Title: A new construction of $c=1$ Virasoro conformal blocks

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Abstract:

The Virasoro conformal blocks are very interesting since they have many connections to other areas of math and physics. For example, when $c=1$, they are related to tau functions of integrable systems of Painlevé equations. They are also closely related to non-perturbative completions in the topological string theories. I will first explain what Virasoro conformal blocks are. Then I will describe a new way to construct Virasoro blocks at $c=1$ on C by using the "abelian" Heisenberg conformal blocks on a branched double cover of C. The main new idea in our work is to use a spectral network and I will show the advantages of this construction. This nonabelianization construction enables us to compute the harder-to-get Virasoro blocks using the simpler abelian objects. It is closely related to the idea of nonabelianization of the flat connections in the work of Gaiotto-Moore-Neitzke and Neitzke-Hollands. This is based on a joint work with Andrew Neitzke.

A new construction of $c = 1$ Virasoro conformal blocks

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(joint work with Andrew Neitzke)
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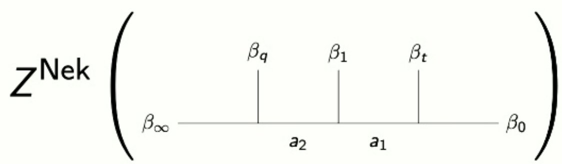
Introduction

Two-dimensional conformal field theory has been an important interface of theoretical physics and mathematics over the past decades.

In this talk, the main object is the $c = 1$ (chiral) Virasoro conformal blocks which is especially interesting because of its relation to

- self-dual Nekrasov partition function [Alday-Gaiotto-Tachikawa], e.g.



- commutative Verlinde line operators [Alday-Gaiotto-Gukov-Tachikawa-Verlinde, Teschner, $\cdots$]
- dual 4d limit of the topological string/ spectral theory correspondence [Grassi-Hatsuda-Marino]
- $\cdots$

A Virasoro block $\Psi \in \mathrm{Conf}(C, \mathrm{Vir}_{c=1})$ is defined as a system of n-point correlation functions

$$\langle T(p_1)^{z_1} \cdots T(p_n)^{z_n} \rangle_\Psi \ , \qquad \forall n \geq 0$$

for the stress energy tensor T and $p_i \in C$, which are holomorphic away from $p_i = p_j$, and at $p_i = p_j$, the singular behavior is given by

$$T(p_i)^z T(p_j)^z = \frac{1/2}{(z(p_i) - z(p_j))^4} + \frac{2T(p_j)^z}{(z(p_i) - z(p_j))^2} + \frac{\partial_{z(p_j)} T(p_j)^z}{z(p_i) - z(p_j)} + \text{reg} \, .$$

Under change of coordinates, T transforms as

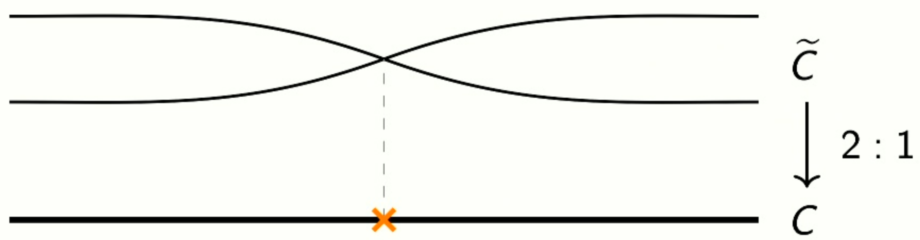
$$T(p)^w = \left(\frac{\mathrm{d}z(p)}{\mathrm{d}w(p)} \right)^2 \left(T(p)^z + \frac{c}{12} \{z, w\} \right) \, .$$

We can also define $\Psi \in \mathrm{Conf}(C, \mathrm{Vir}_{c=1}; W_{h_1}(q_1) \cdots W_{h_n}(q_n))$ with primary fields $W_h(q)$, where h is the conformal weight. In the correlators $\langle T(p_1)^{z_1} \cdots T(p_n)^{z_n} W_{h_1}(q_1) \cdots W_{h_n}(q_n) \rangle_\Psi$ we have extra singularities controlled by

$$T(p)^z W_h(q) = \frac{h W_h(q)}{(z(p) - z(q))^2} + \cdots \, .$$

My talk is to describe a new way to construct the $c = 1$ Virasoro conformal blocks over C.

Our nonabelianization approach constructs harder to compute Virasoro conformal blocks over C using the abelian Heisenberg ones which are easier to be solved exactly, over a branched double cover of C, denoted by $\widetilde{C}$.

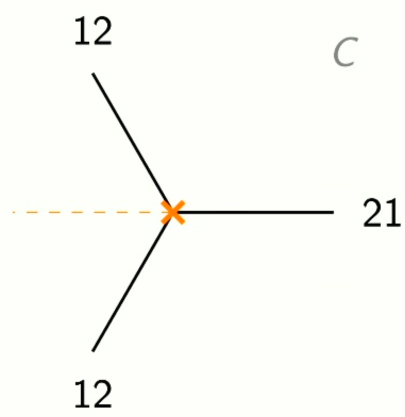


The generator J of the Heisenberg algebra satisfies the OPE $J(p_i)J(p_j) = \frac{1}{(z(p_i)-z(p_j))^2} + \text{reg}$ and $J(p)^z = \frac{\mathrm{d}w(p)}{\mathrm{d}z(p)} J(p)^w$.

The important new idea in our work is to use the spectral network which are defined originally in [Gaiotto-Moore-Neitzke].

A spectral network is a collection of walls on the Riemann surface C. Each branch point is an endpoint of 3 walls, meeting at an angle $\frac{2\pi}{3}$.

Each wall comes with a label ij. The label corresponds to the two sheets of the cover $\widetilde{C}$.



In the previous free field constructions using a double cover, it has been known that there are extra insertions at each branch point. [Dixon-Friedan-Martinec-Shenker, Gavrylenko-Marshakov, Zamolodchikov, $\cdots$]

An advantage of including the spectral network is that it cancels these unwanted insertions at branch points.

Certain $c = 1$ Virasoro blocks/ Nekrasov-Okounkov partition functions are identified with τ-functions of a class of important nonlinear second order ODE, the painlevé equations.

Our construction also manifests the dependence of τ-function on the spectral network which has been noticed before. [Coman-Longhi-Teschner, Iwaki-Marino, Coman-Pomoni-Teschner, Iwaki-Kidwai, Gavrylenko-Grassi-H, $\cdots$]

Outline

Nonabelianization

Space of conformal blocks

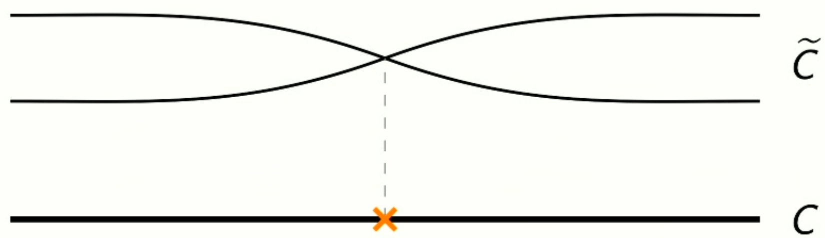
Applications

Future directions

We build the nonabelianization map

$$\mathcal{F}_{\mathcal{W}} : \quad \mathrm{Conf}(\widetilde{C}, \mathrm{Heis}) \quad \to \quad \mathrm{Conf}(C, \mathrm{Vir}_{c=1}) \, ,$$

where $\widetilde{C}$ is a branched double cover of C with $\pi : \widetilde{C} \to C$ and $\mathcal{W}$ stands for a spectral network.



Using the dictionary

$$T(p) \rightsquigarrow \frac{1}{4}\colon (J(p^{(1)}) - J(p^{(2)}))^2\colon,$$

it seems that we can already write down a map of conformal blocks.

Question: Why do we need the spectral network $\mathcal{W}$?

Let's first forget about the spectral network and see what needs to be improved. We try to construct the map

$$\mathcal{F}_\varnothing : \quad \mathrm{Conf}(\widetilde{C}, \mathrm{Heis}) \quad \xrightarrow{?} \quad \mathrm{Conf}(C, \mathrm{Vir}_{c=1})\,,$$

$$\left\langle \cdots \frac{1}{4}\colon \left(J(p^{(1)})^{z^{(1)}} - J(p^{(2)})^{z^{(2)}}\right)^2 \colon \right\rangle_{\widetilde{\Psi}} \overset{?}{\mapsto} \langle \cdots T(p)^z \rangle_{\mathcal{F}_\varnothing(\widetilde{\Psi})}\,.$$

We can test this map in a baby example:

The Riemann surface we consider is $C = \mathbb{CP}^1$, and its double cover is also $\widetilde{C} = \mathbb{CP}^1$, with the projection map

$$\begin{aligned} \pi : \widetilde{C} &\to C \\ z &\mapsto x = z^2 \end{aligned}$$

This map has two branch points, at $z = 0$ and $z = \infty$.

Using $\mathcal{F}_\varnothing$, we can calculate

$$T(p)^x = \frac{1}{16x(p)^2} + \cdots .$$

Note this equation holds in the correlation function. If $\mathcal{F}_\varnothing(\widetilde{\Psi}) \in \mathrm{Conf}(C, \mathrm{Vir}_{c=1})$,

$$\langle T(p)^x \rangle_{\mathcal{F}_\varnothing(\widetilde{\Psi})} = \frac{\langle 1 \rangle_{\mathcal{F}_\varnothing(\widetilde{\Psi})}}{16(x(p) - 0)^2} + \langle \cdots \rangle_{\mathcal{F}_\varnothing(\widetilde{\Psi})} .$$

Recall that the definition of conformal blocks requires that the correlation functions are holomorphic away from the diagonal.

The singularity at $(x(p) - 0)$ means that there must be an insertion at 0, i.e.

$$\mathcal{F}_{\varnothing}(\widetilde{\Psi}) \in \mathrm{Conf}\left(C, \mathrm{Vir}_{c=1}; W_h(0)\right)$$

$$\langle T(p)^x \, W_h(0) \rangle_{\mathcal{F}_{\varnothing}(\widetilde{\Psi})} = \frac{\langle W_h(0) \rangle_{\mathcal{F}_{\varnothing}(\widetilde{\Psi})}}{16(x(p) - 0)^2} + \cdots .$$

The insertion can be determined using the definition of Virasoro blocks with primary fields,

$$T(p)^x W_h(q) = \frac{h W_h(q)}{(x(p) - x(q))^2} + \cdots .$$

$$\mathcal{F}_{\varnothing} : \quad \mathrm{Conf}(\widetilde{C}, \mathrm{Heis}) \quad \to \quad \mathrm{Conf}\left(C, \mathrm{Vir}_{c=1}; W_{\frac{1}{16}}(0) W_{\frac{1}{16}}(\infty)\right) .$$

$$\mathcal{F}_\varnothing : \quad \mathrm{Conf}(\widetilde{C}, \mathrm{Heis}) \quad \to \quad \mathrm{Conf}\left(C, \mathrm{Vir}_{c=1}; W_{\frac{1}{16}}(0) W_{\frac{1}{16}}(\infty)\right) .$$

For experts, these extra insertions of primary fields $W_{\frac{1}{16}}$ are known for a while in the "branched free field" construction.
[Dixon-Friedan-Martinec-Shenker, Gavrylenko-Marshakov, Zamolodchikov, $\cdots$]

However, we still want to have a map to $\mathrm{Conf}\,(C, \mathrm{Vir}_{c=1})$, because for example τ-functions live there.

$W_{\frac{1}{16}}$ at the branch points can be canceled by using a spectral network $\mathcal{W}$. In $\mathcal{F}_{\mathcal{W}}$, we insert an operator $E(\mathcal{W})$ into the correlation functions. The first few correlation functions read

$$\langle 1 \rangle_{\mathcal{F}_{\mathcal{W}}(\widetilde{\Psi})} = \langle E(\mathcal{W}) \rangle_{\widetilde{\Psi}} ,$$

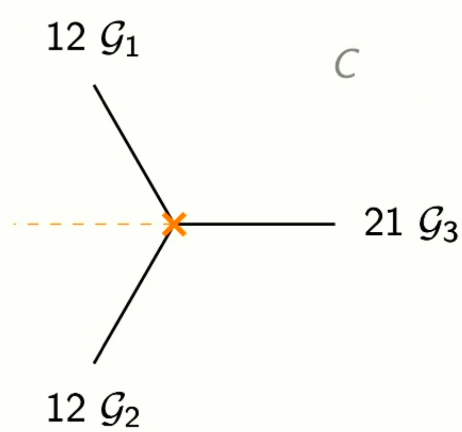
$$\langle T(p)^z \rangle_{\mathcal{F}_{\mathcal{W}}(\widetilde{\Psi})} = \left\langle \frac{1}{4} \!:\! \left(J(p^{(1)})^{z^{(1)}} - J(p^{(2)})^{z^{(2)}} \right)^2 \!:\! E(\mathcal{W}) \right\rangle_{\widetilde{\Psi}} ,$$

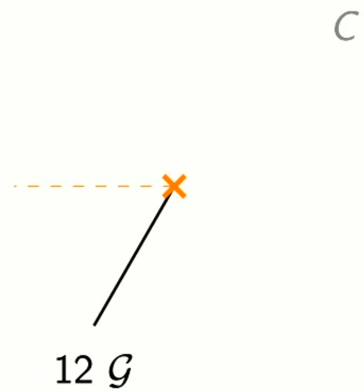
$$\vdots$$

$$E(\mathcal{W}) = \exp\left(\frac{1}{2\pi \mathrm{i}} W(\mathcal{W})\right).$$

We define $W(\mathcal{G}_i)$ for each wall $\mathcal{G}_i \subset \mathcal{W}$, such that

$$W(\mathcal{W}) = \sum_i W(\mathcal{G}_i).$$

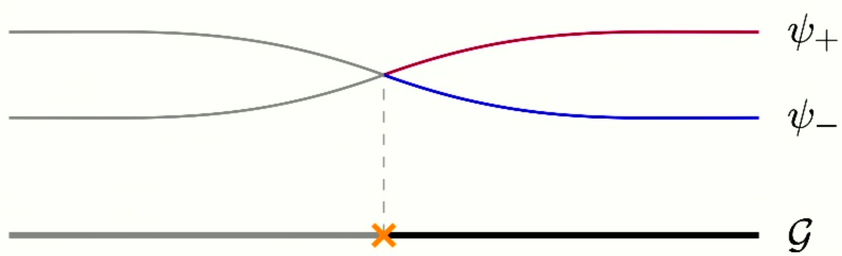




For a single $\mathcal{G}$,

$$W(\mathcal{G}) = \int_{\mathcal{G}} \psi_+(q^{(1)})^{x^{(1)}} \, \psi_-(q^{(2)})^{x^{(2)}} \, \mathrm{d}x(q) \,,$$

where $\psi_\pm$ are usually called free fermions. And $x^{(1)}$ and $x^{(2)}$ are lifts of x to the cover.



With the extra insertion

$$E(\mathcal{W}) = \exp\left(\frac{1}{2\pi\mathrm{i}}\int_{\curlywedge} \psi_+(q^{(1)})^{x^{(1)}}\,\psi_-(q^{(2)})^{x^{(2)}}\,\mathrm{d}x(q)\right)\,,$$

$$T(p)^x = \mathrm{reg}\,.$$

We also checked that the insertion of $E(\mathcal{W})$ won't cause any other singularities of $T(p)$.

This indicates that instead of a map to the conformal block with $W_{\frac{1}{16}}$, $\mathcal{F}_{\mathcal{W}}$ is indeed a map to the space of vacuum conformal blocks $\mathrm{Conf}(C, \mathrm{Vir}_{c=1})$

$$\mathcal{F}_{\mathcal{W}}: \quad \mathrm{Conf}(\widetilde{C}, \mathrm{Heis}) \quad \to \quad \mathrm{Conf}(C, \mathrm{Vir}_{c=1})\,.$$

In general, conformal blocks form an infinite dimensional vector space $\mathrm{Conf}(C, \mathcal{V}; \cdots)$, where $\mathcal{V}$ is the Vertex algebra.

$\mathrm{Conf}(C, \mathrm{Vir}_{c=1}; \cdots)$ is acted on by commutative Verlinde line operators L_γ. [Alday-Gaiotto-Gukov-Tachikawa-Verlinde, Teschner, $\cdots$]

$$L_\gamma : \mathrm{Conf}(C, \mathrm{Vir}_{c=1}; \cdots) \circlearrowleft,$$

where γ is a loop on C.

$\mathrm{Conf}(\widetilde{C}, \mathrm{Heis}; \cdots)$ is acted on by the abelian verison of L_γ.

We use common eigenblocks of (abelian) Verlinde operators which form a 1d subspace of Conf.

The abelian Verlinde operators have the explicit form

$$L_\gamma = \exp(\ell_\gamma), \quad \ell_\gamma = \oint_\gamma J,$$

where $\gamma \in H_1(\widetilde{C}, \mathbb{Z})$. We call ℓ_γ log abelian Verlinde operator.

ℓ_γ acts on conformal block $\widetilde{\Psi}$ as

$$\langle \cdots \rangle_{\ell_\gamma(\widetilde{\Psi})} = \langle \cdots \ell_\gamma \rangle_{\widetilde{\Psi}} \ .$$

We calcuate that

$$[\ell_\gamma, \ell_{\gamma'}] = -2\pi \mathrm{i} \left\langle \gamma, \gamma' \right\rangle ,$$

where $\langle \quad , \quad \rangle$ denotes the intersection number.

Earlier I mentioned diagonalizing L_γ. But actually it's easier to first diagnolize half of ℓ_γ.

Fix a choice of A and B cycles on $\widetilde{C}_{\tilde{g}}$, with the intersection condition $\langle A_i, B_j \rangle = \delta_{ij}$, also fix a vector $a = (a_1, \ldots, a_{\tilde{g}}) \in \mathbb{C}^{\tilde{g}}$. We first define an eigenblock $\widetilde{\Psi}_a$ of ℓ_A, which satisfies

$$\ell_{A_i} \widetilde{\Psi}_a = a_i \widetilde{\Psi}_a \, .$$

Fix the normalization $f(a)$ by further requiring

$$\ell_{B_i} \widetilde{\Psi}_a = 2\pi \mathrm{i} \partial_{a_i} \widetilde{\Psi}_a \, , \qquad \langle 1 \rangle_{\widetilde{\Psi}_{a=0}} = 1 \, .$$

Now $\widetilde{\Psi}_a$ is a point in $\mathrm{Conf}(\widetilde{C}, \mathrm{Heis})$.

We can construct the Heisenberg blocks by hand explicitly, e.g.

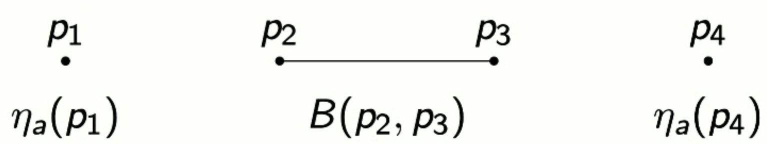
$$\langle 1 \rangle_{\widetilde{\Psi}_a} = \mathrm{e}^{\frac{1}{4\pi \mathrm{i}} a\cdot\tau a},$$

$$\langle J(p) \rangle_{\widetilde{\Psi}_a} = \mathrm{e}^{\frac{1}{4\pi \mathrm{i}} a\cdot\tau a} \left(\overset{p}{\bullet} \right),$$

$$\langle J(p)J(q) \rangle_{\widetilde{\Psi}_a} = \mathrm{e}^{\frac{1}{4\pi \mathrm{i}} a\cdot\tau a} \left(\overset{p}{\bullet} \qquad \overset{q}{\bullet} + \overset{p}{\bullet}\!\!\text{———}\!\!\overset{q}{\bullet} \right),$$

$$\vdots$$

where we use Feynman diagrams



- $\eta_a = \sum_{i=1}^{\tilde{g}} a_i \omega_i$, where $(\omega_1, \ldots, \omega_{\tilde{g}})$ is the basis of holomorphic 1-forms dual to $(A_1, \ldots, A_{\tilde{g}})$.
- $B(p,q)$ is the Bergman kernel $B(p,q)$ on $\widetilde{C}$ normalized on the A cycles.
- $\tau_{ij} = \oint_{B_j} \omega_i$.

We can also construct eigenblock of abelian Verlinde operators $L_{A,B}$ from eigenblock $\widetilde{\Psi}_a$ of log abelian Verlinde operators ℓ_a.

$L_{A,B}$ act on $\widetilde{\Psi}_a$ by

$$L_{A_i}\widetilde{\Psi}_a = \exp(a_i)\widetilde{\Psi}_a, \qquad L_{B_i}\widetilde{\Psi}_a = \widetilde{\Psi}_{a+2\pi \mathrm{i} e_i}\,.$$

Fix parameters $((x_1,\ldots,x_{\tilde{g}}),(y_1,\ldots,y_{\tilde{g}})) \in \mathbb{C}^{2\tilde{g}}$, the eigenblock $\widetilde{\Psi}_{x,y} \in \mathrm{Conf}(\widetilde{C},\mathrm{Heis})$ is defined by

$$\widetilde{\Psi}_{x,y} = \sum_{n\in\mathbb{Z}^{\tilde{g}}} \exp\left(-n\cdot y\right)\widetilde{\Psi}_{a=x+2\pi \mathrm{i} n}\,.$$

with eigenvalues $\exp(x_i)$ and $\exp(y_i)$.

E.g.

$$\langle 1\rangle_{\widetilde{\Psi}_{x,y}} = \sum_{n\in\mathbb{Z}^{\tilde{g}}} \mathrm{e}^{-n\cdot y}\,\langle 1\rangle_{\widetilde{\Psi}_{a=x+2\pi \mathrm{i} n}} = \exp\left(\frac{x\cdot\tau x}{4\pi \mathrm{i}}\right)\Theta\left(\tau, z=\frac{y+\tau x}{2\pi \mathrm{i}}\right)\,.$$

Abelian L_γ generate $\mathsf{Sk}_{-1}(\widetilde{C}, \mathsf{GL}(1))$

$$L_\gamma L_\mu = (-1)^{\langle \gamma, \mu \rangle} L_{\gamma+\mu}.$$

$\mathsf{Sk}_{-1}(\widetilde{C}, \mathsf{GL}(1))$ dually is the algebra $\mathcal{O}(\mathcal{M}(\widetilde{C}, \mathsf{GL}(1)))$

$$(\mathrm{e}^x, \mathrm{e}^y) \in \mathcal{M}(\widetilde{C}, \mathsf{GL}(1)).$$

Similarly, the Verlinde operators generate $\mathsf{Sk}_{-1}(C, \mathsf{SL}(2))$, dually $\mathcal{O}(\mathcal{M}(C, \mathsf{SL}(2)))$. The common eigenvalue of Verlinde operators specifies $\lambda \in \mathcal{M}(C, \mathsf{SL}(2))$.

$\mathcal{F}_\mathcal{W}$ intertwines these two actions of Verlinde and abelian Verlinde operators via the nonabelianization map

$$\mathcal{F}^\flat_\mathcal{W} : \mathcal{M}(\widetilde{C}, \mathrm{GL}(1)) \to \mathcal{M}(C, \mathsf{SL}(2))$$
$$(\mathrm{e}^x, \mathrm{e}^y) \mapsto \lambda\,.$$

[Gaiotto-Moore-Neitzke, Hollands-Neitzke]

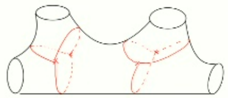
Eigenblocks of abelian Verlinde operators form a line bundle $\widetilde{\mathcal{L}}$ over $\mathcal{M}(\widetilde{C}, \mathrm{GL}(1))$.

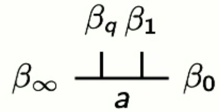
Eigenblocks of Verlinde operators form a line bundle $\mathcal{L}$ over $\mathcal{M}(C, \mathrm{SL}(2))$.

$\mathcal{F}_{\mathcal{W}}$ gives a lift of $\mathcal{F}^{\flat}_{\mathcal{W}}$ to the line bundle.

$$
\begin{array}{ccc}
\widetilde{\mathcal{L}} & \xrightarrow{\mathcal{F}_{\mathcal{W}}} & \mathcal{L} \\
\downarrow & & \downarrow \\
\mathcal{M}(\widetilde{C}, \mathrm{GL}(1)) & \xrightarrow{\mathcal{F}^{\flat}_{\mathcal{W}}} & \mathcal{M}(C, \mathrm{SL}(2))
\end{array}
$$

We expect $\langle 1 \rangle_{\mathcal{F}_{\mathcal{W}_P}(\widetilde{\Psi}_a)}$, for the Fenchel-Nielsen $\mathcal{W}_P$, e.g.



to correspond to a computable form of the conformal block, the Nekrasov partition function, e.g. $Z^{\mathrm{Nek}}($ 

$)$.

We further expect

$$\langle 1 \rangle_{\mathcal{F}_{\mathcal{W}_P}(\widetilde{\Psi}_{x,y})} = \sum_{n \in \mathbb{Z}} \exp(-ny) \langle 1 \rangle_{\mathcal{F}_{\mathcal{W}_P}(\widetilde{\Psi}_{a=x+2\pi \mathrm{i} n})} .$$

to reproduce the Nekrasov-Okunkov partition function Z^{NO}.

This expression of $\langle 1 \rangle_{\mathcal{F}_{\mathcal{W}_P}(\widetilde{\Psi}_{x,y})}$ is known as the Kyiv formula for the $\tau(x, y)$ function of certain integrable system, e.g. Painlevé VI. [Gamayun-Iorgov-Lisovyy]

We give a new expression of the $c = 1$ Virasoro block/ τ function (up to regularization and normalization)

$$\tau = \det(1 + \mathcal{I}_{x,y}) \,,$$

where $\mathcal{I}_{x,y}$ is an integral operator along the spectral network $\mathcal{W}$ with kernel

$$\mathcal{K}(p, q) = \frac{1}{2\pi \mathrm{i}} \frac{\langle \psi_+(p^{(+)})\psi_-(q^{(-)})\rangle_{\widetilde{\Psi}_{x,y}}}{\langle 1 \rangle_{\widetilde{\Psi}_{x,y}}} \,.$$

This can be viewed as a resummation of the NO partition function.

For generic $\mathcal{W}$, we propose it gives a generic conformal block which is conjectured by Goncharov-Shen.

For example, we propose it can give the strong coupling expansion of τ funtions.
[Gamayun-Iorgov-Lisovyy, Coman-Pietro-Teschner, Bonelli-Lisovyy-Maruyoshi-Sciarappa-Tanzini, Gavrylenko-Grassi-H, $\cdots$]

A Fredholm determinant form of τ function comes from the conjectured topological string/ spectral theory correspondence. [Grassi-Hatsuda-Marino, $\cdots$]

$$\sum_{\mathrm{w}\in\mathbb{Z}^{N-1}} \exp\left(F^{\mathrm{top}} + \mathrm{Nonpert}(F^{\mathrm{NS}})\right)(\mathrm{t}+\mathrm{w}) = \det\Big(1 + \sum_{i=1}^{N-1} \kappa_i A_i\Big)$$

E.g., for 5d $SU(2)$ theory, $A_1 = \left(\mathrm{e}^{\hat{p}} + \mathrm{e}^{-\hat{p}} + \xi \mathrm{e}^{-\hat{x}} + \mathrm{e}^{\hat{x}}\right)^{-1}$. The kernel for A_i is only written down for limited special examples.

We expect our construction for 2 irregular singularities can reproduce the proved dual 4d limit of pure $SU(2)$ gauge theory [Gavrylenko-Grassi-H]

$$Z^{\mathrm{NO}} = \tau = \det\left(1 + \frac{(\mathrm{e}^{x} + \mathrm{e}^{-x})}{2\pi}\mathcal{K}\right),$$

where $\mathcal{K}$ is an integral operator on $\mathbb{R}_+$ with the kernel

$$\mathcal{K}(u,v) = \frac{\mathrm{e}^{-4\Lambda\left(u+v^{-1}\right)}}{u+v}.$$

[Bonelli-Grassi-Tanzini, Tracy-Widom, Gavrylenko-Grassi-H]

Future directions

- We hope to produce a way of writing down Fredholm determinants of spectral problems closely related to topological string non-perturbative completion.
  [Tracy-Widom, Codesido-Grassi-Mariño, Gavrylenko-Grassi-H, $\cdots$]
- The generalization of nonabelianization to W_N-algebras at $c = N - 1$. This is an analogy to the $SL(N, \mathbb{C})$ connection. And there conformal blocks are less studied.
- Generalize it to other central charges c.
- $\ldots$

Thank you!