

**Title:** A 3d integrable field theory with 2-Kac-Moody algebra symmetry (Virtual)

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**Abstract:**

This talk is based on my recent joint works arXiv:2405.18625, arXiv:2307.03831 with Joaquin Liniado and Florian Girelli. Based on Lie 2-groups, I will introduce a 3d topological-holomorphic integrable field theory  $W$ , which can be understood as a higher-dimensional version of the Wess-Zumino-Witten model. By studying its higher currents and holonomies, it is revealed that  $W$  is related to both the raviolo VOAs of Garner- Williams --- a type of derived higher quantum algebra --- and the lasagna modules of Manolescu-Walker-Wedrich --- a type of 4d higher-skein invariant. I will then analyze the Noether charges of  $W$ , and prove that its symmetries are encoded by a derived version of the Kac-Moody algebra. If time allows, I will discuss how  $W$  enjoys a certain notion of "higher Lax integrability".

# A 3d topological-holomorphic integrable field theory and its derived Kac-Moody symmetry

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arXiv:2405.18625 w/ Joaquin Liniado  
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# Overview

- Part I:

- ① Reminder: 2d Wess-Zumino-Witten model.
- ② Reminder: Lie 2-groups & Lie 2-algebras.

## Main punchline I.

3d integrable field theory  $\mathcal{W}$  on  $Y$  has infinitely many conserved 2-holonomy charges w/ interesting geometry.

- Part II:

- ① Reminder: Lax integrability; classical  $r$ -matrix method.
- ② Reminder: the Kac-Moody algebra; Lax formulation of 2d WZW model.

## Main punchline II.

3d integrable field theory  $\mathcal{W}$  has global symmetry governed by 2-Kac-Moody algebra  $\widehat{\Sigma}_s \mathfrak{G}$ , leading to its "2-Lax" formulation.

# Part I: the theory $\mathcal{W}$ from derived (super)fields

## Reminder: 2d Wess-Zumino-Witten model

The Wess-Zumino-Witten model [Wess and Zumino 1971; Witten 1983].

Let  $\partial B^3 = \Sigma$  and  $G$  compact with  $\tilde{g} : B^3 \rightarrow G$  lifting  $g : \Sigma \rightarrow G$ .

$$W(g) = \frac{k}{4\pi} \int_{\Sigma} \langle g^{-1} dg, \star g^{-1} dg \rangle + \frac{k}{12\pi} \int_{B^3} \langle \tilde{g}^{-1} d\tilde{g}, [\tilde{g}^{-1} d\tilde{g}, \tilde{g}^{-1} d\tilde{g}] \rangle,$$

where  $k \in \mathbb{Z}$  (level quantization).

- Chiral currents:  $J = \partial g g^{-1}$  &  $\bar{J} = g^{-1} \bar{\partial} g$ ,

$$\text{equations of motion} = \text{holomorphicity} : \quad \bar{\partial} J = 0, \quad \partial \bar{J} = 0.$$

- Symmetries: **Polyakov-Wiegmann relation** [Knizhnik and Zamolodchikov 1984]

$$W(gh^{-1}) = W(g) + W(h) + \frac{k}{4\pi} \int_{\Sigma} \langle g^{-1} \bar{\partial} g, h^{-1} \partial h \rangle.$$

## Reminder: Lie 2-groups & Lie 2-algebras

Definition. (Lie 2-group  $\mathbb{G} = (H, G, M_1, \triangleright)$ ) [Baez and Lauda 2004]

Lie group map  $M_1 : H \rightarrow G$  and smooth action  $\triangleright : G \rightarrow \text{Aut } H$ , s.t.

$$M_1(g \triangleright h) = gM_1(h)g^{-1}, \quad (M_1 h) \triangleright h' = hh'h^{-1}.$$

- Equivalently:  $\mathbb{G} = \text{category in LieGrp}$  [Porst 2008],

$$\Gamma = (H \rtimes G) \begin{array}{c} \xrightarrow{s} \\ \xrightarrow{t} \end{array} G, \quad M_1(h) = s(h)^{-1}t(h)$$

Definition. (Lie 2-algebra  $\mathfrak{G}$ ) [Baez and Crans 2004]

Lie algebra map  $\mu_1 : \mathfrak{h} \rightarrow \mathfrak{g}$  and derivation  $\mu_2 : \mathfrak{g} \rightarrow \text{Der } \mathfrak{h}$  s.t.

$$\mu_1(X \triangleright Y) = \mu_2(X, \mu_1 Y), \quad \mu_2(\mu_1 Y, Y') = [Y, Y'].$$

- "2-Lie theorem" [Chen, Stiénon, and Xu 2013]: can "integrate"  
 $\exp : \mathfrak{G} \rightarrow \mathbb{G}$  such that the dg tangent space  $T_{(1,1_1)}\mathbb{G} \cong \mathfrak{G}$ .

# Building up to $\mathcal{W}$ : derived Lie 2-groups & poly-form fields

Definition. (derived Lie 2-group  $D\mathbb{G}$ ) [Zucchini 2021]

The **derived Lie 2-group**  $D\mathbb{G}$  is the space of maps  $\mathbb{R}[1] \rightarrow \mathbb{G}$  given by

$$(h, \eta) : \alpha \mapsto (h, e^{\alpha \cdot \eta}), \quad h \in G, \eta \in \mathfrak{h}.$$

Namely  $D\mathbb{G}$  is a "dg version" of  $\mathbb{G}$  which inherits its categorical structures.

- "Derived poly-form fields" = elements of  $\Omega^\bullet(Y) \otimes \mathfrak{G}$ :

$$(\Omega^\bullet(Y) \otimes \mathfrak{G})_n = \bigoplus_{p+q=n} \Omega^p(Y) \otimes \mathfrak{G}_q$$

- Our fields in the 3d IFT will be elements of the form

$$(g, \Theta^g) = g \triangleright (1, \Theta) \in (\Omega^\bullet(Y) \otimes D\mathbb{G})_0.$$

# The 3d integrable field theory

Definition. [Chen and Liniado 2024]

Let  $\langle -, - \rangle$  be a deg. 1 pairing on  $\mathfrak{G} = \text{Lie } \mathbb{G}$  and  $J = dgg^{-1}$ :

$$\mathcal{W}[g, \Theta] = -2 \int_Y \langle d_\ell J, \Theta^g \rangle - \frac{1}{2} \langle d_\ell \Theta^g, \mu_1 \Theta^g \rangle.$$

Unit vector  $\ell \in S^2$ , called "**chirality**" (notice  $\text{proj}_\ell \Theta$  is absent!).

- Higher currents  $(L, H)$ :

$$L \stackrel{\triangleleft}{=} -J - \mu_1 \Theta^g \in \Omega^1 \otimes \mathfrak{g}, \quad H = g \triangleright (d\Theta - \frac{1}{2}[\Theta, \Theta]) \in \Omega^2 \otimes \mathfrak{h}.$$

- $\mathcal{W}$  comes from 5d 2-Chern-Simons [ibid.]

$$S_{2\text{CS}_5}[A, B] = \int_{Y \times \mathbb{C}} \frac{dz}{z} \wedge \langle B, F - \frac{1}{2} \mu_1 B \rangle \xrightarrow{\text{Lax reparam.}}$$

$$S_{2\text{CS}_5}[L, H] + \int_Y \Omega[L, H; g, \Theta]|_{z=0, \infty} \xrightarrow{\text{Res}_{z=0}} \mathcal{W}[g, \Theta].$$



# Higher flatness equations and 2-holonomies

Proposition. (2-flatness of  $\mathcal{W}$ -currents) [Chen and Liniado 2024]

Define  $\mathcal{J} = (L_\perp, H_\ell)$  and  $\tilde{\mathcal{J}} = (\tilde{L}_\ell, \tilde{H}_\perp)$ . EOMs of  $\mathcal{W}$  are exactly 2MC<sup>a</sup>

$$\hat{d}\mathcal{J} + \frac{1}{2}[\mathcal{J}, \mathcal{J}] = 0 \iff \hat{d}\tilde{\mathcal{J}} + \frac{1}{2}[\tilde{\mathcal{J}}, \tilde{\mathcal{J}}] = 0 \quad \hat{d} = d - \mu_1.$$

<sup>a</sup>2-Maurer- Cartan, aka. fake- and 2-flatness:  $dL + \frac{1}{2}[L, L] - \mu_1 H = 0, dH + \mu_2(L, H) = 0$  [Chen and Girelli 2022; Martins and Porter 2007; Radenkovic and Vojinovic 2019; Zucchini 2021].

- **2-holonomies** as 2-groupoid map (cf. [Kim and Saemann 2020])

$$2\text{Hol}_{\mathcal{J}} : P^2 Y \rightarrow B\mathbb{G}, \quad (\Sigma, \gamma) \mapsto (V_\Sigma, W_\gamma),$$

from  $H$  as a  $\mathbb{H}$ -connection on *loop space*  $\Omega Y$  [Alvarez, Ferreira, and Sanchez Guillen 1998].

Lemma. (homotopy invariance) [Chen and Liniado 2024]

$2\text{Hol}_{(L,H)}$  descend to homotopy rel. boundary  $\pi P^2 Y \rightarrow B\mathbb{G}$  iff  $(L, H)$  satisfies the 2MC conditions.

# Conservation of surface holonomies

- Changing timeslice = "whiskering" of 2-holonomies:

$$V_{\Sigma_1} = W_{\gamma_u} \triangleright V_{\Sigma_0}, \quad \Sigma_{0,1} \text{ at } u = 0, 1 \text{ timeslice.}$$

- ... hence 2-monodromy matrices are conserved  $\forall (\Sigma, \gamma) \in \pi P^2 Y$ ,

$$\frac{d}{dt} \mathcal{X}(V, W)_{(\Sigma, \gamma)} \cong 0, \quad \forall \mathcal{X} \in \text{"2-char.'s"} \hat{\mathbb{G}}.$$

## Categorical segue: What is... a 2-character?

Take  $(\mathcal{M}, \rho) \in 2\text{Rep}(\mathbb{G})$ . 2-char.'s can be made out of co/ends [Ganter and Usher 2016; Huang, Xu, and Zhang 2024; Sanford 2024]:

$$\mathcal{X}_\rho^{\searrow}(g) = \int_{M \in \mathcal{M}} \text{Hom}_{\mathcal{M}}(M, g \triangleright M), \quad \mathcal{X}_\rho^{\circlearrowleft}(g) = \int^{M \in \mathcal{M}} \text{Hom}_{\mathcal{M}}(M, g \triangleright M),$$

called the "diagonal" or "round" traces<sup>a</sup> (they're the same if  $\mathcal{M}$  is fin. ss.).

<sup>a</sup>See talk <http://www.simonwillerton.staff.shef.ac.uk/ftp/TwoTracesBeamerTalk.pdf> by Willerton.

# Transverse holomorphic foliation (THF)

Definition. [Aganagic et al. 2017; Scárdua and Jurado 2017]

A 3-fold  $Y$  has **transverse holomorphic foliation (THF)** structure (along  $dt = dx_3$ ) when charts  $(w, \bar{w}, x_3) \in U \subset Y$  transform as

$$(w, \bar{w}, x_3) \mapsto (w'(w), \bar{w}'(\bar{w}), x_3'(w, \bar{w}, x_3)).$$

This need *not* align with the chirality  $\ell$ !

- When they misalign (eg.  $dx_\ell = dw$ ):
  - ①  $\mu_1 = 0$  recovers *Chern-Simons/matter coupling* in [Aganagic et al. 2017]

$$\mathcal{W} = -2 \int_Y dw \wedge d\bar{w} \wedge dx_3 [\langle \Theta_{\bar{w}}, \partial_3 \bar{J}_w \rangle - \langle \Theta_3, \partial_{\bar{w}} \bar{J}_w \rangle].$$

- ②  $\mathcal{J}, \tilde{\mathcal{J}}$  are *raviolo fields* on  $(\mathcal{A}^{\bullet, \bullet}, d')$  [Garner and Williams 2023]

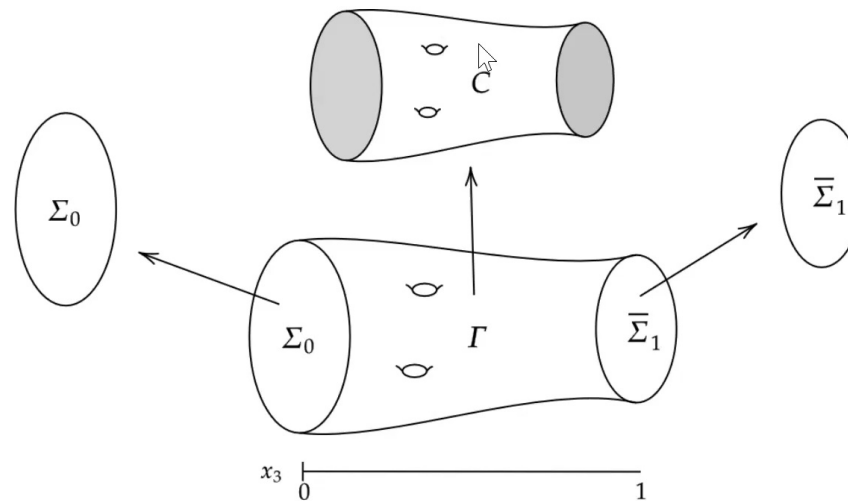
$$L_\perp \in \mathcal{A}^{0,1} \otimes \mathfrak{g}, \quad H_\ell \in \mathcal{A}^{0,2} \otimes \mathfrak{h}, \quad \tilde{L}_\ell \in \mathcal{A}^{1,0} \otimes \mathfrak{g}, \quad \tilde{H}_\perp \in \mathcal{A}^{1,1} \otimes \mathfrak{h}.$$

# Bordism invariance

- When they align  $dx_\ell = dx_3$ :

Theorem. (open bordism invariance) [Chen and Liniado 2024]

Chiral  $2\text{Hol}_\mathcal{J}$ 's descend to *framed open bordism 2-group*  $\Omega_{2,1}^{\text{fr}}(Y) \rightarrow \mathbb{G}$ .



- $2\text{Hol}_\mathcal{J}$  only has support  $\perp dx_3$ ,  $2\text{Hol}_{\tilde{\mathcal{J}}}$  only has support  $\parallel dx_3$ .
- $2\text{Hol}_{\tilde{\mathcal{J}}}$  lives on *lasagna fillings* [Morrison, Walker, and Wedrich 2019]!

# Noether analysis of $\mathcal{W}$

- **Higher Polyakov-Wiegmann:**  $\mathcal{W}$  has left/right symm. by  $(\alpha^{(\prime)}, \Gamma^{(\prime)}) \in (\Omega^\bullet \otimes \mathfrak{G})_0$

$$\Gamma_\ell^{(\prime)} = 0, \quad d_\ell(\alpha, \Gamma) = 0, \quad d_\perp(\alpha', \Gamma') = 0.$$

Proposition. (Noether charges of  $\mathcal{W}$ ) [Chen and Liniado 2024]

The following codim-1 charges

$$q_{(\alpha, \Gamma)} = \int \langle (\alpha, \Gamma), \mathcal{J} \rangle, \quad \tilde{q}_{(\alpha', \Gamma')} = \int \langle (\alpha', \Gamma'), \tilde{\mathcal{J}} \rangle$$

generate left/right global symmetries of  $\mathcal{W}$ .

- **Graded charge algebra  $\hat{\mathfrak{D}}$ :**

$$\begin{aligned} [q_{(\alpha_1, \Gamma_1)}, q_{(\alpha_2, \Gamma_2)}] &= -q_{([\alpha_1, \alpha_2], \mu_2(\alpha_1, \Gamma_2) - \mu_2(\alpha_2, \Gamma_1))} \\ &\quad - \underbrace{\int \langle \Gamma_1, d\alpha_2 + \mu_1 \Gamma_2 \rangle + \langle \alpha_1, d\Gamma_2 \rangle}_{\text{central}}; \end{aligned} \quad (1)$$

$\tilde{q} \in \hat{\mathfrak{D}}$  is analogous.

# Part II: the 2-Kac-Moody algebra and 2-Lax formulation of $\mathcal{W}$

## Reminder: Lax integrability (see also [Meusburger 2021; Olivier Babelon and Talon 2003])

Definition. (Lax pair) [Lax 1968]

Hamiltonian system  $(M, \{-, -\}, H)$  is *Lax integrable* iff there are Lie alg.  $\mathfrak{g}$  valued maps  $L, P : M \rightarrow \mathfrak{g}$  s.t.

$$\dot{L} = \{L, H\} = [P, L] \implies \frac{d}{dt} \text{tr} \rho(L)^i = 0. \quad (2)$$

- Classical Yang-Baxter equations (CYBE) [Belavin and Drinfel'd 1982]:  
 $r \in \mathfrak{g} \otimes \mathfrak{g}$  such that

$$[[r, r]] = [r_{12}, r_{23}] + [r_{13}, r_{23}] + [r_{12}, r_{13}] = 0.$$

Theorem. (canonical Lax pair) [Semenov-Tyan-Shanskii 1983]

A solution  $r \in \mathfrak{g} \otimes \mathfrak{g}$  to the CYBE  $\implies$  *canonical* Lax pair  $(L, P): \mathfrak{g}^* \rightarrow \mathfrak{g}$  on  $(C(\mathfrak{g}^*), \{-, -\}_{KK}, H)$  satisfying

$$\{L, L\}_{KK} = [L \otimes 1 + 1 \otimes L, r^\wedge], \quad \forall \text{ ad-inv. } H \in C^\infty(\mathfrak{g}^*)$$

# Reminder: Kac-Moody algebra and 2d WZW model

Definition. (the Kac-Moody algebra) [Kac 1978]

Simple Lie alg.  $\mathfrak{g}$ , Killing form  $\langle -, - \rangle$ . Take the central extension

$$\mathbb{C} \rightarrow \widehat{\Omega_k \mathfrak{g}} \rightarrow \Omega \mathfrak{g}, \quad k(X, X') = 2k \int_{S^1} \langle X, \frac{d}{d\tau} X' \rangle$$

of the loop algebra  $\Omega \mathfrak{g}$  w/ class  $k \in H^2(\Omega \mathfrak{g}, \mathbb{C}) \cong \mathbb{C}$ .

- $L, P \in \widehat{\Omega_k \mathfrak{g}} \implies$  flatness of **Lax connection** [Olivier Babelon and Talon 2003],

$$\dot{L} = [P, L] \implies \partial_t A_x - \partial_x A_t + [A_x, A_t] = 0, \quad A = Ldx + Pdt.$$

- *Currents in 2d WZW model are Lax connections* [Hoare 2022; Knizhnik and Zamolodchikov 1984]!

$$J \in \mathbb{C}[z] \otimes \mathfrak{g}, \quad \bar{J} \in \mathbb{C}[\bar{z}] \otimes \mathfrak{g}, \quad (\text{for } z \in S^1)$$

Noether currents live in Kac-Moody  $\implies \widehat{\Omega_k \mathfrak{g}}$ -symmetry of  $W$ .



## 2-Lax integrability

- How to "categorify" ?
  - Try replacing  $\mathfrak{g}$  (resp. (2)) with Lie 2-alg.  $\mathfrak{G}$  (resp. graded equation).
- Works on *dg Poisson mflds*  $(C(M)_1 \rightarrow C(M)_0, \{-, -\})$ .

Theorem. (canonical 2-Lax pair) [Chen and Girelli 2023]

A solution  $R \in (\mathfrak{G} \otimes \mathfrak{G})_1$  to the 2-CYBE [Bai, Sheng, and Zhu 2013]  $\implies$  a *canonical split 2-Lax pair* on  $(C(\mathfrak{G}^*[1]), \{-, -\}_{2KK}, H)$ ,

$$L \in (C(\mathfrak{G}^*[1]) \otimes \mathfrak{G})_1, \quad P \in (C(\mathfrak{G}^*[1]) \otimes \mathfrak{G})_0,$$

s.t.  $\{L, L\}_{2KK} = [L \otimes 1 + 1 \otimes L, R^\wedge]$  and the "bulk-boundary relation":

$$\underline{L} = \mu_1^* L_0 = \mu_1(L_1) \quad \text{is a Lax oper. on } C(\mathfrak{h}^* )!$$

- Conserved charges: take 2-rep  $\rho : \mathfrak{G} \rightarrow \mathfrak{gl}(V)$  of  $\mathfrak{g}$  [Angulo 2018], then

$$\frac{d}{dt} \mathcal{X}_\rho(L)^i = 0, \quad \forall \text{ "2-chars." } \mathcal{X}_\rho \text{ of } \rho.$$

# Higher derived current algebras

- Take a Riemann surface  $\Sigma$  and its **tangent complex** [Ševera 2005]

$$T[1]\Sigma \iff C(T[1]\Sigma) = \Omega^\bullet(\Sigma).$$

- Form the *dg current algebra*  $\Sigma\mathfrak{G} \equiv C(T[1]\Sigma, \mathfrak{G})$  [Faonte, Hennion, and Kapranov 2019; Kapranov 2021].
- Consider  $\Sigma\mathfrak{G} = C(T[1]\Sigma, \mathfrak{G})$ . It has components

grade 0	grade 1	grade 2
$\Omega^0(\Sigma, \mathfrak{g})$	$\Omega^1(\Sigma, \mathfrak{g})$	$\Omega^2(\Sigma, \mathfrak{g})$
$\Omega^1(\Sigma, \mathfrak{h})$	$\Omega^2(\Sigma, \mathfrak{h})$	0

- ... but  $\Omega^2(\Sigma)$  is a *superalgebra* [Kac 1977], and I want to keep the supergrading.
- Idea: secretly use the Hodge star to send  $\Omega^2 \xrightarrow{\sim} \Omega^0$ :

$$\Sigma\mathfrak{G} \cong (\Omega^0(\Sigma, \mathfrak{g}) \oplus \Omega^1(\Sigma, \mathfrak{g})) \oplus (\Omega^0(\Sigma, \mathfrak{h}) \oplus \Omega^1(\Sigma, \mathfrak{h})).$$

# The 2-Kac-Moody algebra

- Let  $\mathbb{C}^{1|1} = \mathbb{C}^{1|0} \oplus \mathbb{C}^{0|1}$  denote the superline.

Definition/Theorem. (2-Kac-Moody algebra  $\widehat{\Sigma_s \mathfrak{G}}$ ) [Chen and Girelli 2023]

$\widehat{\Sigma_s \mathfrak{G}}$  is a  $\mathbb{C}^{1|1}[1]$ -extension of  $\Sigma \mathfrak{G} \cong C(\mathcal{T}[1]\Sigma, \mathfrak{G})$  (cf. [Zhang and Liu 2014]) by

$$s(\mathcal{X}, \mathcal{Y}) = \int_{\Sigma} \langle \mathcal{X}, \hat{d}\mathcal{Y} \rangle \in \mathbb{C}^{1|1}[1], \quad \hat{d} = d - \mu_1,$$

and equipped with a pairing ( $\langle -, - \rangle =$  an invariant deg-1 pairing on  $\mathfrak{G}$ )

$$\langle \mathcal{X} \oplus \xi, \mathcal{Y} \oplus \zeta \rangle = \int_{\Sigma} \langle \mathcal{X} \cdot \mathcal{Y} \rangle + \langle \xi, \zeta \rangle,$$

where  $\langle - \cdot - \rangle$  (contains) the Hodge pairing of forms.

- $s \in Z^2(\Sigma \mathfrak{G}, \mathbb{C}^{1|1}[1])$  is a Lie 2-algebra 2-cocycle [Angulo 2018]!

# Zero 2-curvature formulation of 2-Lax integrability

- Q: can we see 2-Lax integrability as a 2MC condition?
- Answer: yes!

① Take  $L, P \in C^\infty(\mathbb{R}, \widehat{\Sigma}_s \mathfrak{G})$ , where

$$\begin{aligned} L_0 &\in \Omega^1(\Sigma, \mathfrak{g}), & P_0 &\in \Omega^0(\Sigma, \mathfrak{g}) \\ L_1 &\in \Omega^0(\Sigma, \mathfrak{h}), & P_1 &\in \Omega^1(\Sigma, \mathfrak{h}), \end{aligned}$$

② rename ( $\tau = x_3$ )

$$\begin{aligned} L_0 &= A_w dw + A_{\bar{w}} d\bar{w}, & L_1 &= B_w \bar{w}, \\ P_0 &= A_\tau, & P_1 &= B_{w\tau} dw + B_{\bar{w}\tau} d\bar{w}, \end{aligned}$$

③ then 2-Lax equation  $\iff$  2-flatness of  $(A, B)$ .

## Coisotropy

The red  $\Sigma \mathfrak{G}_+$  and blue  $\Sigma \mathfrak{G}_-$  subspaces are coisotropic

$$\langle \Sigma \mathfrak{G}_\pm, \Sigma \mathfrak{G}_\pm \rangle = 0, \quad \langle \Sigma \mathfrak{G}_\pm, \Sigma \mathfrak{G}_\mp \rangle \neq 0.$$

## 2-Lax formulation of $\mathcal{W}$

- Q: The currents  $\mathcal{J} = (L_\perp, H_\ell)$ ,  $\tilde{\mathcal{J}} = (\tilde{L}_\ell, \tilde{H}_\perp)$  in  $\mathcal{W}$  are also 2-flat, so are there "hidden" 2-Lax pairs in  $\mathcal{W}$ ?
- Answer: yes, when  $dx_\ell = d\tau$  align, and they are coisotropic!

$$\begin{cases} L_0 = L_\perp, & L_1 = \star_Y H_\ell, & P = 0 \\ \bar{L} = 0, & \bar{P}_0 = \tilde{L}_\ell, & \bar{P}_1 = \star_Y \tilde{H}_\perp. \end{cases}$$

- Must use the 3d Hodge star  $\star_Y$  on  $Y$ .

Proposition. (2-Lax connections in  $\mathcal{W}$ ) [Chen and Girelli 2023]

EOMs of  $\mathcal{J}, \tilde{\mathcal{J}} \iff$  2-Lax equations for  $(L, P)$  and  $(\bar{L}, \bar{P})$ .

- We get *two* 2-Lax pairs from  $\mathcal{W}$ , one cannot exist without the other, similar to the anti-/chiral Lax connections  $J, \bar{J}$  in WZW.
- But what of the symmetries?

# The 2-Kac-Moody symmetry of $\mathcal{W}$

Theorem (the moment map). [Chen and Girelli 2023]

If  $dx_\ell = d\tau$  align, then there is a 2-graded Lie algebra homomorphism

$$\hat{\mathcal{D}} \oplus \hat{\mathcal{D}} \rightarrow \Sigma\mathcal{G}_+ \oplus \Sigma\mathcal{G}_- \cong \widehat{\Sigma_s\mathcal{G}},$$

identifying symmetry Noether charges of  $\mathcal{W}$  with the 2-Kac-Moody algebra.

- The proof goes through the 2-Lax formulation!

—§—

- The "derived 2-Kac-Moody group  $\widehat{\Sigma_s\mathbb{G}}$ " (cf. [Henriques 2006]):
  - Extension class  $s$  may arise from "categorical surface transgression"

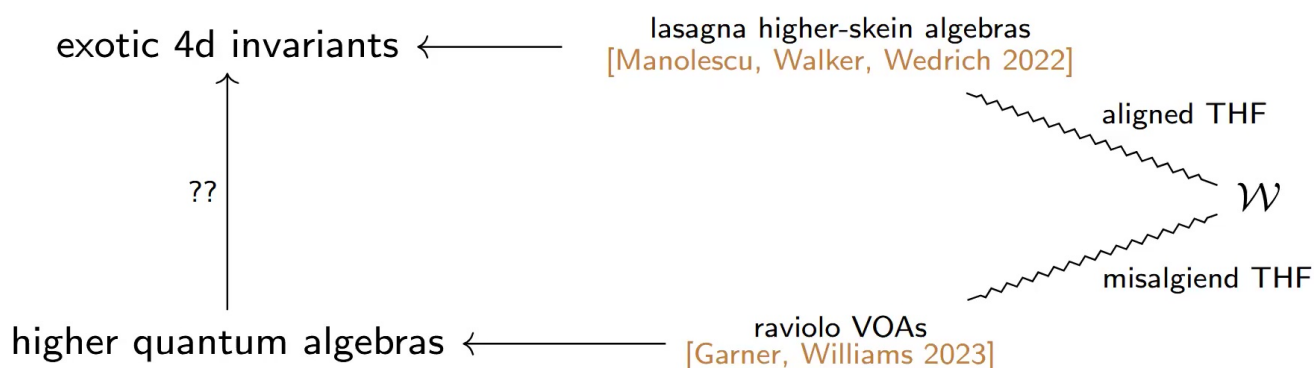
$$H^3(\mathbb{G}, \mathbb{C}^{1|1}[1]^\times) \rightarrow H^2(\Sigma\mathbb{G}, \mathbb{C}^{1|1}[1]^\times)$$

analogous to CS/WZW level transgression [Carey, Murray, and Wang 1997].

- May be an example of a raviolo VOA [Garner and Williams 2023].

# Summary

- The 3d IFT  $\mathcal{W}$  has
  - ① higher holonomies  $2\text{Hol}_{\mathcal{J}, \tilde{\mathcal{J}}}$  with interesting geometry,
  - ② global "2-Kac-Moody" symmetry  $\widehat{\Sigma}_s \mathfrak{G} \circlearrowleft \mathcal{W}$ .
- Outlook:
  - ① **Quantizing  $\mathcal{W}$ :** unitary representations of  $\widehat{\Sigma}_s \mathfrak{G}$ .
    - Alignment with the THF (on-going w/ Joaquin & Leon & more?):



- Higher Kazhdan-Lusztik (cf. [Kazhdan and Lusztig 1994]) with categorical quantum groups [Chen 2025]?
- ② Other choices of  $\omega$  [Schenkel and Vicedo 2024]:
  - Can get spectral parameter  $\implies$  2d integrable spin systems?

The End