Title: A 3d integrable field theory with 2-Kac-Moody algebra symmetry (Virtual)

Speakers: Hank Chen

Collection/Series: Mathematical Physics

Subject: Mathematical physics

Date: January 23, 2025 - 11:00 AM

URL: https://pirsa.org/25010078

Abstract:

This talk is based on my recent joint works arXiv:2405.18625, arXiv:2307.03831 with Joaquin Liniado and Florian Girelli.
Based on Lie 2-groups, I will introduce a 3d topological-holomorphic integrable field theory W, which can be understood as a higher-dimensional version of the Wess-Zumino-Witten model. By studying its higher currents and holonomies, it is revealed that W is related to both the raviolo VOAs of Garner- Williams --- a type of derived higher quantum algebra --- and the lasagna modules of Manolescu-Walker-Wedrich --- a type of 4d higher-skein invariant. I will then analyze the Noether charges of W, and prove that its symmetries are encoded by a derived version of the Kac-Moody algebra. If time allows, I will discuss how W enjoys a certain notion of "higher Lax integrability".

A 3d topological-holomorphic integrable field theory and its derived Kac-Moody symmetry

Hank Chen

arXiv:2405.18625 w/ Joaquin Liniado
arXiv:2307.03831 w/ Florian Girelli

Perimeter Institute, now BIMSA

北京雁栖湖应用数学研究院
Beijing Institute of Mathematical Sciences and Applications

PERIMETER INSTITUTE

Overview

- Part I:
  1. Reminder: 2d Wess-Zumino-Witten model.
  2. Reminder: Lie 2-groups & Lie 2-algebras.

Main punchline I.

3d integrable field theory $\mathcal{W}$ on Y has infinitely many conserved 2-holonomy charges w/ interesting geometry.

- Part II:
  1. Reminder: Lax integrability; classical r-matrix method.
  2. Reminder: the Kac-Moody algebra; Lax formulation of 2d WZW model.

Main punchline II.

3d integrable field theory $\mathcal{W}$ has global symmetry governed by *2-Kac-Moody algebra* $\widehat{\Sigma_s\mathfrak{G}}$, leading to its "2-Lax" formulation.

Part I: the theory $\mathcal{W}$ from derived (super)fields

Reminder: 2d Wess-Zumino-Wittem model

The Wess-Zumino-Witten model [Wess and Zumino 1971; Witten 1983].

Let $\partial B^3 = \Sigma$ and G compact with $\tilde{g} : B^3 \to G$ lifting $g : \Sigma \to G$.

$$W(g) = \frac{k}{4\pi} \int_\Sigma \langle g^{-1} dg, \star g^{-1} dg \rangle + \frac{k}{12\pi} \int_{B^3} \langle \tilde{g}^{-1} d\tilde{g}, [\tilde{g}^{-1} d\tilde{g}, \tilde{g}^{-1} d\tilde{g}] \rangle,$$

where $k \in \mathbb{Z}$ (level quantization).

- Chiral currents: $J = \partial g g^{-1}$ & $\bar{J} = g^{-1} \bar{\partial} g$,

$$\text{equations of motion} = \text{holomorphicity} : \qquad \bar{\partial} J = 0, \qquad \partial \bar{J} = 0.$$

- Symmetries: **Polyakov-Wiegmann relation** [Knizhnik and Zamolodchikov 1984]

$$W(gh^{-1}) = W(g) + W(h) + \frac{k}{4\pi} \int_\Sigma \langle g^{-1} \bar{\partial} g, h^{-1} \partial h \rangle.$$

Reminder: Lie 2-groups & Lie 2-algebras

Definition. (Lie 2-group $\mathbb{G} = (\mathrm{H}, G, M_1, \rhd)$) [Baez and Lauda 2004]

Lie group map $M_1 : \mathrm{H} \to G$ and smooth action $\rhd : G \to \mathrm{Aut}\,\mathrm{H}$, s.t.

$$M_1(g \rhd h) = gM_1(h)g^{-1}, \qquad (M_1 h) \rhd h' = hh'h^{-1}.$$

- Equivalently: $\mathbb{G}$ = category in LieGrp [Porst 2008],

$$\Gamma = (\mathrm{H} \rtimes G) \overset{s}{\underset{t}{\rightrightarrows}} G, \qquad M_1(h) = s(h)^{-1}t(h)$$

Definition. (Lie 2-algebra $\mathfrak{G}$) [Baez and Crans 2004]

Lie algebra map $\mu_1 : \mathfrak{h} \to \mathfrak{g}$ and derivation $\mu_2 : \mathfrak{g} \to \mathrm{Der}\,\mathfrak{h}$ s.t.

$$\mu_1(X \rhd Y) = \mu_2(X, \mu_1 Y), \qquad \mu_2(\mu_1 Y, Y') = [Y, Y'].$$

- "2-Lie theorem" [Chen, Stiénon, and Xu 2013]: can "integrate" $\exp : \mathfrak{G} \to \mathbb{G}$ such that the dg tangent space $T_{(1,\mathbf{1}_1)}\mathbb{G} \cong \mathfrak{G}$.

Building up to $\mathcal{W}$: derived Lie 2-groups & poly-form fields

Definition. (derived Lie 2-group $\mathrm{D}\mathbb{G}$) [Zucchini 2021]

The **derived Lie 2-group** $\mathrm{D}\mathbb{G}$ is the space of maps $\mathbb{R}[1] \to \mathbb{G}$ given by

$$(h, \eta) : \alpha \mapsto (h, e^{\alpha \cdot \eta}), \qquad h \in G, \ \eta \in \mathfrak{h}.$$

Namely $\mathrm{D}\mathbb{G}$ is a "dg version" of $\mathbb{G}$ which inherits its categorical structures.

- *"Derived poly-form fields"* = elements of $\Omega^\bullet(Y) \otimes \mathfrak{G}$:

$$(\Omega^\bullet(Y) \otimes \mathfrak{G})_n = \bigoplus_{p+q=n} \Omega^p(Y) \otimes \mathfrak{G}_q$$

- Our fields in the 3d IFT will be elements of the form

$$(g, \Theta^g) = g \rhd (1, \Theta) \in (\Omega^\bullet(Y) \otimes \mathrm{D}\mathbb{G})_0.$$

The 3d integrable field theory

Definition. [Chen and Liniado 2024]

Let $\langle -, - \rangle$ be a deg. 1 pairing on $\mathfrak{G} = \mathrm{Lie}\,\mathbb{G}$ and $J = dgg^{-1}$:

$$\mathcal{W}[g, \Theta] = -2 \int_Y \langle d_\ell J, \Theta^g \rangle - \frac{1}{2} \langle d_\ell \Theta^g, \mu_1 \Theta^g \rangle.$$

Unit vector $\ell \in S^2$, called "**chirality**" (notice $\mathrm{proj}_\ell\, \Theta$ is absent!).

- Higher currents (L, H):

$$L \doteq -J - \mu_1 \Theta^g \in \Omega^1 \otimes \mathfrak{g}, \qquad H = g \rhd (d\Theta - \frac{1}{2}[\Theta, \Theta]) \in \Omega^2 \otimes \mathfrak{h}.$$

- $\mathcal{W}$ comes from 5d 2-Chern-Simons [ibid.]

$$S_{2\mathrm{CS}_5}[A, B] = \int_{Y \times \mathbb{C}} \frac{dz}{z} \wedge \langle B, F - \frac{1}{2} \mu_1 B \rangle \xrightarrow{\text{Lax reparam.}}$$

$$S_{2\mathrm{CS}_5}[L, H] + \int_Y \Omega[L, H; g, \Theta]|_{z=0,\infty} \xrightarrow{\mathrm{Res}_{z=0}} \mathcal{W}[g, \Theta].$$

Higher flatness equations and 2-holonomies

Proposition. (2-flatness of $\mathcal{W}$-currents) [Chen and Liniado 2024]

Define $\mathcal{J} = (L_\perp, H_\ell)$ and $\tilde{\mathcal{J}} = (\tilde{L}_\ell, \tilde{H}_\perp)$. EOMs of $\mathcal{W}$ are exactly 2MC[a]

$$\hat{d}\mathcal{J} + \frac{1}{2}[\mathcal{J},\mathcal{J}] = 0 \iff \hat{d}\tilde{\mathcal{J}} + \frac{1}{2}[\tilde{\mathcal{J}},\tilde{\mathcal{J}}] = 0 \qquad \hat{d} = d - \mu_1.$$

[a] 2-Maurer- Cartan, aka. fake- and 2-flatness: $dL + \frac{1}{2}[L, L] - \mu_1 H = 0$, $dH + \mu_2(L, H) = 0$ [Chen and Girelli 2022; Martins and Porter 2007; Radenkovic and Vojinovic 2019; Zucchini 2021] .

- **2-holonomies** as 2-groupoid map (cf. [Kim and Saemann 2020])

$$2\mathrm{Hol}_{\mathcal{J}} : P^2Y \to B\mathbb{G}, \qquad (\Sigma, \gamma) \mapsto (V_\Sigma, W_\gamma),$$

from H as a $\mathbb{H}$-connection on *loop space* ΩY [Alvarez, Ferreira, and Sanchez Guillen 1998].

Lemma. (homotopy invariance) [Chen and Liniado 2024]

$2\mathrm{Hol}_{(L,H)}$ descend to homotopy rel. boundary $\pi P^2 Y \to B\mathbb{G}$ iff (L, H) satisfies the 2MC conditions.

Conservation of surface holonomies

- Changing timeslice = "whiskering" of 2-holonomies:

$$V_{\Sigma_1} = W_{\gamma_u} \triangleright V_{\Sigma_0}, \qquad \Sigma_{0,1} \text{ at } u = 0,1 \text{ timeslice.}$$

- ...hence 2-monodromy matrices are conserved $\forall\ (\Sigma, \gamma) \in \pi P^2 Y$,

$$\frac{d}{dt}\mathcal{X}(V, W)_{(\Sigma,\gamma)} \cong 0, \qquad \forall\ \mathcal{X} \in \text{"2-char.'s" } \widehat{\mathbb{G}}.$$

Categorical segue: What is... a 2-character?

Take $(\mathcal{M}, \rho) \in 2\mathrm{Rep}(\mathbb{G})$. 2-char.'s can be made out of co/ends [Ganter and Usher 2016; Huang, Xu, and Zhang 2024; Sanford 2024]:

$$\mathcal{X}_\rho^{\searrow}(g) = \int_{M \in \mathcal{M}} \mathrm{Hom}_{\mathcal{M}}(M, g \triangleright M), \qquad \mathcal{X}_\rho^{\circlearrowleft}(g) = \int^{M \in \mathcal{M}} \mathrm{Hom}_{\mathcal{M}}(M, g \triangleright M),$$

called the "diagonal" or "round" traces[a] (they're the same if $\mathcal{M}$ is fin. ss.).

[a] See talk http://www.simonwillerton.staff.shef.ac.uk/ftp/TwoTracesBeamerTalk.pdf by Willerton.

Transverse holomorphic foliation (THF)

Definition. [Aganagic et al. 2017; Scárdua and Jurado 2017]

A 3-fold Y has **transverse holomorphic foliation (THF)** structure (along $dt = dx_3$) when charts $(w, \bar{w}, x_3) \in U \subset Y$ transform as

$$(w, \bar{w}, x_3) \mapsto (w'(w), \bar{w}'(\bar{w}), x_3'(w, \bar{w}, x_3)).$$

This need *not* align with the chirality ℓ!

- When they misalign (eg. $dx_\ell = dw$):
  1. $\mu_1 = 0$ recovers *Chern-Simons/matter coupling* in [Aganagic et al. 2017]

$$\mathcal{W} = -2 \int_Y dw \wedge d\bar{w} \wedge dx_3 \left[\langle \Theta_{\bar{w}}, \partial_3 \bar{J}_w \rangle - \langle \Theta_3, \partial_{\bar{w}} \bar{J}_w \rangle \right].$$

  2. $\mathcal{J}, \tilde{\mathcal{J}}$ are *raviolo fields* on $(\mathcal{A}^{\bullet,\bullet}, d')$ [Garner and Williams 2023]

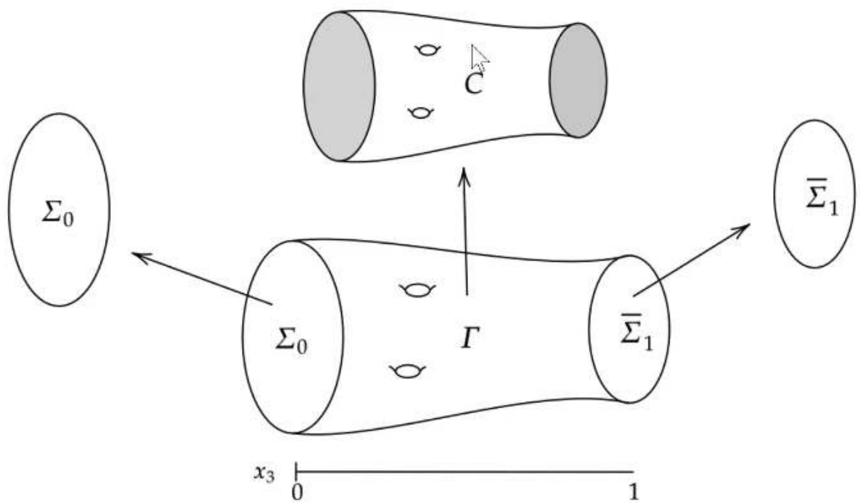
$$L_\perp \in \mathcal{A}^{0,1} \otimes \mathfrak{g}, \qquad H_\ell \in \mathcal{A}^{0,2} \otimes \mathfrak{h}, \qquad \tilde{L}_\ell \in \mathcal{A}^{1,0} \otimes \mathfrak{g}, \qquad \tilde{H}_\perp \in \mathcal{A}^{1,1} \otimes \mathfrak{h}.$$

Bordism invariance

- When they align $dx_\ell = dx_3$:

Theorem. (open bordism invariance) [Chen and Liniado 2024]

Chiral $2\mathrm{Hol}_{\mathcal{J}}$'s descend to *framed open bordism 2-group* $\Omega^{\mathrm{fr}}_{2,1}(Y) \to \mathbb{G}$.



- $2\mathrm{Hol}_{\mathcal{J}}$ only has support $\perp dx_3$, $2\mathrm{Hol}_{\tilde{\mathcal{J}}}$ only has support $|| \; dx_3$.
- $2\mathrm{Hol}_{\tilde{\mathcal{J}}}$ lives on *lasagna fillings* [Morrison, Walker, and Wedrich 2019]!

Noether analysis of $\mathcal{W}$

- **Higher Polyakov-Wiegmann**: $\mathcal{W}$ has left/right symm. by $(\alpha^{(\prime)}, \Gamma^{(\prime)}) \in (\Omega^\bullet \otimes \mathfrak{G})_0$

$$\Gamma_\ell^{(\prime)} = 0, \qquad d_\ell(\alpha, \Gamma) = 0, \qquad d_\perp(\alpha', \Gamma') = 0.$$

Proposition. (Noether charges of $\mathcal{W}$) [Chen and Liniado 2024]

The following codim-1 charges

$$q_{(\alpha,\Gamma)} = \int \langle (\alpha, \Gamma), \mathcal{J} \rangle, \qquad \tilde{q}_{(\alpha',\Gamma')} = \int \langle (\alpha', \Gamma'), \tilde{\mathcal{J}} \rangle$$

generate left/right global symmetries of $\mathcal{W}$.

- **Graded charge algebra** $\hat{\mathfrak{D}}$:

$$[q_{(\alpha_1,\Gamma_1)}, q_{(\alpha_2,\Gamma_2)}] = -q_{([\alpha_1,\alpha_2],\mu_2(\alpha_1,\Gamma_2)-\mu_2(\alpha_2,\Gamma_1))} - \underbrace{\int \langle \Gamma_1, d\alpha_2 + \mu_1 \Gamma_2 \rangle + \langle \alpha_1, d\Gamma_2 \rangle}_{\text{central}}; \qquad (1)$$

$\tilde{q} \in \hat{\hat{\mathfrak{D}}}$ is analogous.

Part II: the 2-Kac-Moody algebra and 2-Lax formulation of $\mathcal{W}$

Reminder: Lax integrability (see also [Meusburger 2021; Olivier Babelon and Talon 2003])

Definition. (Lax pair) [Lax 1968]

Hamiltonian system $(M, \{-,-\}, H)$ is *Lax integrable* iff there are Lie alg. $\mathfrak{g}$ valued maps $L, P : M \to \mathfrak{g}$ s.t.

$$\dot{L} = \{L, H\} = [P, L] \quad \Longrightarrow \quad \frac{d}{dt} \operatorname{tr} \rho(L)^i = 0. \tag{2}$$

- Classical Yang-Baxter equations (CYBE) [Belavin and Drinfel'd 1982]: $r \in \mathfrak{g} \otimes \mathfrak{g}$ such that

$$[\![r, r]\!] = [r_{12}, r_{23}] + [r_{13}, r_{23}] + [r_{12}, r_{13}] = 0.$$

Theorem. (canonical Lax pair) [Semenov-Tyan-Shanskii 1983]

A solution $r \in \mathfrak{g} \otimes \mathfrak{g}$ to the CYBE $\Longrightarrow$ *canonical* Lax pair (L, P):$\mathfrak{g}^* \to \mathfrak{g}$ on $(C(\mathfrak{g}^*), \{-,-\}_{KK}, H)$ satisfying

$$\{L, L\}_{KK} = [L \otimes 1 + 1 \otimes L, r^\wedge], \qquad \forall \ \text{ad-inv. } H \in C^\infty(\mathfrak{g}^*)$$

Reminder: Kac-Moody algebra and 2d WZW model

Definition. (the Kac-Moody algebra) [Kac 1978]

Simple Lie alg. $\mathfrak{g}$, Killing form $\langle -,- \rangle$. Take the central extension

$$\mathbb{C} \to \widehat{\Omega_k \mathfrak{g}} \to \Omega\mathfrak{g}, \qquad k(X, X') = 2\mathrm{k} \int_{S^1} \langle X, \frac{d}{d\tau} X' \rangle$$

of the loop algebra $\Omega\mathfrak{g}$ w/ class $k \in H^2(\Omega\mathfrak{g}, \mathbb{C}) \cong \mathbb{C}$.

- $L, P \in \widehat{\Omega_k \mathfrak{g}} \implies$ flatness of **Lax connection** [Olivier Babelon and Talon 2003],

$$\dot{L} = [P, L] \implies \partial_t A_x - \partial_x A_t + [A_x, A_t] = 0, \qquad A = Ldx + Pdt.$$

- *Currents in 2d WZW model are Lax connections* [Hoare 2022; Knizhnik and Zamolodchikov 1984]!

$$J \in \mathbb{C}[z] \otimes \mathfrak{g}, \qquad \bar{J} \in \mathbb{C}[\bar{z}] \otimes \mathfrak{g}, \qquad (\text{for } z \in S^1)$$

Noether currents live in Kac-Moody $\implies$ $\widehat{\Omega_k \mathfrak{g}}$-symmetry of W.

2-Lax integrability

- How to "categorify"?
  - Try replacing $\mathfrak{g}$ (resp. (2)) with Lie 2-alg. $\mathfrak{G}$ (resp. graded equation).
- Works on *dg Poisson mflds* $(C(M)_1 \to C(M)_0, \{-,-\})$.

Theorem. (canonical 2-Lax pair) [Chen and Girelli 2023]

A solution $R \in (\mathfrak{G} \otimes \mathfrak{G})_1$ to the 2-CYBE [Bai, Sheng, and Zhu 2013] $\implies$ a *canonical split* 2-Lax pair on $(C(\mathfrak{G}^*[1]), \{-,-\}_{2KK}, H)$,

$$L \in (C(\mathfrak{G}^*[1]) \otimes \mathfrak{G})_1, \qquad P \in (C(\mathfrak{G}^*[1]) \otimes \mathfrak{G})_0,$$

s.t. $\{L, L\}_{2KK} = [L \otimes 1 + 1 \otimes L, R^\wedge]$ and the **"bulk-boundary relation"**:

$$\underline{L} = \mu_1^* L_0 = \mu_1(L_1) \quad \text{is a Lax oper. on } C(\mathfrak{h}^*)!$$

- Conserved charges: take 2-rep $\rho : \mathfrak{G} \to \mathfrak{gl}(V)$ of $\mathfrak{g}$ [Angulo 2018], then

$$\frac{d}{dt}\mathcal{X}_\rho(L)^i = 0, \qquad \forall \text{ "2-chars." } \mathcal{X}_\rho \text{ of } \rho.$$

Higher derived current algebras

- Take a Riemann surface Σ and its **tangent complex** [Ševera 2005]
$$T[1]\Sigma \iff C(T[1]\Sigma) = \Omega^\bullet(\Sigma).$$
- Form the *dg current algebra* $\Sigma\mathfrak{G} \equiv C(T[1]\Sigma, \mathfrak{G})$ [Faonte, Hennion, and Kapranov 2019; Kapranov 2021].
- Consider $\Sigma\mathfrak{G} = C(T[1]\Sigma, \mathfrak{G})$. It has components

grade 0	grade 1	grade 2
$\Omega^0(\Sigma, \mathfrak{g})$	$\Omega^1(\Sigma, \mathfrak{g})$	$\Omega^2(\Sigma, \mathfrak{g})$
$\Omega^1(\Sigma, \mathfrak{h})$	$\Omega^2(\Sigma, \mathfrak{h})$	0

- ... but $\Omega^2(\Sigma)$ is a *super*algebra [Kac 1977], and I want to keep the supergrading.
- Idea: secretly use the Hodge star to send $\Omega^2 \xrightarrow{\sim} \Omega^0$:
$$\Sigma\mathfrak{G} \cong (\Omega^0(\Sigma, \mathfrak{g}) \oplus \Omega^1(\Sigma, \mathfrak{g})) \oplus (\Omega^0(\Sigma, \mathfrak{h}) \oplus \Omega^1(\Sigma, \mathfrak{h})).$$

The 2-Kac-Moody algebra

- Let $\mathbb{C}^{1|1} = \mathbb{C}^{1|0} \oplus \mathbb{C}^{0|1}$ denote the superline.

Definition/Theorem. (2-Kac-Moody algebra $\widehat{\Sigma_s \mathfrak{G}}$) [Chen and Girelli 2023]

$\widehat{\Sigma_s \mathfrak{G}}$ is a $\mathbb{C}^{1|1}[1]$-extension of $\Sigma\mathfrak{G} \cong C(T[1]\Sigma, \mathfrak{G})$ (cf. [Zhang and Liu 2014]) by

$$s(\mathcal{X}, \mathcal{Y}) = \int_\Sigma \langle \mathcal{X}, \hat{d}\mathcal{Y} \rangle \in \mathbb{C}^{1|1}[1], \qquad \hat{d} = d - \mu_1,$$

and equipped with a pairing ($\langle -, - \rangle$ = an invariant deg-1 pairing on $\mathfrak{G}$)

$$(\!|\mathcal{X} \oplus \xi, \mathcal{Y} \oplus \zeta|\!) = \int_\Sigma \langle \mathcal{X} \cdot \mathcal{Y} \rangle + (\xi, \zeta),$$

where $\langle - \cdot - \rangle$ (contains) the Hodge pairing of forms.

- $s \in Z^2(\Sigma\mathfrak{G}, \mathbb{C}^{1|1}[1])$ is a Lie 2-algebra 2-cocycle [Angulo 2018]!

Zero 2-curvature formulation of 2-Lax integrability

- Q: can we see 2-Lax integrability as a 2MC condition?
- Answer: *yes*!
  1. Take $L, P \in C^\infty(\mathbb{R}, \widehat{\Sigma_s \mathfrak{G}})$, where
  $$L_0 \in \Omega^1(\Sigma, \mathfrak{g}, \qquad P_0 \in \Omega^0(\Sigma, \mathfrak{g})$$
  $$L_1 \in \Omega^0(\Sigma, \mathfrak{h}), \qquad P_1 \in \Omega^1(\Sigma, \mathfrak{h}),$$
  2. rename $(\tau = x_3)$
  $$L_0 = A_w dw + A_{\bar{w}} d\bar{w}, \qquad L_1 = B_{w\bar{w}},$$
  $$P_0 = A_\tau, \qquad P_1 = B_{w\tau} dw + B_{\bar{w}\tau} d\bar{w},$$
  3. then 2-Lax equation $\iff$ 2-flatness of (A, B).

Coisotropy

The red $\Sigma\mathfrak{G}_+$ and blue $\Sigma\mathfrak{G}_-$ subspaces are coisotropic

$$(\!|\Sigma\mathfrak{G}_\pm, \Sigma\mathfrak{G}_\pm|\!) = 0, \qquad (\!|\Sigma\mathfrak{G}_\pm, \Sigma\mathfrak{G}_\mp|\!) \neq 0.$$

2-Lax formulation of $\mathcal{W}$

- Q: The currents $\mathcal{J} = (L_\perp, H_\ell), \tilde{\mathcal{J}} = (\tilde{L}_\ell, \tilde{H}_\perp)$ in $\mathcal{W}$ are also 2-flat, so are there "hidden" 2-Lax pairs in $\mathcal{W}$?
- Answer: *yes*, when $dx_\ell = d\tau$ align, and they are coisotropic!

$$\begin{cases} L_0 = L_\perp, & L_1 = \star_Y H_\ell, & P = 0 \\ \bar{L} = 0, & \bar{P}_0 = \tilde{L}_\ell, & \bar{P}_1 = \star_Y \tilde{H}_\perp. \end{cases}$$

  - Must use the 3d Hodge star $\star_Y$ on Y.

Proposition. (2-Lax connections in $\mathcal{W}$) [Chen and Girelli 2023]

EOMs of $\mathcal{J}, \tilde{\mathcal{J}} \iff$ 2-Lax equations for (L, P) and $(\bar{L}, \bar{P})$.

- We get *two* 2-Lax pairs from $\mathcal{W}$, one cannot exist without the other, similar to the anti-/chiral Lax connections $J, \bar{J}$ in WZW.
- But what of the symmetries?

The 2-Kac-Moody symmetry of $\mathcal{W}$

Theorem (the moment map). [Chen and Girelli 2023]

If $dx_\ell = d\tau$ align, then there is a 2-graded Lie algebra homomorphism

$$\hat{\hat{\mathfrak{D}}} \oplus \hat{\mathfrak{D}} \to \Sigma\mathfrak{G}_+ \oplus \Sigma\mathfrak{G}_- \cong \widehat{\Sigma_s\mathfrak{G}},$$

identifying symmetry Noether charges of $\mathcal{W}$ with the 2-Kac-Moody algebra.

- The proof goes through the 2-Lax formulation!

——§——

- The "**derived 2-Kac-Moody group** $\widehat{\Sigma_s\mathbb{G}}$" (cf. [Henriques 2006]):
  - Extension class s may arise from "categorical surface transgression"

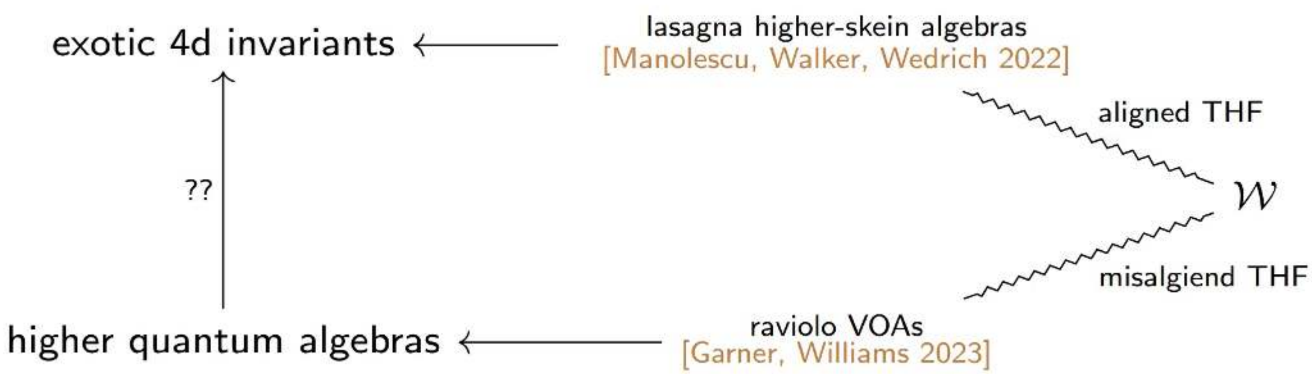
    $$H^3(\mathbb{G}, \mathbb{C}^{1|1}[1]^\times) \to H^2(\Sigma\mathbb{G}, \mathbb{C}^{1|1}[1]^\times)$$

    analogous to CS/WZW level transgression [Carey, Murray, and Wang 1997].
  - May be an example of a raviolo VOA [Garner and Williams 2023].

Summary

- The 3d IFT $\mathcal{W}$ has
  1. higher holonomies $2\mathrm{Hol}_{\mathcal{J},\tilde{\mathcal{J}}}$ with interesting geometry,
  2. global "2-Kac-Moody" symmetry $\widehat{\Sigma_s\mathfrak{G}} \circlearrowleft \mathcal{W}$.

- Outlook:
  1. **Quantizing** $\mathcal{W}$: unitary representations of $\widehat{\Sigma_s\mathbb{G}}$.
     - Alignment with the THF (on-going w/ Joaquin & Leon & more?):



     - Higher Kazhdan-Lusztik (cf. [Kazhdan and Lusztig 1994]) with categorical quantum groups [Chen 2025]?
  2. Other choices of ω [Schenkel and Vicedo 2024]:
     - Can get spectral parameter $\implies$ 2d integrable spin systems?

The End