Title: Lecture - Gravitational Physics, PHYS 636

Speakers: Ruth Gregory

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Lecture 12 Perturbations of Black Holes

Recall that writing $g_{ab} = g_{0ab} + h_{ab}$ gives the Lichnerowicz eqn

$$\Delta_L h_{ab} = \Box h_{ab} + 2R_{acbd}h^{cd} - 2R_{(a}{}^{c} h_{b)c} - 2\nabla_{(a}\nabla^{e}\bar{h}_{b)e} \quad \left(\bar{h}_{ab} = h_{ab} - \tfrac{1}{2} h g_{0ab}\right)$$

Lecture 12 Perturbations of Black Holes

Recall that writing $g_{ab} = g_{0ab} + h_{ab}$ gives the Lichnerowicz eqn

$$\Delta_L h_{ab} = \Box h_{ab} + 2R_{acbd}h^{cd} - 2R_{(a}{}^{c}h_{b)c}$$
$$- 2\nabla_{(a}\underbrace{\nabla^{e}\bar{h}_{b)e}}_{=0 \text{ gauge De Donder}} = -2\delta R_{ab}.$$

$$\left(\bar{h}_{ab} = h_{ab} - \tfrac{1}{2}h^{c}{}_{c}\,g_{0ab}\right)$$

Common gauge $\nabla^e \bar{h}_{eb} = 0$

"transverse tracefree" $h^a{}_a = 0 = \nabla^a h_{ab}$

Regge & Wheeler 1957 Phys Rev 108

g_{ab} is chosen with symmetries, so we can decompose h_{ab} into a basis of eigenfns — need self-adjointness

R_{ab}.

SCH: SO(3) & ∂_t

ang-mom e-fns ℓ

fourier modes in time $e^{-i\omega t}$

Eigenfns corr to tensor spherical harmonics, & parity eg odd parity $(m=0)$

$$h_{ab} = \begin{bmatrix} 0 & 0 & 0 & h_0(r) \\ 0 & 0 & 0 & h_1(r) \\ 0 & 0 & 0 & 0 \\ h_0 & h_1 & 0 & 0 \end{bmatrix} e^{-i\omega t} \times \sin\theta\, \partial_\theta P_\ell(\cos\theta)$$

where $$\frac{d^2}{dr^{*2}}\left(\frac{f h_1}{r}\right) + \omega^2 h - \underbrace{\left[\ell(\ell+1) - 6GM/r\right]\frac{f}{r^2}}_{V_\ell(r).} h = 0$$

$= -2\delta R_{ab}$.
Order

we can decompose ...
of eigenfns — need self-adjointness

Application Black string instability

$$ds^2_{BS} = \left(1 - \frac{r_4}{r}\right)dt^2 - \frac{dr^2}{\left(1-\frac{r_4}{r}\right)} - r^2 d\Omega^2 - dz^2$$

$$M = \frac{r_4 L}{2G_5} = r_4/2G_4$$

$$S_{BS} = \frac{\pi r_4^2 L}{G_5} = 4\pi G_4 M^2$$

— compare to 5D black hole

$$ds^2_{BH} = \left(1 - \frac{\rho_5^2}{\rho^2}\right)dt^2 - \frac{d\rho^2}{1-\rho_5^2/\rho^2} - \rho^2 d\Omega^2_{\text{III}}$$

Mass $M = \left.\frac{(D-2)}{16\pi G} A_{D-2}\, \rho_D^{D-3}\right|_{D=5}$ (Myers Perry)

$$= \frac{3\pi r_5^2}{8G_5}$$

Entropy $S_{BH} = \frac{2\pi^2 r_5^3}{4G_5} = \frac{\pi^2}{2G_5}\left(\frac{8G_5 M}{3\pi}\right)^{3/2}$

$V_e(r)$.

Compare at fixed mass

$$\frac{S_{BH}}{S_{BS}} = \frac{\pi^2}{2G_5}\left(\frac{8G_5 M}{3\pi}\right)^{3/2} \Big/ 4\pi G_4 M^2$$

$$= \sqrt{\frac{3}{27\pi}}\,\sqrt{\frac{L}{G_4 M}}$$

Indicates for

the black hole

preferred.

Indicates for $L > \frac{27\pi}{8} G_4 M$

the black hole is entropically preferred.

Look at s-wave ptn of string

$$h_{ab} = \left[\begin{array}{cc|cc|c} h_{tt}(r) & h_{tr}(r) & 0 & & h_{tz} \\ h_{tr} & h_{rr}(r) & & & h_{rz} \\ \hline & & K & 0 & \\ 0 & & 0 & K\sin^2\theta & 0 \\ \hline h_{tz} & h_{rz} & 0 & & h_{zz} \end{array}\right] e^{\Omega t} e^{i\mu z}$$

- h_{zz}

$e^{i\mu z}$

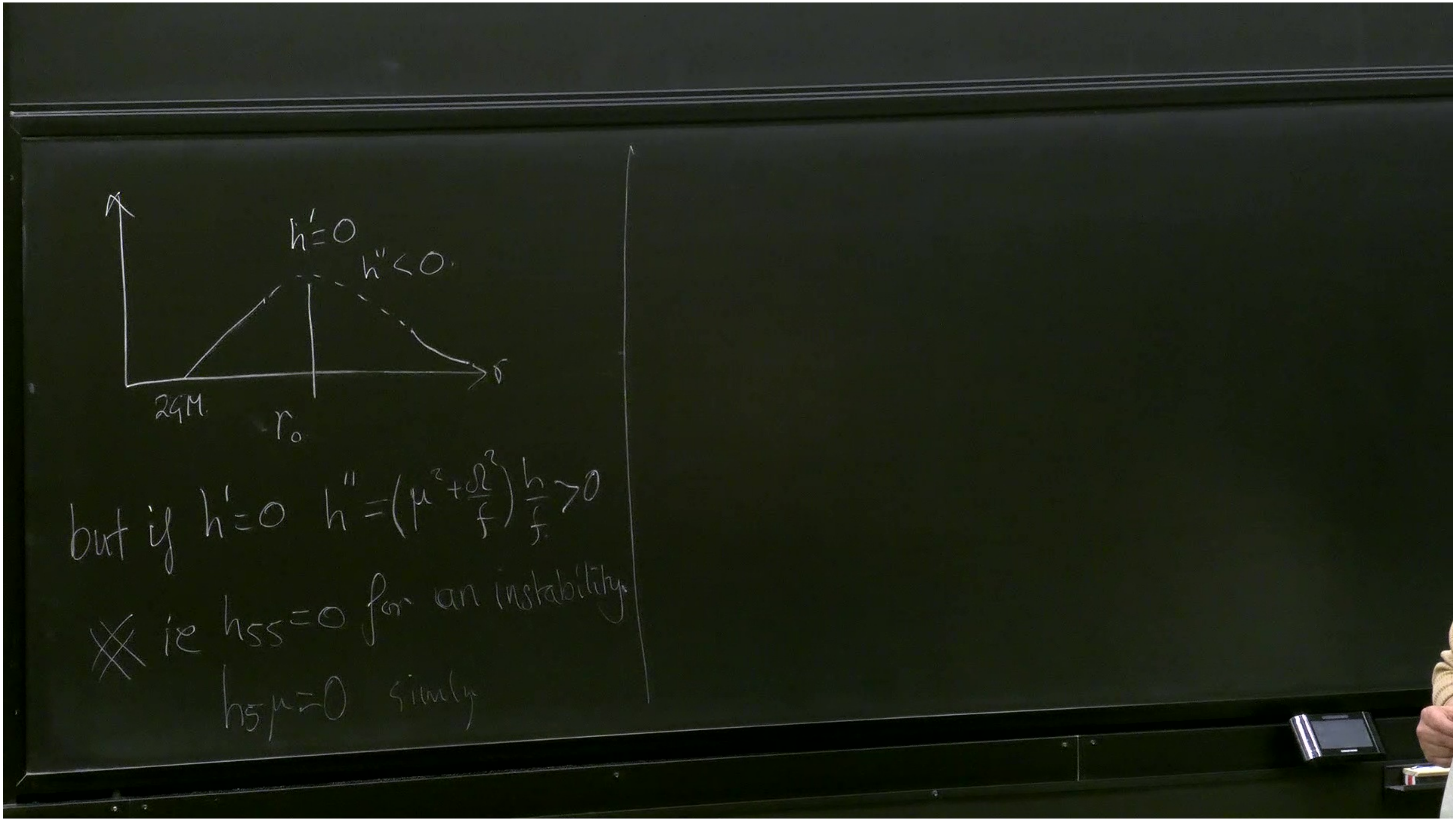
• h_{ss} $\quad \Delta_L h_{ss} = (\Box_4 + \mu^2)\, h_{zz}$,

$$= -f h''_{ss} - \frac{2}{r^2}(r - GM)\, h_{ss} + \left(\mu^2 + \frac{\Omega^2}{f}\right) h_{ss}$$

$r \to \infty \quad h_{ss} \propto e^{\pm\sqrt{\mu^2+\Omega^2}\, r}$

$r \to 2GM \quad h_{ss} \propto (r - 2GM)^{\pm 2GM\Omega}$

Normalizable $\Rightarrow$ $-$ root at ∞ & $+$ root at $2GM$


h' = 0
h'' < 0
r
2GM
r_0
but if h' = 0 h'' = (μ² + Ω²/f) h/f > 0
ie h55 = 0 for an instability
h5μ = 0 simily

Indicates for $L > \frac{27\pi}{8} G_4 M$

the black hole is entropically preferred.

Look at s-wave ptn of string

$$h_{ab} = \begin{bmatrix} h_{tt}(r) & h_{tr}(r) & 0 & h_{tz} \\ h_{tr} & h_{rr}(r) & & h_{rz} \\ 0 & & \begin{matrix} K & 0 \\ 0 & K\sin^2\theta \end{matrix} & 0 \\ h_{tz} & h_{rz} & 0 & h_{ss} \end{bmatrix} e^{\Omega t} e^{i\mu z}$$

• h_{ss} $\quad \Delta_L h_{ss} = (\Box_4 + \mu^2) h_{zz}$

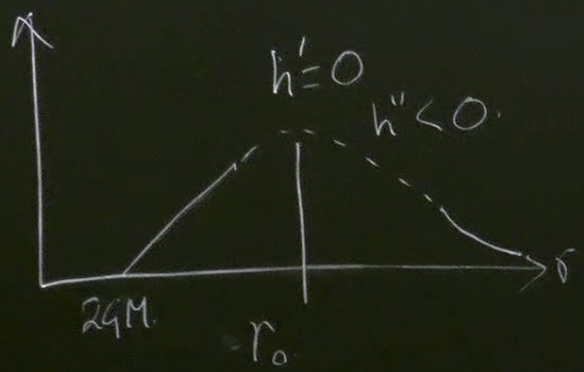
$$= -f h''_{ss} - \frac{2}{r^2}(r - 4M)$$

$+ ($

$r \to \infty \quad h_{ss} \propto e^{\pm\sqrt{\mu^2+\Omega^2}\, r}$

$r \to 2GM \quad h_{ss} \propto (r - 2GM)^{\pm 2GM\Omega}$

Normalizable $\Rightarrow$ $-$ root at ∞ & $+$ root



but if $h'=0$ $\quad h''=\left(\mu^2+\frac{\Omega^2}{f}\right)\frac{h}{f}>0$

※ ie $h_{55}=0$ for an instability

$h_{5\mu}=0$ similarly

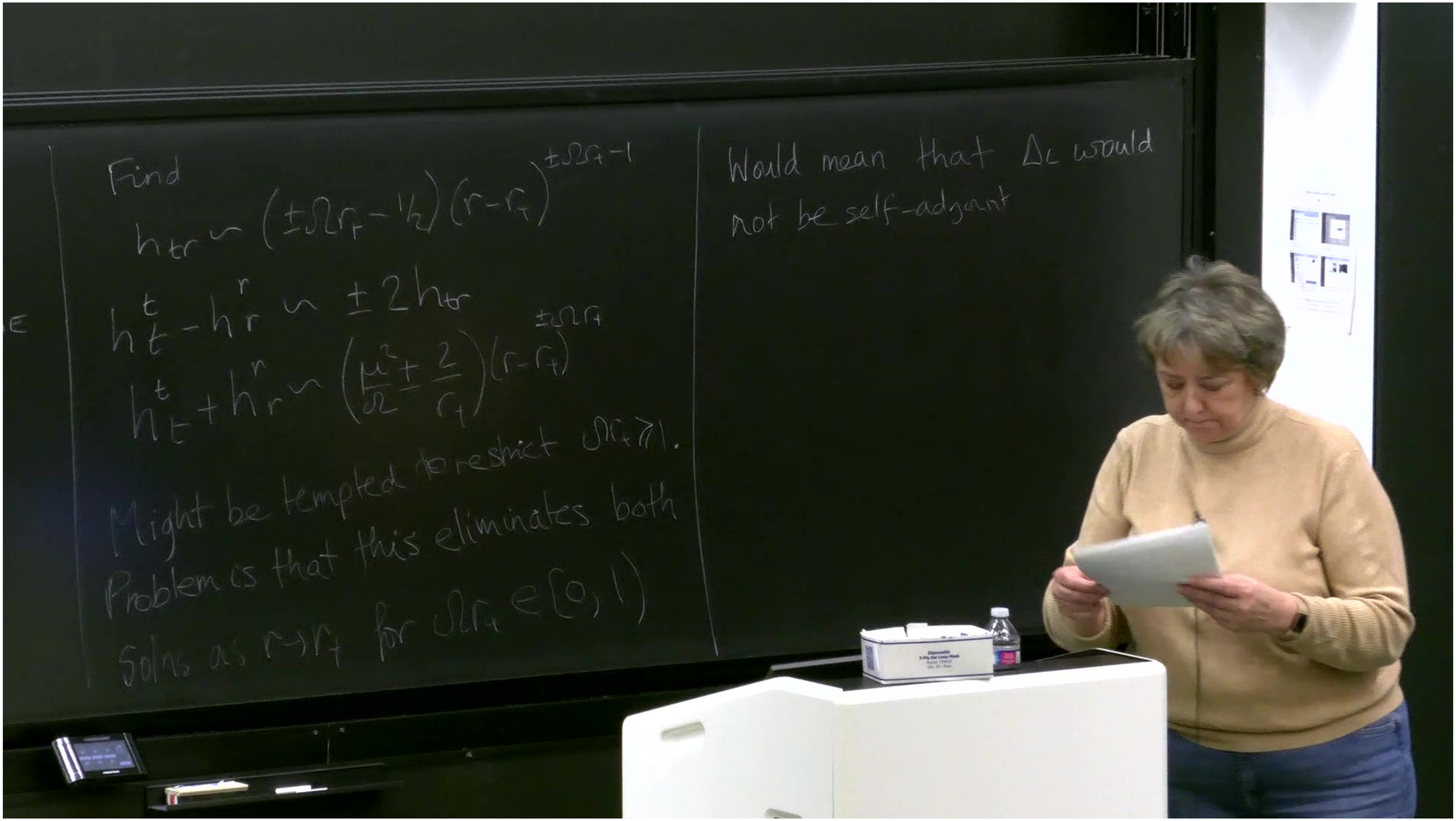
Left with $K, h_{tr}, h_{rr}, h_{tt}$

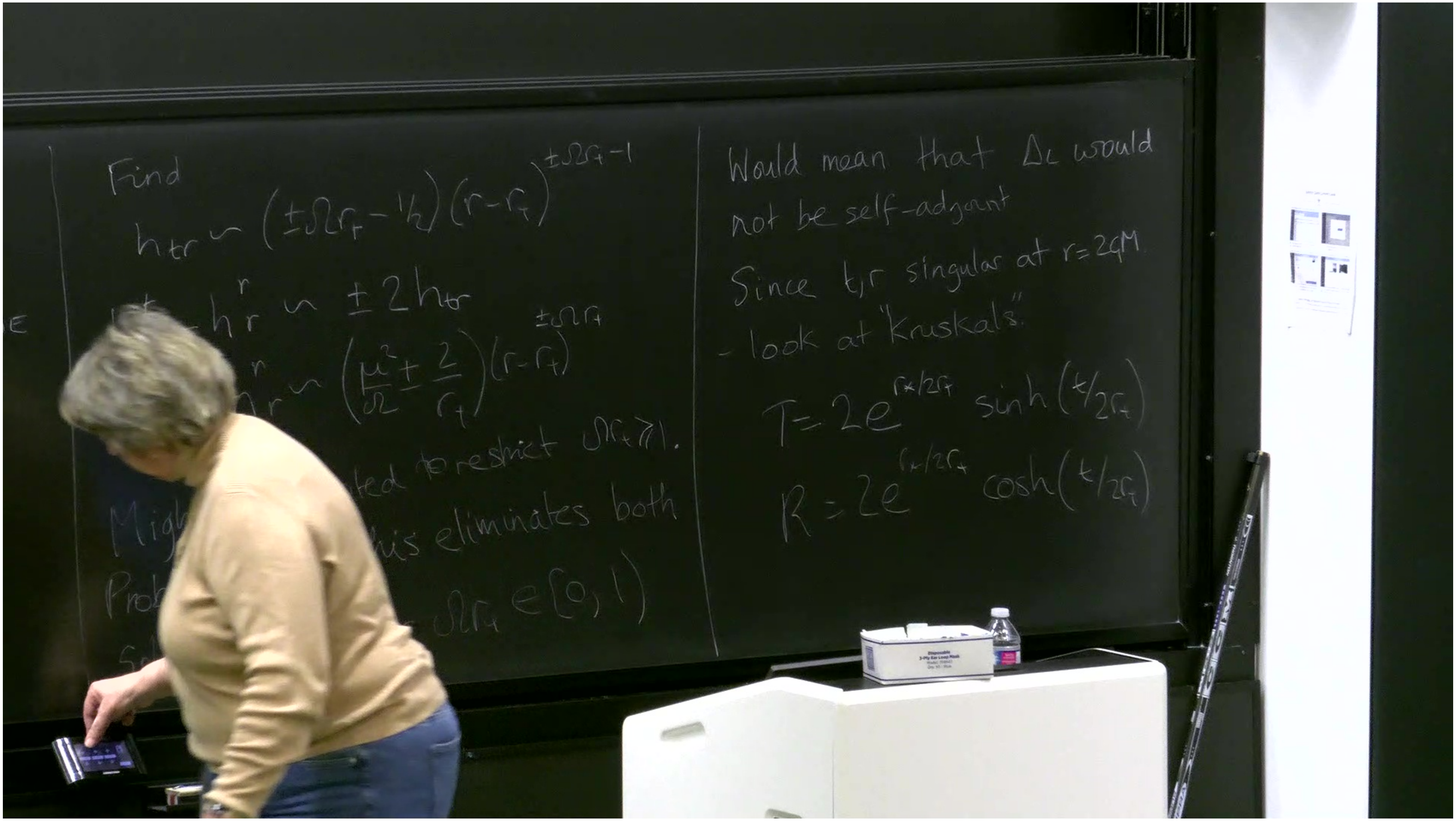
have $h^a{}_{b;a}=0=h^a{}_a$

Can rewrite as a 2^{nd} order ODE in 1 variable. As expected

$$h \propto e^{\pm\sqrt{\Omega^2+\mu^2}\,r}$$

As $r\to r_+=2GM$.


Find
$h_{tr} \sim (\pm\Omega r_t - 1/2)(r-r_t)^{\pm\Omega r_t - 1}$
$h^t_t - h^r_r \sim \pm 2 h_{tr}$
$h^t_t + h^r_r \sim \left(\frac{\mu^2}{\Omega} \pm \frac{2}{r_t}\right)(r-r_t)^{\pm\Omega r_t}$
Might be tempted to restrict $\Omega r_t \geq 1$.
Problem is that this eliminates both
solns as $r \to r_t$ for $\Omega r_t \in [0,1)$
Would mean that Δ_L would
not be self-adjoint


Find
h_tr ~ (±Ωr+ − 1/2)(r − r+)^(±Ωr+ − 1)
± 2h_tr
to restrict Ωr+ ≫ 1.
this eliminates both
Ωr+ ∈ [0, 1)
Would mean that Δ_L would
not be self-adjoint
Since t, r singular at r = 2GM
- look at "Kruskals"
T = 2e^(r*/2r+) sinh(t/2r+)
R = 2e^(r*/2r+) cosh(t/2r+)

Use: $\dfrac{\partial t}{\partial T} = \dfrac{2 r_+ R}{R^2 - T^2} \qquad \dfrac{\partial r}{\partial T} = -\dfrac{2 r_+ T}{R^2 - T^2} f$

e.g. $$h_{TT} = \left(\frac{\partial t}{\partial T}\right)^2 h_{tt} + 2 \frac{\partial t}{\partial T}\frac{\partial r}{\partial T} h_{tr} + \left(\frac{\partial r}{\partial T}\right)^2 h_{rr}$$

$$= \frac{4 r_+^2}{(R^2 - T^2)^2}\left(R^2 h_{tt} - 2 f R T h_{tr} + f^2 T^2 h_{rr}\right)$$

But $R^2 - T^2 = 4 e^{r_*/r_+} = 4 e^{(r - r_+)/r_+} \dfrac{(r - r_+)}{r_+} \sim 4 f$

f

$\left(\frac{\partial r}{\partial T}\right)^2 h_{rr}$

$f^2 T^2 h_{rr})$

$(r_+) \sim 4f$

$$\Rightarrow h_{TT} = \frac{r_+^2}{(R^2-T^2)}\left[R^2 h^t_t - 2RT h_{tr} - T^2 h^r_r\right]$$

$$= \frac{r_+^2}{R^2-T^2} h_{tr} \Big[\underbrace{\pm (R^2+T^2) - 2RT}_{\pm(R\mp T)^2}\Big]$$

$$\text{Since } R - T \approx 2 e^{r_*/2r_+} e^{-t/2r_+} \propto U \to 0 \text{ at } r_+.$$

$\neq$ root

h_{TT} is

$\ldots T h_{tr} - T^2 h^r_r]$

$\ldots T^2) - 2RT]$

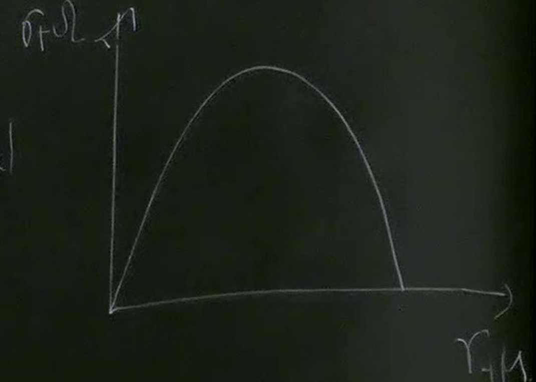
$(\mp T)^2$

$e^{-t/2r_+} \;\; \alpha\, U \to 0$ at r_+.

+ root

h_{TT} is regular for + root.

Finding solns becomes a numerical e-val problem



But $R^2 - T^2 = 4e^{r_*/r_+} = 4e^{(r-r_+)/r_+}\left(\frac{r-r_+}{r_+}\right) \sim 4f$

Since $R - T = 2e$

A complex frequency $\omega = \omega_R + i\omega_I$

tone/pitch

$\omega_I < 0$

decaying amplitude

Normal modes eg. violin string

modes labelled by n

$\delta \ddot{x}'' + \omega^2 \delta x = \ldots$

In reality, string loses energy. — damped h.o

For a b.h.

at r_+.

(QNM)

For a b.h. energy can only "fall" into the b.h. & the notion of the b.h. settling to vacuum soln means energy flows out at ∞. Expect complex frequency solns.

eg Scalar field

$$Q'' + \omega^2 Q \sim V_{eff}\, Q$$

with $Q \sim e^{-i\omega(t-r)}$ $\quad r\to\infty \quad$ outgoing

$\qquad\quad e^{-i\omega(t+r^*)}$ $\quad r\to r_+ \quad$ ingoing

$\frac{d}{dr_*^2}(r) \quad \underbrace{\qquad r^2}_{V_\ell(r).}$

Separating variables for a scalar gives " $'$ " $= \frac{d}{dr_*}$

$$V_\ell = f\left[\frac{\ell(\ell+1)}{r^2} + \frac{2r_+}{r^3}\right]$$

$$\varphi \sim \begin{cases} e^{-i\omega r_*} & \text{at } r_+ \\ e^{i\omega r_*} & \text{at } \infty \end{cases}$$

Although numerical, illustrate eval concept with toy model

$$V_{PT} = \frac{V_0}{\cosh^2\alpha(r_* - r_{*0})} \qquad \text{Poschl-Teller}$$

where $V_\varphi'(r_{*0}) = 0$

$$V_0 = V_\ell(r_{*0}) \quad \alpha^2 = -\frac{V_\varphi''(r_{*0})}{2V_0}$$

strate eval

Poschl-
Teller

$\alpha^2 = -\frac{V_\varphi''(r_{*0})}{2V_0}$

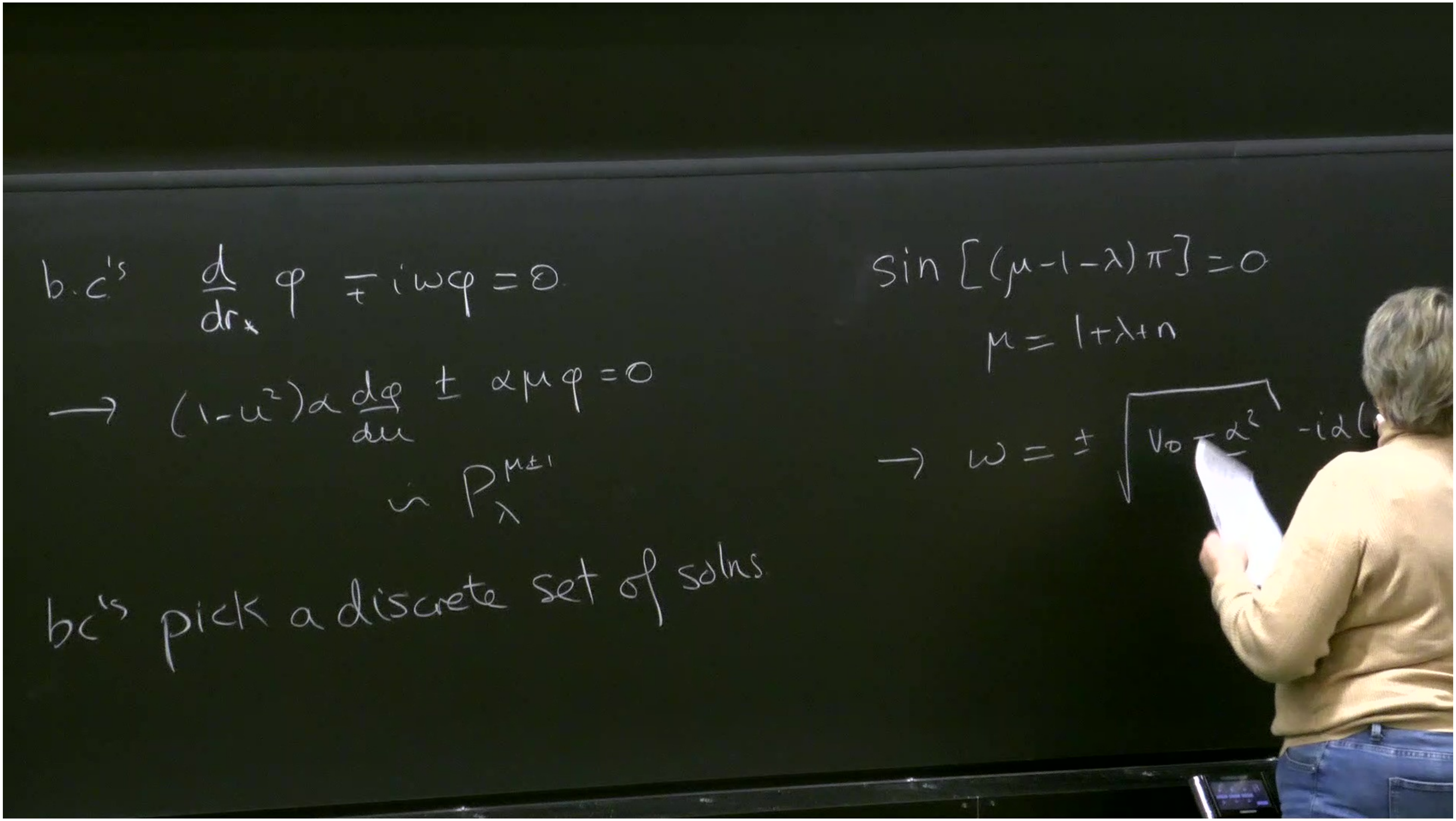
Define $u = \tanh\alpha(r_* - r_{*0})$

$$\lambda = -\frac{1}{2} \pm \frac{1}{2}\sqrt{1 - \frac{4V_0^2}{\alpha^2}}$$

$$\mu = i\omega/\alpha$$

then $(1-u^2)\varphi_{uu} - 2u\varphi_u + \lambda(\lambda+1)\varphi - \frac{\mu^2}{1-u^2}\varphi = 0$

— Legendre eqn


b.c's $\frac{d}{dr_*}\varphi \mp i\omega\varphi = 0$
$\longrightarrow (1-u^2)\alpha\frac{d\varphi}{du} \pm \alpha\mu\varphi = 0$
$\sim P_\lambda^{\mu\pm1}$
bc's pick a discrete set of solns
$\sin[(\mu-1-\lambda)\pi] = 0$
$\mu = 1+\lambda+n$
$\rightarrow \omega = \pm\sqrt{V_0 - \alpha^2} - i\alpha($

$$\sin\left[(\mu - 1 - \lambda)\pi\right] = 0$$

$$\mu = 1 + \lambda + n$$

$$\rightarrow \quad \omega = \pm \sqrt{V_0 - \frac{\alpha^2}{4}} - i\alpha\left(n + \tfrac{1}{2}\right)$$

tower

quasi-normal as dissipate energy.

$$\sin[(\mu-1-\lambda)\pi]=0$$

$$\mu = 1+\lambda+n$$

$$\to \quad \omega = \pm\sqrt{V_0-\frac{\alpha^2}{4}}\ - i\alpha(n+1/2)$$

tower

Quasi-normal as dissipate energy.

$\ell\to\infty$ (eikonal).

$V'=0$ at $3GM \leftrightarrow$ light ring

Ringdown frequency $\sqrt{V_0-\frac{\alpha^2}{4}} \sim \sqrt{\frac{4\ell^2}{27r_s^2}-\frac{8}{12r_s^2}}$

related to M.