Title: Lecture - Gravitational Physics, PHYS 636

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Subject: Cosmology, Strong Gravity

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In varying S_{EH}, get
non-standard Euler-Lagrange

$$\delta R_{ab} = \nabla_c \delta \Gamma^c_{ab} - \nabla_a \delta \Gamma^c_{bc}$$

$$= \frac{1}{2} \Box (\delta g^{-1})_{ab} + \frac{1}{2} \nabla_a \nabla_b (\delta g^{-1})^c{}_c$$

$$- \nabla_c \nabla_{(a} \delta g^{-1\,c}{}_{b)}$$

Note, am ta

of <u>inverse</u> metr

Note, am taking variation of <u>inverse</u> metric in action

$$(\delta g^{-1})_{ab} = g_{ac} g_{bd} \delta g^{cd}$$
$$= -\delta g_{ab}$$

So that

$$\delta S_{EH} = -\frac{1}{16\pi G} \int d^4x \sqrt{g} \left[R_{ab} \delta g^{ab} - \frac{1}{2} R g_{ab} \delta g^{ab} + \delta R_{ab} g^{ab} \right.$$

am taking variation

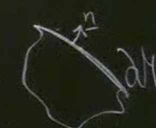
se metric in action

$$\ldots_{b} = g_{ac} g_{bd}\, \delta g^{cd}$$

$$= -\delta g_{ab}$$

Therefore the variation has the boundary term

$$\int_{\partial M} d^3x \sqrt{g}\, n_a \left[\nabla^a (\delta g^{-1})^c_c - \nabla_b \delta g^{ab} \right]$$



$$= -\frac{1}{16\pi G} \int d^4x \sqrt{g} \left[R_{ab}\delta g^{ab} - \frac{1}{2} R g_{ab} \delta g^{ab} + \underline{\delta R_{ab}}\, g^{ab} \right]$$

$$\left(\nabla_a V^a = \frac{1}{\sqrt{g}} \partial_a \sqrt{g} V^a \right)$$

Since bdry term has normal derivs, cannot (strictly) set to zero

The boundary

Look for a geometric boundary term that accounts for this extra piece in δS_{EH}.

al

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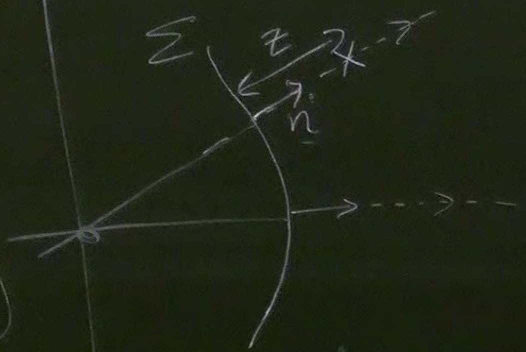
dary

xtra

Take Gaussian Normal gauge



codim 1

z = proper distance from Σ.

Valid within radius of curvature of Σ.

$$\underline{n} = \frac{\partial}{\partial z}$$

$$n^z = 1 = -n_z$$

$$ds^2 = h_{\mu\nu} dx^\mu dx^\nu - dz^2$$

$h_{\mu\nu}(0)$ 1st f.f. of Σ

Therefore $\delta g^{ab} = \delta h^{ab}$, $\delta n^a = 0$.

$$\delta g_{az} = 0.$$

Look at $K_{ab} = \nabla_a n_b = \Gamma^z_{ab} = g_{ab,z}$

$$\delta K_{ab} = \delta \Gamma^z_{ab} = -\frac{1}{2} n^c \left(\nabla_a \delta g_{bc} + \nabla_b \delta g_{ac} - \nabla_c \delta g_{ab} \right)$$

(note δg here) $\quad = \frac{1}{2} \delta g_{bc} \nabla_a n^c \; \ldots\ldots$ etc piece in δS_{EH}.

$$\delta K_{ab} = K_{(a}{}^{c}\, \delta g_{b)c} + \frac{1}{2} \nabla_n \delta g_{ab}.$$

$$= -K_{c(a}\, \delta g^{-1\,c}{}_{b)} - \frac{1}{2} \nabla_n (\delta g^{-1})_{ab}.$$

$$\Rightarrow h^{ab} \delta K_{ab} = -K_{ac} (\delta g^{-1})^{ac} - \underline{\frac{1}{2} \nabla_n (\delta g^{-1})^{c}{}_{c}}$$

Also, $\quad n_a \nabla_b \delta g^{-1\,ab} = \nabla_b (\cancel{n_a \delta g^{-1\,ab}}) - K_{ab} \delta g^{-1\,ab}.$

$\nabla_n \delta g_{ab}.$

$\frac{1}{2}\nabla_n(\delta g^{-1})_{ab}.$

$\frac{1}{2}\nabla_n(\delta g^{-1})^c_c$

$g^{-1ab}) - K_{ab}\delta g^{-1\,ab}.$

$$h^{ab}\delta K_{ab} + K_{ab}\delta h^{ab}$$

$$= \delta(h^{ab}K_{ab}) = -\frac{1}{2}\nabla_n(\delta g^{-1})^c_c$$

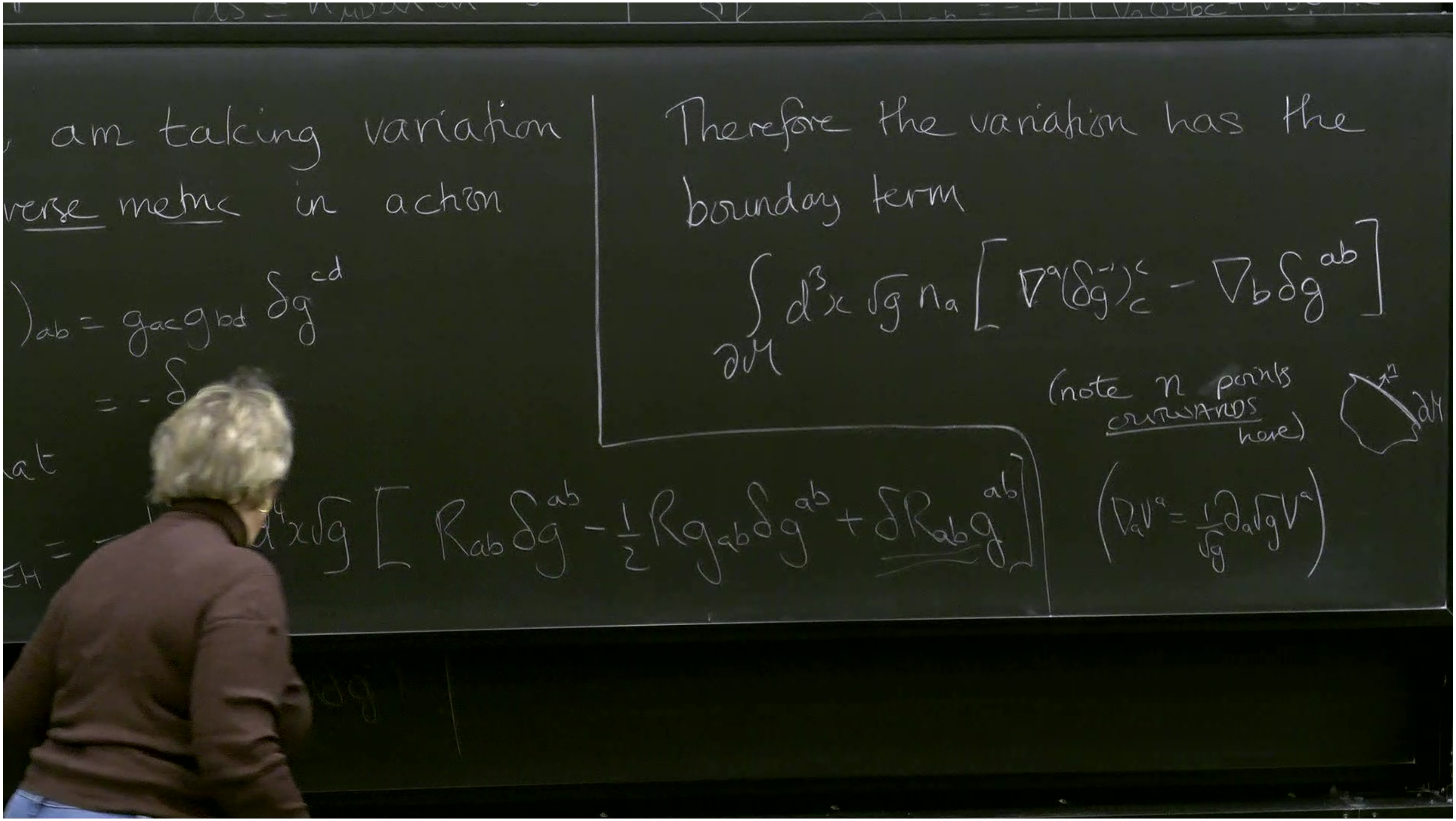
$$\Rightarrow \delta(K\sqrt{h}) = -\frac{1}{2}\nabla_n(\delta g^{-1})^c_c\sqrt{h}$$

$$-\frac{1}{2}K\,h_{ab}(\delta g^{-1})^{ab}.$$

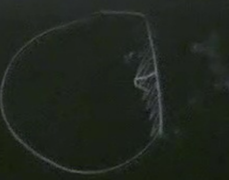
So unwanted boundary term can be expressed as

$$\nabla_n \delta g^{-1} - n_a \nabla_b \delta g^{-1\,ab} = -2\frac{\delta(K\sqrt{h})}{\sqrt{h}} - K h_{ab}\delta h^{ab}$$

$$+ K_{ab}\delta h^{ab}$$


am taking variation
verse metric in action
$)_{ab} = g_{ac} g_{bd} \delta g^{cd}$
$= -\delta$
at
$_{EH} = \int d^4x \sqrt{g} \left[R_{ab} \delta g^{ab} - \frac{1}{2} R g_{ab} \delta g^{ab} + \delta R_{ab} g^{ab} \right]$
Therefore the variation has the
boundary term
$\int_{\partial M} d^3x \sqrt{g}\, n_a \left[\nabla^a (\delta g^{-1})^c_{\ c} - \nabla_b \delta g^{ab} \right]$
(note n points OUTWARDS here)
n
∂M
$\left(\nabla_a V^a = \frac{1}{\sqrt{g}} \partial_a \sqrt{g} V^a \right)$

Thus if we add a boundary term to S_{EH}



$$-\frac{1}{16\pi G}\int d^4x\sqrt{g}\,R + \frac{1}{8\pi G}\int d^3x\sqrt{h}\,K$$

Gibbons-Hawking boundary term

conventionally normal points <u>INWARDS</u>

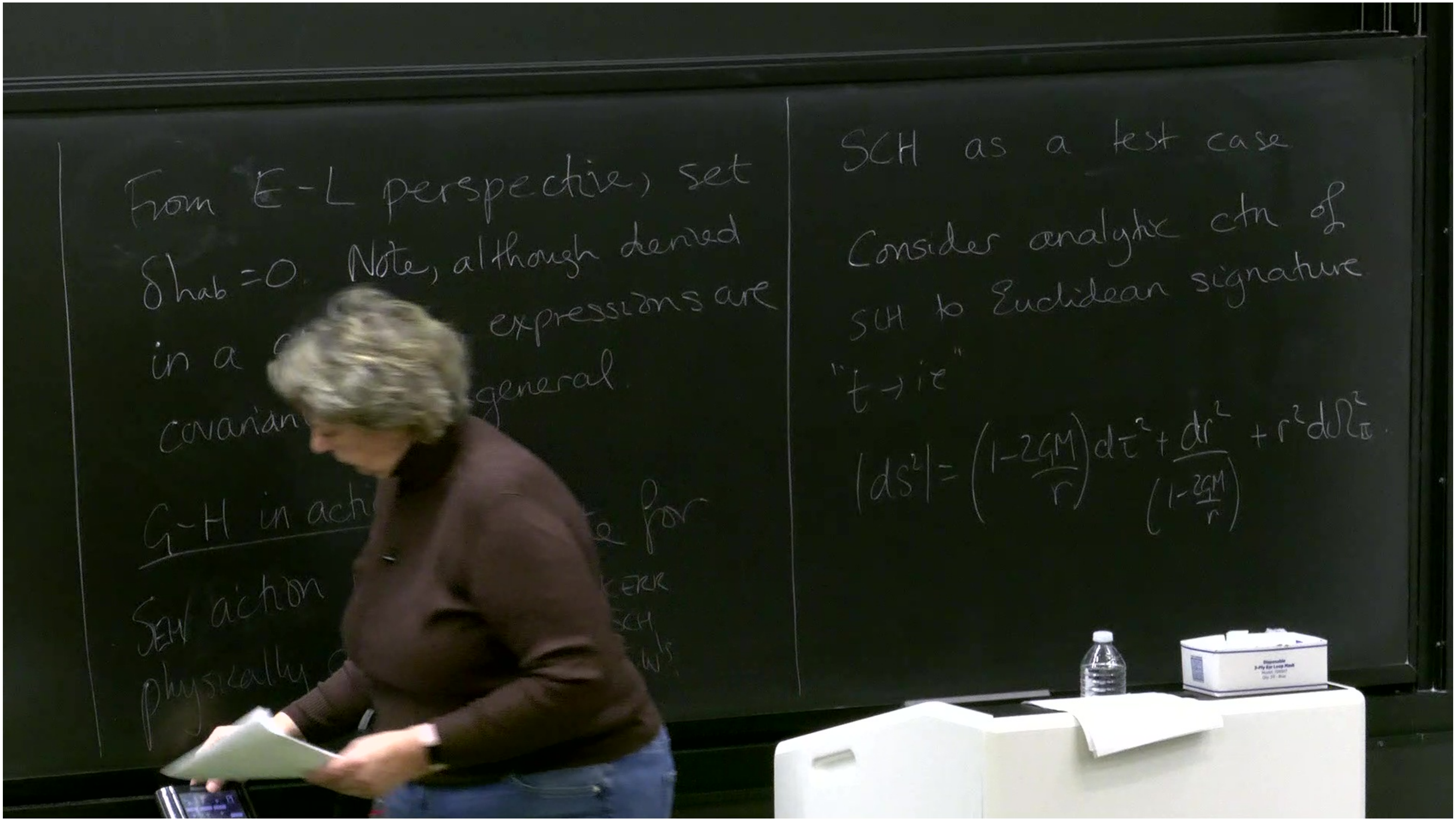
our variation becomes

$$-\frac{1}{16\pi G}\int d^4x\sqrt{g}\,G_{ab}\,\delta g^{ab} + \frac{1}{8\pi G}\int d^3x\sqrt{h}\,\delta h^{ab}\left(K_{ab}-K h_{ab}\right)$$

From E-L perspective, set
$\delta h_{ab} = 0$. Note, although derived
in a GN gauge, expressions are
covariant, hence general.

G-H in action

SEH action was degenerate for
physically distinct solns: KERR
SCH
GW's


From E-L perspective, set
$\delta h_{ab} = 0$. Note, although derived
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general.
G-H in acti
for
SEH action
KERR
SCH
physically
SCH as a test case
Consider analytic ctn of
SCH to Euclidean signature
"$t \to i\tau$"
$|ds^2| = \left(1 - \frac{2GM}{r}\right) d\tau^2 + \frac{dr^2}{\left(1 - \frac{2GM}{r}\right)} + r^2 d\Omega_{II}^2$.

$r \to 2GM$

$$\frac{dr^2}{\left(1-\frac{2GM}{r}\right)} \simeq \frac{2GM\,dr^2}{(r-2GM)} \propto d\sqrt{r-2GM}^{\,2}$$

Try $\rho^2 = \lambda(r-2GM)$

$\Rightarrow 2\rho\,d\rho = \lambda\,dr$

$4\rho^2 d\rho^2 = 4\lambda(r-2GM)d\rho^2 = \lambda^2 dr^2$

$$\frac{2GM\,dr^2}{(r-2GM)} = \underbrace{2GM \cdot \frac{4}{\lambda}}\, d\rho^2$$

$\lambda = 8GM$

ρ = proper distance from $2GM$

$$\left(1-\frac{2GM}{r}\right)d\tau^2 = \frac{\rho^2}{8GM}\frac{d\tau^2}{2GM}$$

$$= \rho^2 d\left(\frac{\tau}{4GM}\right)^2$$

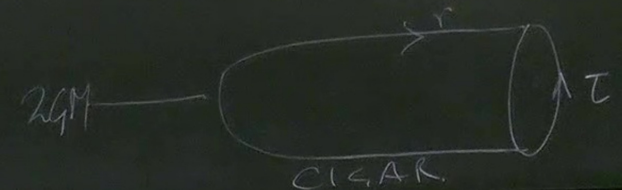
physically distinct sol^ns: KERR
SCH
GW's

Thus if $\tau \sim 4GM\theta$
is identified with
periodicity
$$\Delta\tau = \beta = 8\pi GM$$

$$ds^2_{r\tau} \to \underbrace{d\rho^2 + \rho^2 d\theta^2}_{\text{origin of polars.}}$$

i.e. $r = 2GM$ is regular &
τ is periodic.

SCH gives thermal interpretation
at temp
$$T = 1/\beta = \frac{1}{8\pi GM}$$



$$\Rightarrow \quad 2\rho\, d\rho = \ldots$$

$$4\rho^2 d\rho^2 = 4\lambda(r-2GM)\,d\rho^2 = X^2 dr^2$$

$$= \rho^2 d\left(\frac{\tau}{4GM}\right)^2$$

Compute action:

Set boundary at large $r = R_0$.

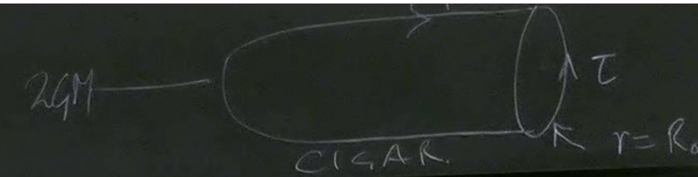
∂M is an $S^2 \times S^1$ at $r = R_0$.

$$ds^2_{\partial M} = \left(1 - \frac{2GM}{R_0}\right) d\tau^2 + R_0^2\, d\Omega^2_{II}.$$

with inward pointing normal $\underline{n} = -\sqrt{1 - \frac{2GM}{R_0}}\,\frac{\partial}{\partial r}$

$$= \rho^2 d\left(\frac{\tau}{4GM}\right)^2$$

origin of polars.



$$K = \nabla_a n^a = \frac{1}{r^2}\partial_a r^2 n^a \Big|_{r=R_0}$$

$$= \left[\frac{2}{r} + \frac{GM}{r(r-2GM)}\right] n^r \Big|_{r=R_0}$$

$$\sqrt{h} = R_0^2 \sin\theta \sqrt{1-\frac{2GM}{R_0}}$$

$$K\sqrt{h} = -R_0^2 \sin\theta \left[\frac{2}{R_0}\left(1-\frac{2GM}{R_0}\right) + \frac{GM}{R_0^2}\right]$$

$$= -\sin\theta\,[2R_0 - 3GM]$$

– now we have a problem

$$\Rightarrow \int K_0 \sqrt{h_0}\, d^3x = -4\pi \beta_0 \cdot 2R_0$$

$\uparrow$ periodicity of τ_0

also divgt!

To subtract, must match ∂M's

$$\tau_0 = \sqrt{1 - \frac{2GM}{R_0}}\, \tau$$

$$\Rightarrow \beta_0 = \sqrt{1 - \frac{2GM}{R_0}}\, \beta = \left(1 - \frac{GM}{R_0} + \cdots\right)\beta$$

$$\ldots = -n_z$$

$$\ldots \sqrt{h_0}\, d^3x = -4\pi \beta_0 \cdot 2R_0$$

$\uparrow$ periodicity of τ_0

...divgt!

...ract, must match ∂M's

$$\ldots = \sqrt{1-\frac{2GM}{R_0}}\,\tau$$

$$\beta_0 = \sqrt{1-\frac{2GM}{R_0}}\,\beta = \left(1-\frac{GM}{R_0}+\cdots\right)\beta$$

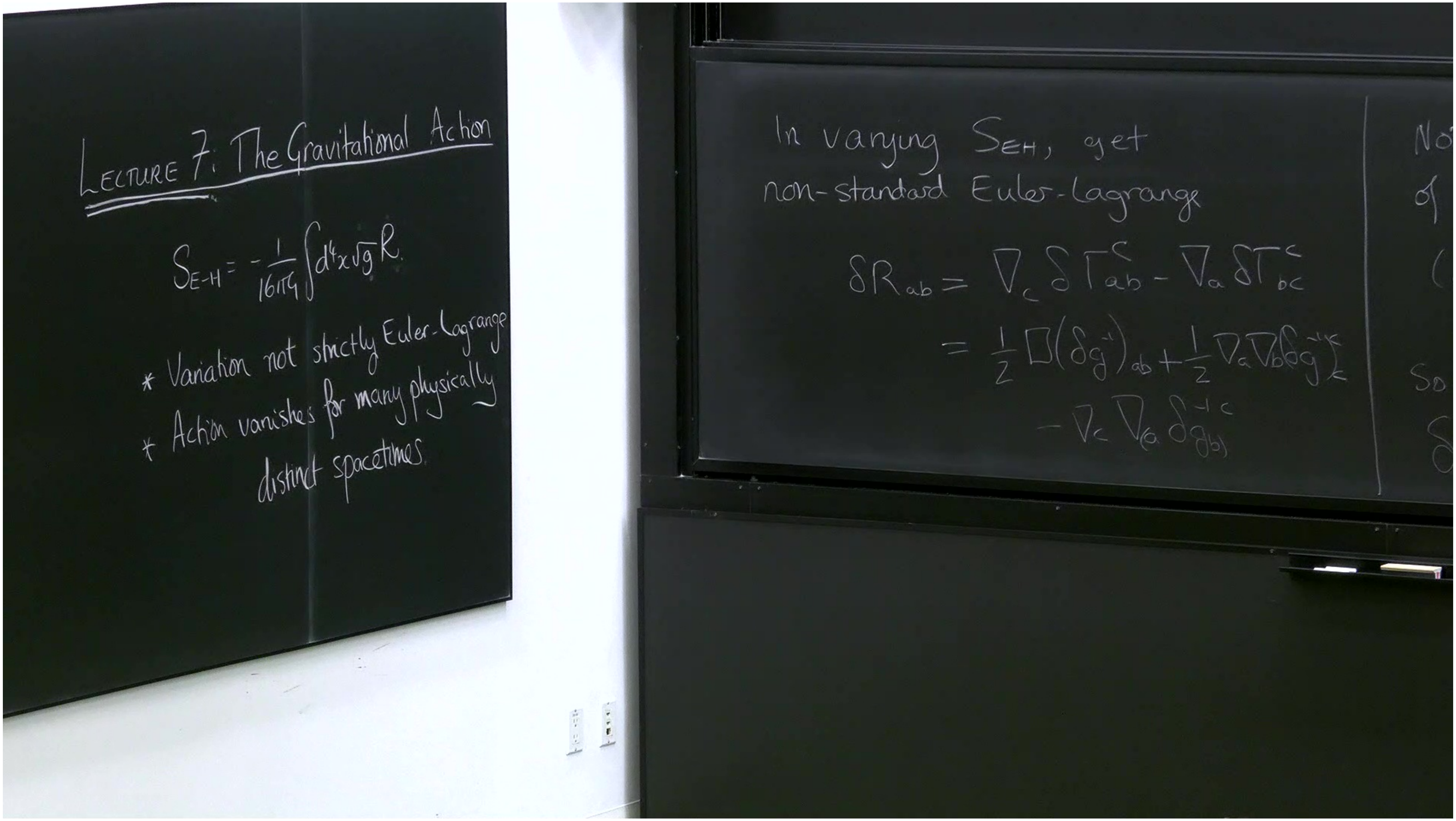
Hence

$$\int K\sqrt{h}\Big|_0 = -4\pi\beta_0 \cdot 2R_0$$

$$= -8\pi R_0 \beta\left(1-\frac{GM}{R_0}\right)$$

$$\Rightarrow S_{GH,SCH} - S_{GH,0} = -\frac{4\pi\beta}{8\pi G}\left[2R_0 - 3GM\right] + \frac{8\pi\beta}{8\pi G}\left[R_0 - GM\right]$$

$$= \frac{\beta}{2G}\cdot GM = \beta M/2$$



LECTURE 7: The Gravitational Action
$S_{E\text{-}H} = -\frac{1}{16\pi G}\int d^4x\sqrt{g}\,R$
* Variation not strictly Euler-Lagrange
* Action vanishes for many physically distinct spacetimes
In varying S_{EH}, get
non-standard Euler-Lagrange
$\delta R_{ab} = \nabla_c \delta\Gamma^c_{ab} - \nabla_a \delta\Gamma^c_{bc}$
$= \frac{1}{2}\Box(\delta g^{-1})_{ab} + \frac{1}{2}\nabla_a\nabla_b(\delta g^{-1})^c_c$
$- \nabla_c \nabla_{(a}\delta g^{-1\,c}_{b)}$