Title: Is the no-cloning theorem truly quantum? Topological Obstructions to Cloning in Classical Mechanics

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Abstract:

the no-cloning theorem is an essential result in quantum information on top of which many quantum cryptography protocols are built. In this talk we examine the cloning question in the context of classical mechanics/Hamiltonian mechanics. We find the answer is quite subtle: whether a mechanical system can be cloned depends on the topological structure of its phase space. In particular, for a system to be clonable, its phase space must be contractible. This means certain systems (e.g. particle moving on a line) is clonable, while others (e.g. the simple pendulum) cannot be cloned. We explain the idea of the proof, which uses tools from algebraic topology (homotopy groups and Whitehead's theorem). Finally we discuss the physical interpretations of this result: how do we reconcile this theorem with the experience that generally speaking, classical information is clonable? Can we use this no-cloning theorem to build secure communication protocols in classical systems instead of quantum ones?

Is the No-Cloning Theorem Truly Quantum?
Non-Clonable Phase Spaces in Classical Mechanics

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January 27, 2025

Contents

Introduction: The Quantum No-Cloning Theorem

The Quantum No-Cloning Theorem

- The no-cloning theorem in quantum mechanics is an essential result in quantum information on top of which many quantum cryptography protocols are built.
- The version of the theorem we will need is the following: we cannot build a machine that copies the (unknown) state of a quantum system onto another identical quantum system.

The Quantum No-Cloning Theorem

Theorem

[Fen12; Sca+05] Let $\mathcal{H}$ and $\mathcal{H}'$ both be finite dimensional complex Hilbert spaces, with $\dim \mathcal{H} > 1$. There cannot exist vectors $|b\rangle \in \mathcal{H}, |r\rangle \in \mathcal{H}'$ and an unitary map $U : \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}' \to \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}'$ taking

$$|\psi\rangle \otimes |b\rangle \otimes |r\rangle \to |\psi\rangle \otimes |\psi\rangle \otimes |r'\rangle$$

for every $|\psi\rangle \in \mathcal{H}$. Here $|r'\rangle \in \mathcal{H}'$ depends on $|\psi\rangle$

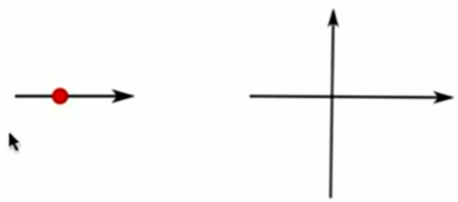
Application to Secure Communication

- This no-cloning theorem is used construct secure communication protocols between Alice and Bob. The idea is that if they exchange quantum states, then Eve cannot make a perfect copy of the quantum state without Alice and Bob noticing.

Cloning in Classical Mechanics

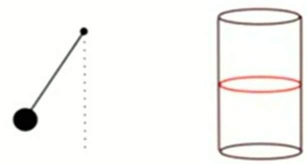
- What should be the analogue of the cloning process in classical mechanics?
- Replace Hilbert space $\mathcal{H}$ with the phase space T^*M of the system.
- Phase space: if a particle is confined to move on a geometric space M, the phase space is the space of all its possible states by specifying both its position and (generalized) momentum. It's the classical analogue of the Hilbert space of states.

- Example. If a particle is confined to move on the real line $\mathbb{R}$, the phase space is $T^*\mathbb{R} \cong \mathbb{R}^2$, consisting $(x, p) \in \mathbb{R}^2$ that specifies its position and momentum.



Cloning in Classical Mechanics ct'd

- Example. If a particle is confined to move in a circle S^1 (for example a simple pendulum). Then its phase space is $T^*S^1 = \mathbb{R} \times S^1$, which is a 2 dimensional cylinder. The variables $(L, \theta) \in \mathbb{R} \times S^1$. L specifies its angular momentum, and θ is its angular position.



- Generally speaking for a particle confined to move in M, its phase space is the cotangent bundle of M, which we write as T^*M. It's a space that specifies both the position and momentum of the particle. The cotangent bundle comes with a natural symplectic structure, which helps specify the dynamics of the system (but we will gloss over this point for now).

Cloning in Classical Mechanics ct'd: dynamics

- To specify dynamics on a phase space T^*M, we need to specify a Hamiltonian $H : T^*M \to \mathbb{R}$, or it may be time dependent which we write as $H_t : T^*M \to \mathbb{R}$. Morally speaking a Hamiltonian assigns an energy to each state.
- The dynamics is specified by solving Hamilton's equations. Let the pair $(x, p) \in \mathbb{R}^n \times \mathbb{R}^n$ denote (canonical) local coordinates on our phase space, the time evolution, after starting at point (x, p), is given by

$$x'(t) = \frac{\partial H}{\partial p}, \quad p'(t) = -\frac{\partial H}{\partial x}.$$

- These assemble into the notion of a Hamiltonian flow on the entire phase space $\phi_t : T^*M \to T^*M$. Given a point $(x, p) \in T^*M$, $\phi_t(x, p)$ is by definition which state it will evolve into after time t.
- Then obviously $\phi_0 = Id$, since no time has passed, and $\phi_t : T^*M \to T^*M$ is continuous, and depends continuously on t.

Definition of cloning in classical mechanics

- We want to capture the notion that cloning is a process where " we use a machine that copies the (unknown) state of one particle onto another identical particle in some initial state".
- Recipe: Replace Hilbert spaces with phase spaces; replace tensor products of Hilbert spaces with products of phase spaces; replace unitary maps with time-1 Hamiltonian flow.

Definition

A classical cloning process is given by

- A phase space T^*M of the system to be cloned and a phase space T^*N of the cloning machine.
- A point $b \in T^*M$, a point $r \in T^*N$, a (potentially time dependent) Hamiltonian on $T^*M \times T^*M \times T^*N$ whose time-1 flow is a map that sends

$$\phi_1 : (y, b, r) \to (y, y, h(y, b, r))$$

  for all $y \in T^*M$. Here $h(-, b, r)$ is some smooth function from T^*M to T^*N.

If the above data exists for the the phase space T^*M, then we say it is **clonable**.

So which phase spaces can be cloned?

- So which phase spaces can be cloned?

Theorem

[Fen12] The phase space $T^*\mathbb{R}^n \cong \mathbb{R}^{2n}$ is clonable. In other words, a particle moving in Euclidean space is clonable.

- In fact you can write down such a Hamiltonian relatively explicitly [Fen12; RSS19]. This mirrors our experience classical information is clonable.

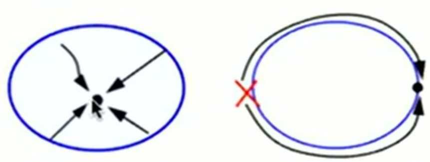
Go to page 93

So which phase spaces cannot be cloned?

Theorem

[Yao23] If a phase space T^*M is clonable, then it must be contractible.

- What is contractible? Consider the space X. Contractible means being able to find a 1 parameter family of continuous maps $f_t : X \to X$ that starts at $f_0 = Id$, and ends at the constant map: for all $x \in X$ we have $f_1(x) = p$ for some fixed $p \in X$.
- In other words, in a contractible space I can simultaneously move all points in the space X to a given point $p \in X$ in a continuous fashion.
- A disk is contractible, a circle is not contractible.



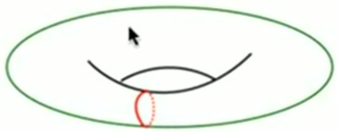
So which phase spaces cannot be cloned?

- In particular the phase space of the simple pendulum is a cylinder, hence not contractible!
- Another way to say the no cloning theorem is as follows: If you try to build a machine that tries to clone a non-contractible phase space T^*M, no matter how complicated your cloning machine is, I can always find some initial conditions in T^*M in which your cloning process fails (and we shall see, fails by a very large amount).

Intuition for the Proof

- Before we launch into an explanation of the technical tools required for the proof, we give an intuitive sketch for the case of the simple pendulum and why it cannot be cloned.
- The phase space of the simple pendulum is T^*S^1 (which is a cylinder), and the phase space of the cloning machine is T^*N, so the phase space of the combined system is $T^*S^1 \times T^*S^1 \times T^*N$. For technical reasons we can forget the third factor and simply think about $T^*S^1 \times T^*S^1$.
- Topologically, the cylinder is not so different from the circle, so we can further replace total phase space with $S^1 \times S^1$. Since we got rid of the $\mathbb{R}$ factor, we can pretend we are only thinking of states with zero angular momentum.

- We think of the first S^1 factor as the state space of the first pendulum, and the second S^1 as the state space of the second particle.
- Topologically, $S^1 \times S^1$ is just a two dimensional torus. The first S^1 factor wraps around the meridian direction, and the second factor wraps around the longitudinal direction.



Intuition for the Proof, ctd

- After all these reductions and simplifications, the cloning map is one that takes the set

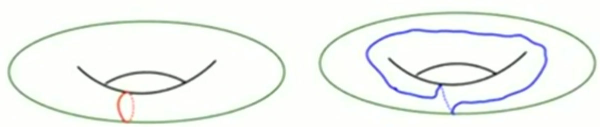
$$R := \{(x, b) \subset S^1 \times S^1 | x \in S^1\}$$

into the set

$$B := \{(x, x) \in S^1 \times S^1 | x \in S^1\}$$

(i.e. we clone the state of the first particle onto the second particle.)

- The set R is represented by the red curve on the torus, and B is represented by the blue curve on the torus.



(a) red curve on a torus (b) blue curve on a torus

Intuition for the Proof, ctd

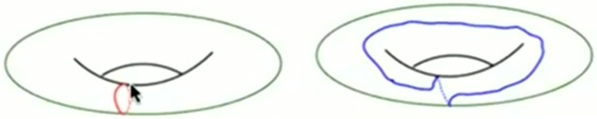
- The point is that cloning must come from continuous Hamiltonian time evolution. At $t = 0$ the Hamiltonian flow is the identity map (since no time has passed), and the Hamiltonian flow at $t = 1$ we have the cloning map. So there must be a continuous deformation taking the set

$$R := \{(x, b) \subset S^1 \times S^1 | x \in S^1\}$$

into

$$B := \{(x, x) \in S^1 \times S^1 | x \in S^1\}.$$

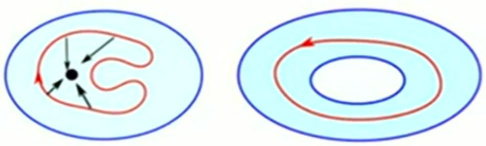
- Then as shown in the figure below, this implies a continuous deformation from the red curve to the blue curve, which is not possible.



A crash course on algebraic topology

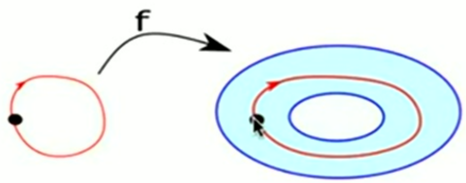
Introduction to homotopy theory

- The correct way to get references for this material is as follows: walk into the nearest math department; grab onto the first person you see; and don't let go until they explain homotopy groups to you.
- The point is this: how do we distinguish an annulus from a disk? How do we make precise the notion that the annulus has "a hole" in the middle?
- The key insight is that on the disk any loop you draw can be continuously collapsed to a point, the same is not true for the annulus.



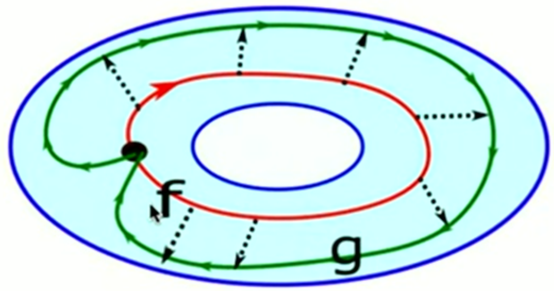
The fundamental group

- Let X denote a geometric space (for example an annulus). We fix a point $p \in X$, and consider the pair (X, p).
- We consider the set of continuous maps from the circle to X, $f : S^1 \to X$ that sends $0 \in S^1$ to $p \in X$.



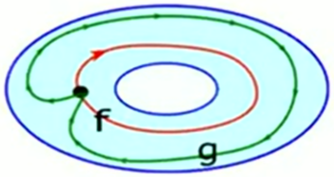
The fundamental group, ct'd

- We say two maps $f, g : S^1 \to X$ are equivalent to each other if they can be deformed to each other while keeping the condition $0 \in S^1$ gets mapped to $p \in X$.



The fundamental group, ct'd

- We then consider the set of equivalence class of maps. It has one operation called composition, which is the concatenation of loops. We write it as $f * g$. Here $*$ is a binary operation that takes two loops and spits out a third loop which is their concatenation.



The fundamental group, ct'd

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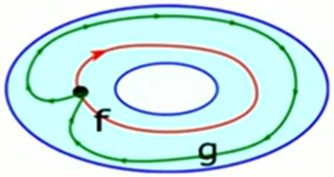


- The set of equivalence classes, combined with the operation $*$, is called the fundamental group. We write it as $\pi_1(X, p)$.
- The fundamental group of the annulus is isomorphic to $(\mathbb{Z}, +)$, the integers with the addition operation. The integer is the winding number of the loop, and when we concatenate two loops their winding numbers add.

The simple pendulum cannot be cloned, revisited

- Using the language of fundamental groups, the proof that the simple pendulum cannot be cloned looks something like this. It's a rephrasing of the heuristic sketch we gave using mathematical language.
- Recall the phase space of the pendulum is T^*S^1, and that of the cloning machine is T^*N.
- Consider $\pi_1(T^*S^1 \times T^*S^1 \times T^*N) \cong \pi_1(T^*S^1) \times \pi_1(T^*S^1) \times \pi_1(T^*N)$.
- Consider the map $f : S^1 \to T^*S^1 \times T^*S^1 \times T^*N$ that has winding number 1 in the first copy of T^*S^1, and is constant on the remaining two factors. This represents the element

$$(1,0,0) \in \mathbb{Z} \times \mathbb{Z} \times \pi_1(T^*N) \cong \pi_1(T^*S^1 \times T^*S^1 \times T^*N).$$

The simple pendulum cannot be cloned, revisited

- Composing this with the cloning map, $\phi_1 : T^*S^1 \times T^*S^1 \times T^*N \to T^*S^1 \times T^*S^1 \times T^*N$.
- The resulting composition is $\phi_1 \circ f : S^1 \to T^*S^1 \times T^*S^1 \times T^*N$. Since the cloning map came from a continuous time evolution it cannot change the element of the fundamental group, so $\phi_1 \circ f$ should still be the same element in the fundamental group

$$(1, 0, 0) \in \mathbb{Z} \times \mathbb{Z} \times \pi_1(T^*N) \cong \pi_1(T^*S^1 \times T^*S^1 \times T^*N).$$

- However, the condition that $\phi_1(y, b, r) = (y, y, h(y, b, r))$ means $\phi_1 \circ f$ should represent

$$(1, 1, *) \in \mathbb{Z} \times \mathbb{Z} \times \pi_1(T^*N) \cong \pi_1(T^*S^1 \times T^*S^1 \times T^*N).$$

- The two above statements cannot both be true, so cloning is not possible.

The general case and higher homotopy groups

- More generally to get clonable phase spaces are contractible, it is not enough to consider the space of loops (i.e. the fundamental group) mapping into the phase space. We should also consider the space of higher dimensional spheres mapping into the phase space. They are known as higher homotopy groups.
- The same argument as before shows all homotopy groups of the phase space must vanish.
- Whitehead's theorem tells us if we have a space all of whose homotopy groups vanish, then the space is contractible. From this we get clonable phase spaces are contractible.

Approximate cloning

- Since our techniques are purely topological, it is robust against small perturbations/noise. This can be formalized as a no-approximate cloning theorem.

Definition

A approximate cloning process is given by

- A phase space T^*M of the system to be cloned and a phase space T^*N of the cloning machine.
- A point $b \in T^*M$, a point $r \in T^*N$, a (potentially time dependent) Hamiltonian on $T^*M \times T^*M \times T^*N$ whose time-1 flow is a map that sends

$$\phi_1 : (y, b, r) \to (y + \epsilon_1(y), y + \epsilon_2(y), h(y, b, r))$$

for all $y \in T^*M$ and $\epsilon_1(y), \epsilon_2(y)$ are suitably small error term. Here $h(-, b, r)$ is some smooth function from T^*M to T^*N.

If the above data exists for the the phase space T^*M, then we say it is **approximately clonable**.

Approximate cloning

Theorem

For suitably defined small error terms $\epsilon(y)$, if a phase space is approximately clonable, then it is contractible.

- Informally speaking, the no-approximate cloning theorem says you cannot setup a cloning machine that approximately clones a noncontractible phase space. For every apparatus you build that tries to approximately clone a noncontractible phase space, I can always find some initial state for which your cloning process will be outrageously wrong: error much greater than allowed in the approximate cloning scheme.

Approximate cloning

- This form of the theorem is what really collides with our intuition in the real world - you can no longer say the no cloning theorem does not apply to real world settings because it's too idealized and in the real world there's no such thing as cloning without error.
- No, something very strange is going on with this no-cloning theorem. We examine next the physical implications of these theorems.

Puzzle 1: we expect classical information is clonable

- I figured out parts of how to reconcile these no cloning theorems with the real world, some are still very mysterious to me. I will start by explaining the parts I figured out and understand.
- Our intuition about the real world is that generally speaking, classical information is clonable. For example a computer is able to copy a string of 0s and 1s without issue.
- We imagine building a mechanical analogue of a computer, maybe made out of pulleys and springs, so we can be in the setting of classical mechanics.
- A mechanical analogue of 0 or 1 could be a coin on a table, 0 representing tails, and 1 representing heads.

Puzzle 1: we expect classical information is clonable

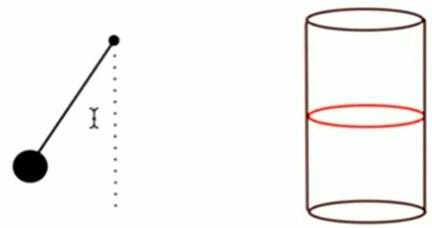
- The answer is yes. Even though we don't apriori know the coin is heads or tails, we apriori know it has be to head or tails. In other words, the coin is confined to discrete points in the phase space. Discrete points are contractible, and clonable by a classical theorem in symplectic geometry.

Puzzle 1: we expect classical information is clonable

- The point is, typically when we think of copying information, we already know the information to be copied is confined to a contractible region of phase space. Hence we can copy without worrying about topology.

Puzzle 2: Why can't I just clone a pendulum?

- Imagine walking into a lab with a swinging pendulum.



I ask my experimentalist friend to clone it. He first measures the position and momentum of the swinging pendulum, then he pushes an identical pendulum into the same position and momentum. There, cloned.

Puzzle 2: Why can't I just clone a pendulum?

- Against since our theorem is about classical mechanics, we build a mechanical representation of the problem.
- We can think of measuring the position and momentum of the swinging pendulum as scattering a large number of massless particles onto the swinging pendulum. Then the position and momentum of the pendulum is encoded in the scattering data of the massless particles.
- Now we need to turn on a Hamiltonian that extracts the information from the scattered particles and pass it onto the identical pendulum.
- Which of these steps fail? Can we quantify the way it fails?
- This is perhaps not so easy to understand since an experimenter has 10^{23} particles in them. Can we find a smaller system in which we can understand why cloning fails?

- Here is another thought experiment. Again in the case of the simple pendulum. Suppose we really want to clone it, we can view it as a particle moving in $\mathbb{R}^2$, and clone it as a particle moving in $\mathbb{R}^2$ onto a test particle moving in $\mathbb{R}^2$.
- Our no cloning theorem then implies that during this cloning process the pendulum cannot remain on its original circle, the length must change (and at some point shrink to zero).

Puzzle 3: can we make the no cloning theorem useful?

- Then this brings us to the final point of discussion. The no cloining theorem is true, but can we make it useful? Or is it just a mathematical curiosity?
- The no cloning theorem in quantum mechanics is used to establish secure communication protocols. Can we do something similar using the classical no cloning theorem?
- Find a system large enough to be not affected by quantum mechanics, however small enough so that we can't just "measure position and momentum by hand" and clone it by hand using an experimenter with 10^{23} particles in them, like cloning a simple pendulum in the lab.
- Conceptually, we need an axiomatic framework for measurement and extracting information in context of classical mechanics (we already have one for quantum mechanics). After all, if we can't clone, what does it mean to measure anymore?

References

[Fen12] Aaron Fenyes. "Limitations on cloning in classical mechanics". In: *Journal of Mathematical Physics* 53.1 (2012), p. 012902. DOI: 10.1063/1.3676295. eprint: https://doi.org/10.1063/1.3676295. URL: https://doi.org/10.1063/1.3676295.

[RSS19] Anirudh Reddy, Joseph Samuel, and Supurna Sinha. "A physical perspective on classical cloning". In: *Physics Letters A* 383.27 (2019), p. 125846. ISSN: 0375-9601. DOI: https://doi.org/10.1016/j.physleta.2019.125846. URL: https://www.sciencedirect.com/science/article/pii/S0375960119306346.

[Sca+05] Valerio Scarani et al. "Quantum cloning". In: *Rev. Mod. Phys.* 77 (4 Nov. 2005), pp. 1225–1256. DOI: 10.1103/RevModPhys.77.1225. URL: https://link.aps.org/doi/10.1103/RevModPhys.77.1225.

[Yao23] Yuan Yao. "Phase spaces that cannot be cloned in classical mechanics". In: *J. Math. Phys.* 64.10 (2023), p. 102901. DOI: 10.1063/5.0150024. arXiv: 2303.00255 [math-ph].