**Title:** Lecture - Numerical Methods, PHYS 777

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# Quasi-Linear PDE

$U$: state vector

$\partial\partial U \qquad \partial U \qquad U$

linear in
highest deriv.

Principal part

$$\boxed{\alpha \partial_t u} + \boxed{\beta \partial_x^2 u} + (\partial_x u)^2 + \sin u = 0$$

$$u(t,x) = c^{\omega k} e^{i\omega t} e^{ikx}$$

$$\partial_t u = -i\omega u$$

$$\partial_x u = -ik u$$

$$\partial_x^2 u = -k^2 u$$

Principal part

$\boxed{\partial_t^2 u} + (\partial_x u)^2 + \sin u = 0$

$= c^{wk}\, e^{iwt}\, e^{ikx}$

elliptic $(\partial_x^2 + \partial_y^2)u$ $A^{ij}\partial_i \partial_j u + \cdots$

parabolic

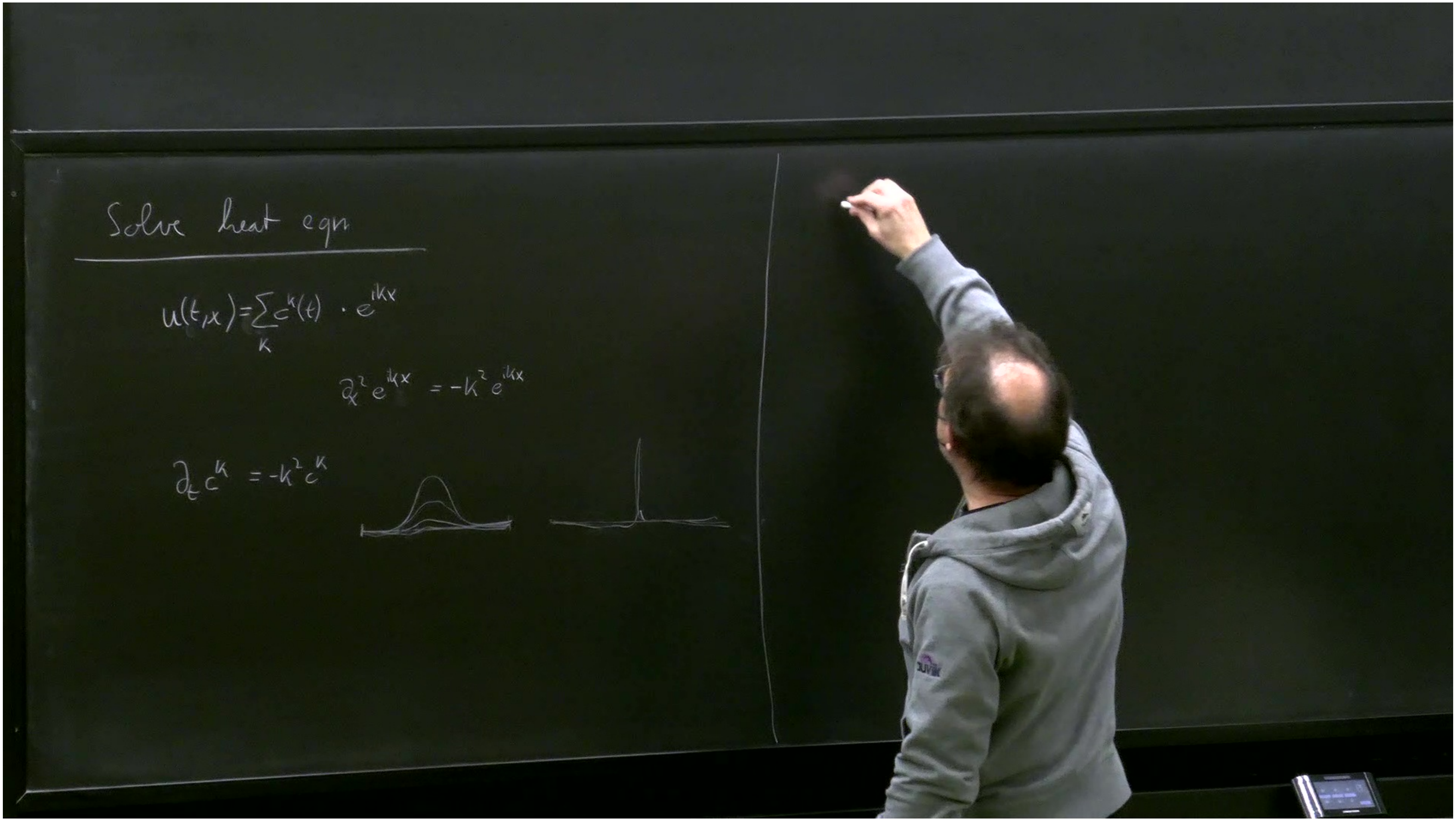
hyperbolic $(-\partial_t^2 + \partial_x^2)u$

$\hookrightarrow$ all ev $> 0$ : elliptic : stationary

$\left.\begin{array}{l}\text{one ev} < 0 \\ \text{others} > 0\end{array}\right\}$ : hyperbolic : time evolution

Heat equation (parabolic)

$\partial_t u - \partial_x^2 u = 0$


Solve heat eqn
$u(t,x) = \sum_k c^k(t) \cdot e^{ikx}$
$\partial_x^2 e^{ikx} = -k^2 e^{ikx}$
$\partial_t c^k = -k^2 c^k$

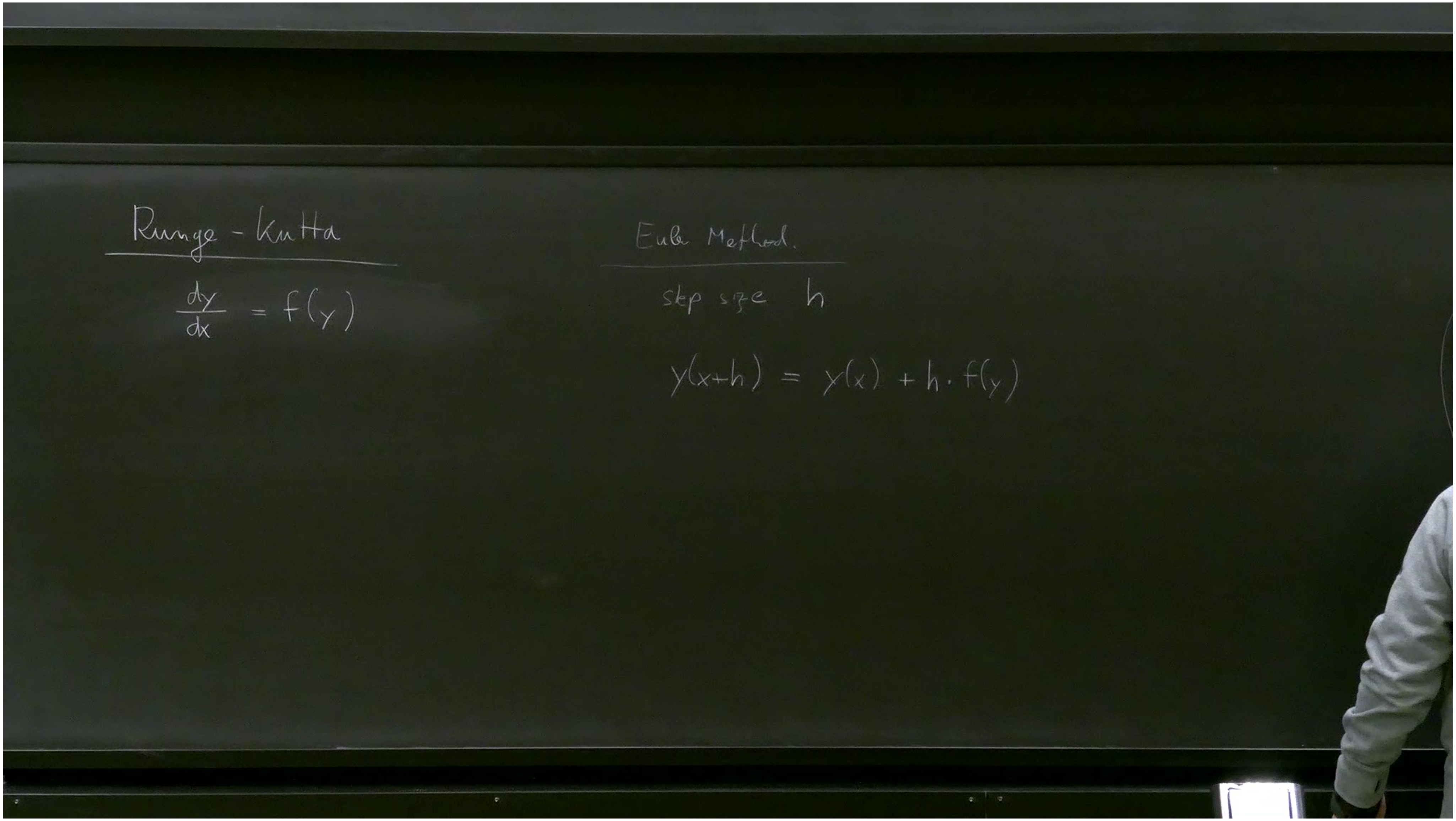
Backwards heat eqn

$$(\partial_t + \partial_x^2)u = 0$$

$$\partial_t c^k = k^2 c^k$$

ill-posed

well-posed problem: depends continuously on initial conditions


Runge - Kutta
$\frac{dy}{dx} = f(y)$
Euler Method.
step size h
$y(x+h) = y(x) + h \cdot f(y)$

## Runge - Kutta

$$\frac{dy}{dx} = f(y)$$

## Euler Method.

step size $h$

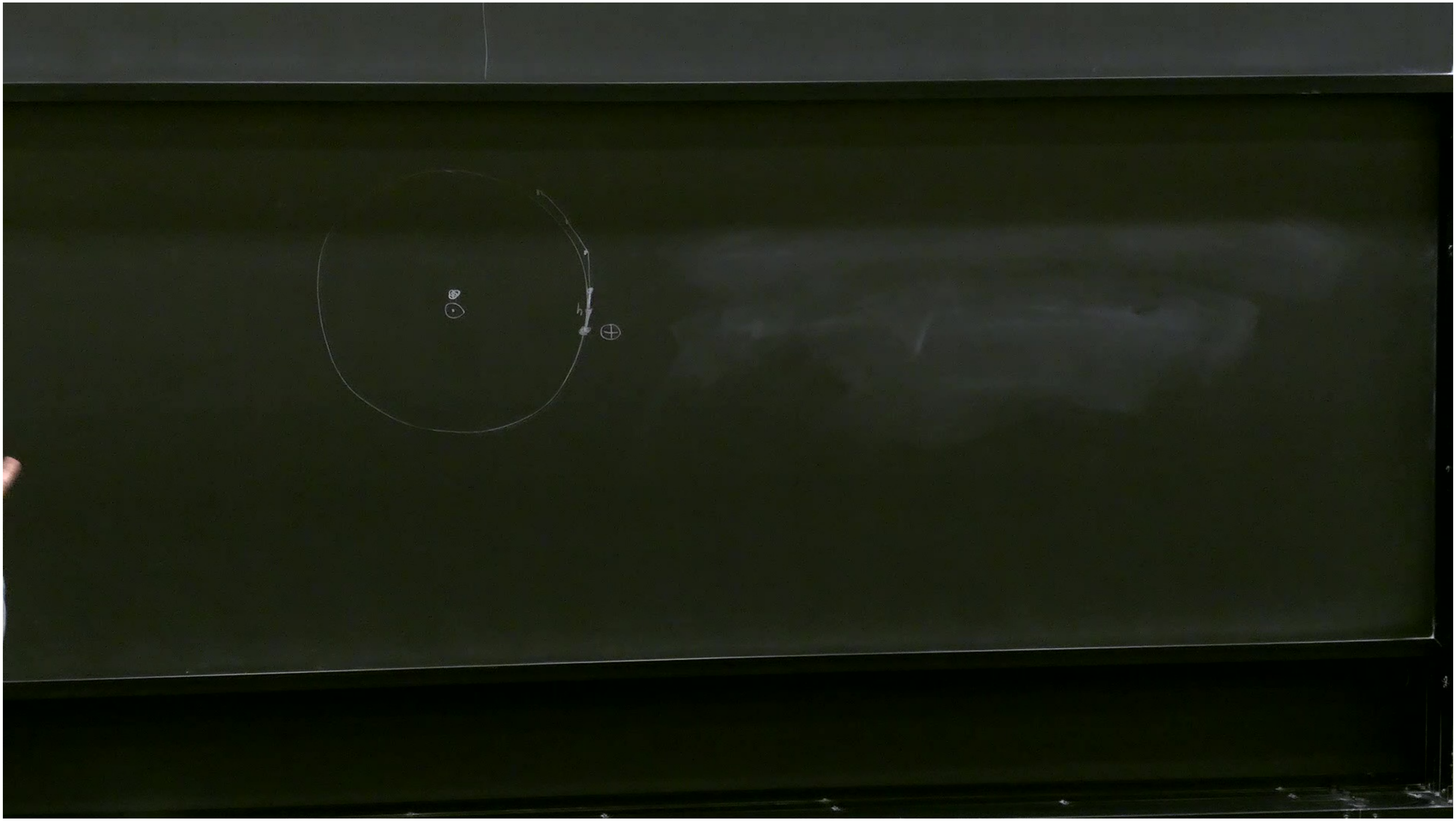
$$y(x+h) = y(x) + h \cdot f(y)$$

## Midpoint Rule

$$y(x) =: y_0$$

$$y_1 = y_0 + \frac{h}{2} f(y_0)$$

$$y_2 = y_0 + h\, f(y_1)$$



## Method of Lines:

PDE: $u(t,x)$ $\quad \partial_t^2 u = \partial_x^2 u$

① discretize space $\quad u^i(t)$

$$\partial_t^2 u^i = \frac{u^{i+1} - 2u^i + u^{i-1}}{h^2}$$

ODEs

② use ODE solver $\quad u^{n,i}$

## Convergence

$f(x)$

resolution $h$ $\qquad f_h(x)$

residual $\qquad \| f_h(x) - f(x) \|$

$$f_h(x) = f(x) + E\, h^p + O(h^{p+1})$$

$$f_{h_1} - f_{h_2} = E\left(h_1^p - h_2^p\right) + O(h^{p+1})$$

$$f_h(x) = f(x) + E\,h^p + O(h^{p+1})$$

$$f_{h_1} - f_{h_2} = E\left(h_1^p - h_2^p\right) + O(h^{p+1})$$

$$= E\,h_1^p\left(1 - \frac{1}{2^p}\right) + \cancel{O(h^{p+1})}$$

$$f_{h_2} - f_{h_3} = \cdots$$

$$\boxed{\begin{aligned} h_2 &= \tfrac{1}{2} h_1 \\ h_3 &= \tfrac{1}{2} h_2 \end{aligned}}$$

$$\frac{f_{h_1} - f_{h_2}}{f_{h_2} - f_{h_3}} = \frac{1 - \frac{1}{\cancel{2^p}}}{\cancel{1} - \frac{1}{2^p}} \cdot 2^p = 2^p$$