Title: Lecture - Gravitational Physics, PHYS 636

Speakers: Ruth Gregory

Collection/Series: Gravitational Physics (Elective), PHYS 636, January 6 - February 5, 2025

Subject: Cosmology, Strong Gravity

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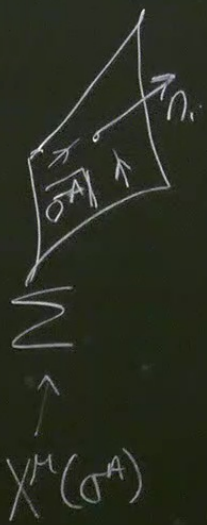
Gauss-Codazzi: describes properties of lower dim^{l} submanifolds

The curvature & connection "split" into parallel & perpendicular parts.

Let $\Sigma \subset M$ be a submanifold of dim d, $\dim(M) = D$

...folds
...arts.
... of



Defn: The co-dimension of Σ in M is $n = D - d$

There exist n lin. indep. vectors $\underline{n}_i \in T_p(M)$ s.t.

$\underline{n}_i(\sigma^A) = 0 \quad \forall$ coord fns σ^A on Σ.

$g_{\mu\nu} n_i^\nu$

eg S^2

...mension

= D-d

indep.

s.t.

...rd fns

on Σ.

$$g_{\mu\nu}\, n_i^\nu \frac{\partial X^\mu}{\partial \sigma^A} = 0$$

eg S^2 : $x^2 + y^2 + z^2 = 1$ $\quad \sigma^A = \theta, \varphi$

$$X^\mu(\sigma^A) = \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix}$$

$$X^\mu_{,\theta} = \begin{pmatrix} \cos\theta\cos\varphi \\ \cos\theta\sin\varphi \\ -\sin\theta \end{pmatrix} \qquad X^\mu_{,\varphi} = \begin{pmatrix} -\sin\theta\sin\varphi \\ \sin\theta\cos\varphi \\ 0 \end{pmatrix}$$

$\dim d, \quad \dim(\mathcal{M}) = D$

$\underline{n}_i(\sigma) = 0$

$$\underline{n} = \frac{\partial}{\partial r} = \frac{x^i}{r}\frac{\partial}{\partial x^i} \longleftrightarrow \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix}$$

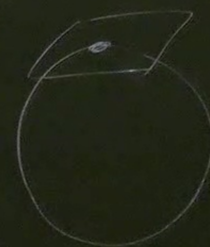
Clearly $\underline{n} \cdot \underline{X}_{,A} = 0$

$\underline{n}$ does not need to be normalised,

but for convenience take $n_{i\mu} n_j{}^{\mu} = \epsilon_i \delta_{ij}$

$+1$ timelike
-1 spacelike

$$\underline{n} = \frac{\partial}{\partial r} = \frac{x^i}{r}\frac{\partial}{\partial x^i} \longleftrightarrow \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix}$$



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+1 timelike

−1 spacelike

Def

h

— th

Not

Defn: The 1st fundamental form or induced metric on Σ is

$$h_{ab} = g_{ab} + n_{ia} n_{ib} \quad \left(\begin{array}{c}\text{take } \underline{n} \\ \text{spacelike}\end{array}\right)$$

– the metric Σ inherits from M

Note h lies in the cotangent bundle of M

$= \epsilon_i \delta_{ij}$

$+1$ timelike

-1 spacelike

ntal form

on Σ is

(take $\underline{n}$ spacelike)

from $\mathcal{M}$

angent bundle of $\mathcal{M}$

Can also represent as an intrinsic metric

$$\gamma_{AB} = g_{\mu\nu} \frac{\partial X^\mu}{\partial \sigma^A} \frac{\partial X^\nu}{\partial \sigma^B}$$

γ_{AB} in cotangent bundle of Σ

but for convenience take $n_{i\mu} n_j^{\ \mu} = \epsilon\, \delta_{ij}$

ϵ: $+1$ timelike, -1 spacelike

Note h lies in th

eg. S^2: $h_{ij} = \delta_{ij} - n_i n_j$

$$= \begin{bmatrix} 1 - \frac{x^2}{r^2} & -\frac{xy}{r^2} & -\frac{xz}{r^2} \\ \cdot & 1 - y^2/r^2 & -\frac{yz}{r^2} \\ \cdot & - & 1 - z^2/r^2 \end{bmatrix}$$

(in cartesians)

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sin^2\theta \end{bmatrix}$$ sph polars

or $\gamma_{AB} = \frac{\partial x^i}{\partial \sigma^A} \frac{\partial x^j}{\partial \sigma^B} \delta_{ij}$

$$= \begin{bmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{bmatrix}$$

$(d\theta^2 + \sin^2\theta\, d\varphi^2)$

... lies in the cotangent bundle of M | γ_{AB} in cotangent bundle of Σ

$\frac{\partial X^0}{\partial \sigma^B} \delta_{ij}$

0

$\sin^2\theta \Big]$

<u>Defn</u>: The <u>2nd fundamental form</u> or <u>extrinsic curvature</u> of Σ measures how Σ curves in M.

$$K_{i\mu\nu} = h_\mu^{\ \sigma} h_\nu^{\ \lambda} \nabla_\sigma n_{i\lambda}$$

or $K_{iAB} = X^\mu_{\ ,A} X^\nu_{\ ,B} \nabla_\mu n_{i\nu} = -n_{i\mu} D_A\left(\frac{\partial X^\mu}{\partial \sigma^A}\right)$

where D_A is the connection on Σ.

$${}^{(d)}\nabla_\mu V_\nu = h_\mu{}^\lambda h_\nu{}^\sigma \, {}^{(D)}\nabla_\lambda V_\sigma$$

Finally, the <u>normal fundamental forms</u> measure how the normals "twist" around Σ.

$$\beta_{\mu i j} = n_{i\nu} \nabla_\mu n_j{}^\nu$$

– connection on normal bundle of Σ.

e.g. S^2 $\quad n^i = \frac{x^i}{r}$

$$\frac{1}{r} h_{ij}$$

$$n_{i,j} = \frac{\delta_{ij}}{r} - \frac{x_i x_j}{r^3}$$

$$K_{ij} = h_i^{\ k} h_j^{\ l} n_{k,l} = \frac{1}{r}(\delta_{ik} - n_i n_k)(\delta_{jl} - n_j n_l)(\delta_{kl} - n_k n_l)$$

$$= \frac{1}{r}(\delta_{ik} - n_i n_k)\left(\delta_{jk} - n_j n_k - \cancel{n_j n_k} + \cancel{n_j n_k}\right) = \frac{1}{r}(\delta_{ij} - n_i n_j)$$

$\frac{1}{r} h_{ij}$

$n_l)(\delta_{kl} - n_k n_l)$

$n_k) = \frac{1}{r}(\delta_{ij} - n_i n_j)$

The extrinsic curvature is related to the intrinsic curvature of Σ, but is distinct

$$^{(d)}R^{a}{}_{bcd} = {}^{(D)}R^{a'}{}_{b'c'd'}\, h^{a}{}_{a'}\, h^{b'}{}_{b}\, h^{c'}{}_{c}\, h^{d'}{}_{d}$$

$$+ \sum \epsilon_i \left[K_i{}^{a}{}_{c} K_{ibd} - K_i{}^{a}{}_{d} K_{ibc} \right]$$

For codim 1. Take n^a to be geodesically extended off Σ.

$$n^a \nabla_a n^b = 0 \quad \& \quad \nabla_a n_b = K_{ab}$$

$$^{(d)}R^a{}_{bcd} V^b = \left({}^{(d)}\nabla_c {}^{(d)}\nabla_d - c \leftrightarrow d \right) V^a$$

$$= h_c^{c'} h_d^{d'} h^a_{a'} \left(\nabla_{c'} {}^{(d)}\nabla_{d'} V^{a'} - c \leftrightarrow d \right)$$

$$= h_c^{c'} h_d^{d'} h^a_{a'} \left(\nabla_{c'} \left[h^e_{d'} h^d_f \nabla_e V^f \right] - c \leftrightarrow d \right)$$

h

$$h^{c'}_{c}\, h^{d'}_{d}\, h^{a}_{a'} \Big[h^{e}_{d'}\, h^{a'}_{f}\, \nabla_{c'} \nabla_{e} V^{f} + \nabla_{e} V^{f} \big((\nabla_{c'} h^{e}_{d'})\, h^{a'}_{f} + (\nabla_{c'} h^{a'}_{f})\, h^{e}_{d'} \big) \Big]$$

Recall $\nabla_{c'}(h^{e}_{d'}) = \nabla_{c'}(\cancel{\delta^{e}_{d'}} + n^{e} n_{d'})$

$$= n^{e} K_{c'd'} + n_{d'} K_{c'}{}^{e}$$

$$\Rightarrow \; = h^{c'}_{c}\, h^{d'}_{d}\, h^{a}_{a'} \Big[h^{e}_{d'}\, h^{a'}_{f}\, \nabla_{c'} \nabla_{e} V^{f} + h^{a'}_{f} K_{c'd'}\, n^{e} \nabla_{e} V^{f} + \underbrace{K_{c'}{}^{a'}\, n_{f}\, h^{e}_{d'} \nabla_{e} V^{f}}_{-K_{c'}{}^{a'}\, h^{e}_{d'} V^{f} K_{ef}} \Big]$$

$$= h_c{}^{c'} h_d{}^{d'} h^a{}_{a'} \left(\nabla_{c'} \nabla_{d'} V^{a'} - c \leftrightarrow d \right) \quad \Rightarrow \; = h_c \, h_d \, h_{a'} \, [\; h \ldots$$

$$= h_c{}^{c'} h_d{}^{d'} h^a{}_{a'} \left(\nabla_{c'} \left[h^e{}_{d'} h^d{}_f \nabla_e V^f \right] - c \leftrightarrow d \right)$$

$$= h_c{}^{c'} h_d{}^{d'} h^a{}_{a'} \nabla_{c'} \nabla_{d'} V^{a'} + K^a{}_c K_{df} V^f - c \leftrightarrow d$$

$$= R^{a'}{}_{b'c'd'} V^{b'} h^a{}_{a'} h^{c'}{}_c h^{d'}{}_d - \left(K^a{}_c K_{db} - K^a{}_d K_{bc} \right) V^b$$

eg S^2 $\quad {}^{(D)}R^a{}_{bcd} = 0 \quad (\mathbb{R}^3)$

$$R^A{}_{BCD} = K^A{}_C K_{BD} - K^A{}_D K_{BC}$$

$$= \frac{1}{r^2} \left(\gamma^A{}_C \gamma_{BD} - \gamma^A{}_D \gamma_{BC} \right).$$

but for convenience take $n_{i\mu} n_j^{\mu} = \epsilon_i \delta_{ij}$

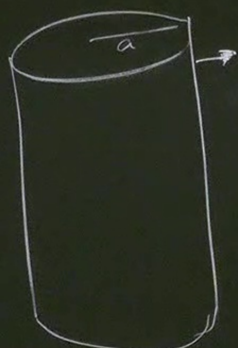
+1 timelike

−1 spacelike

– the metric Σ inherits

Note h lies in the coto

eg Cylinder



$$x^2+y^2=a^2 \subset \mathbb{R}^3.$$

$$\underline{n} = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}$$

$$h_{ij} = \begin{pmatrix} \sin^2\theta & -\sin\theta\cos\theta & 0 \\ -\sin\theta\cos\theta & \cos^2\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

or $\begin{pmatrix} 0 & 0 & 0 \\ 0 & a^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ in cyl polars, $\underbrace{\begin{pmatrix} a^2 & 0 \\ 0 & 1 \end{pmatrix}}_{\gamma_{AB}}$

in the cotangent bundle of M

γ_{AB} in cotangent bundle of Σ

$\begin{matrix} 0 & 0 \\ & 0 \\ & 1 \end{matrix}$

cyl polars

$K_{ij} = n_{i;j}$ in cartesians

$= \Gamma^r_{ij}$ in polars

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$K_{AB} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$$

$$R^A{}_{BCD} = 0.$$

Cylinder is Ricci flat.