly degenerate, an indicator of **spontaneous symmetry breaking!**

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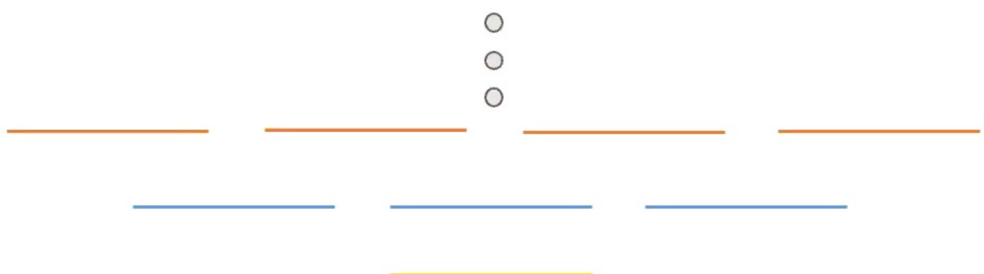
BACK TO ALGEBRAS AND SYMMETRIES

TO CREATE AND TO ANNIHILATE

Can we derive transtatistics in an algebraic way, similarly than we can do with bosonic and fermionic out of the bilinear algebras of Canonical Commutation and Canonical Anticommutation relations? **YES WE CAN**

Remark 1: notice that the coefficients in the single mode partition function $Z(x) = \sum_{i=0}^{\infty} a_i x^i$ are the dimensions of the „number“ sectors, i.e., $a_i = \dim(V_i)$



2 $a_2 = 4$

1 $a_1 = 3$

0 $a_0 = 1$

TO CREATE AND TO ANNIHILATE

Remark 2: This means that the single mode partition function is the Hilbert-Poincaré series of the $U(1)$ representation that we have in each mode seen as a graded vector space.

$$\underbrace{V = \bigoplus_{i\in\mathbb{N}} V_i}_{\text{Graded vector space}} \qquad \underbrace{\sum_{i\in\mathbb{N}} \dim_K(V_i)t^i}_{\text{Hilbert-Poincaré series of V}}$$

Hilbert-Poincaré (HP) series reflect „well-behaviour“, i.e., when the graded vector space verify certain symmetries the HP series will be constrained, e.g. the Hilbert-Serre theorem: if V is a representation of a Noetherian ring the the HP series is a polynomial divided by an specific denominator.

KOSZUL PROPERTY

The Koszul property is a notion of simplicity in graded vector spaces. It indicates that higher degrees are determined quickly by lower degrees. Examples of spaces with the Koszul property are **the fermionic and bosonic state spaces.** This is a projection of the idea that creation-annihilation algebras are bilinear and generate „efficiently“ the whole vector space. We have the following theorem

Theorem (Sam, Vandebogert; 2024): there is a graded vector space with the Koszul property such that its Hilbert-Poincare series is of the form $f(x) = \frac{\prod(1+\alpha_i x)}{\prod(1-\beta_j x)}$, with the same conditions as in the Edrei-Thoma theorem.

Using a classic result on quadratic algebras we can prove:

Lemma: Transtatistics can be generated by bilinear relations

Several alternative statistics are defined with higher-order relations! E.g. Green‘s parastatistics. **We do not need that!**

SUMMARY SO FAR

- Transtatistics are different than other (generalized) statistics, e.g. parastatistics, quons, fractal statistics etc.
- Transtatistics comes from the unitary group symmetry and a locality assumption on counting
- Ordinary statistics come from irreducibility (no hidden symmetry)
- Transtatistics can be generated by bilinear relations (well-behaved creation-annihilation algebras)

Some comments/outlook:

(1) Why are transtatistics non-physical?

(2) Interesting maths (algebraic combinatorics ⟺ particle statistics)

(3) Relation to thermodynamics (character ⟺ partition function, via complex rotation)

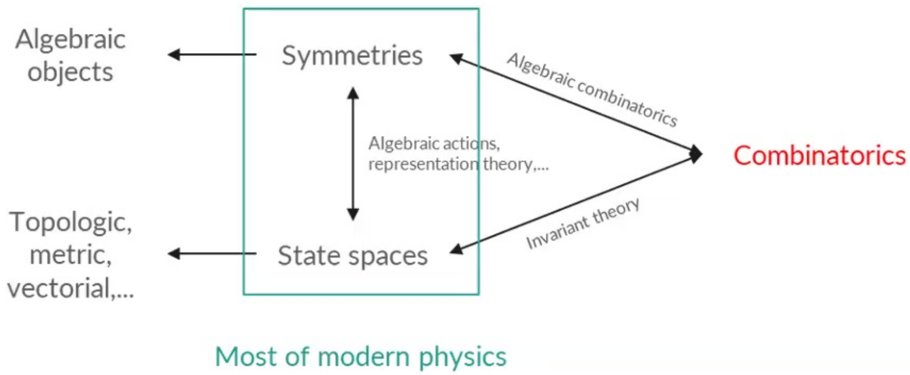
(4) Novel integrable models (generalized Jordan-Wigner transformation)?

(5) Quantum computing perspective (intermediate models):

Bosonic linear optics	Fermionic linear optics	Transtatistics
(Bosons sampling)	(Matchgate computations)	???
hard	**easy**	**???**

INTRODUCTION TO COMBINATORIAL OPERATIONALISM (PERSPECTIVES)

THE PHILOSOPHY OF COMBINATORIAL OPERATIONALISM



*An agent can only **select**, **arrange** and **combine** finite data*

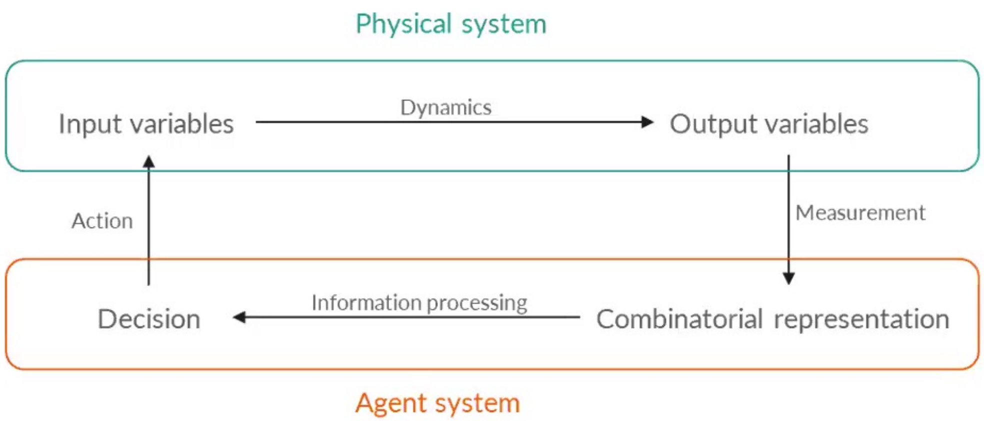
THE RECONSTRUCTION PROGRAM

Equivalence class of symmetric spaces ↕ Formal theories

Combinatorial object ↕ Agent-based theories

Operational reconstructions of physical theories = When is this map 'covariant?'

PHYSICAL CYBERNETICS



PROJECTS

- Combinatorial reconstruction of projectivity and unitarity
- Combinatorial reconstruction of causality and geometry
- The quantum-to-classical transition

THANK YOU!

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