Title: Lecture - Gravitational Physics, PHYS 636

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Lecture 4: Connections, Curvature, & Cartan

Reminder $\quad \underset{\sim}{\nabla}\, \underset{\sim}{e}_a = \Gamma^c{}_{ba}\, \underset{\sim}{e}_c\, \underset{\sim}{\omega}^b \qquad \left(\langle \underset{\sim}{\omega}^b | \underset{\sim}{e}_a \rangle = \delta^b_a\right)$

or $\quad \Gamma^a{}_{bc} = \langle \underset{\sim}{\omega}^a | \nabla_b\, \underset{\sim}{e}_c \rangle$

$\underset{\sim}{\nabla}$ is a derivation that (i) commutes with contractions (ii) Leibnitzian, & (iii) reduces to $\underset{\sim}{d}$ on fns.

$\underline{e}_a \rangle = \delta^b_a)$

...ctions

On a vector:

$$\underline{\nabla}\, \underline{V} = \underline{\nabla}(V^a \underline{e}_a) = (\underline{\nabla} V^a)\underline{e}_a + V^a \underline{\nabla}\underline{e}_a$$

$$= \underline{d}V^a \underline{e}_a + V^a \Gamma^b_{\ ca} \underline{e}_b \underline{\omega}^c$$

$$= (V^a_{\ ,c} + \Gamma^a_{\ cb} V^b)\, \underline{e}_a \underline{\omega}^c.$$

In GR take a metric torsion-free connection.

$$\nabla \underline{g} = 0.$$

$$\underline{T}(\underline{u},\underline{v}) = \nabla_{\underline{u}}\underline{v} - \nabla_{\underline{v}}\underline{u} - [\underline{u},\underline{v}] = 0$$

$$\to T^a_{\ bc} = \Gamma^a_{\ bc} - \Gamma^a_{\ cb} - C^a_{\ bc}$$

where $C^a_{\ bc} = \langle \underline{\omega}^a | [\underline{e}_b, \underline{e}_c] \rangle$

Defn The connection 1-forms.

or spin-connection are:

$$\underset{\sim}{\theta}^{a}{}_{c} = \Gamma^{a}_{(b)c}\, \underline{\omega}^{(b)}$$

so that

$$\underline{\nabla}\, \underline{e}_{b} = \underset{\sim}{\theta}^{c}{}_{b} \otimes \underset{\sim}{e}_{c}$$

The connection 1-forms.

... connection are:

$\underset{\sim}{\theta}^{a}{}_{c} = \Gamma_{(b)}{}^{a}{}_{c}\, \underline{\omega}^{(b)}$

labels

...that

$\nabla \underline{e}_b = \underset{\sim}{\theta}^{c}{}_{b} \otimes \underline{e}_c$

Note $\underline{d}\, g_{ab} = \underset{\sim}{\theta}_{ab} + \underset{\sim}{\theta}_{ba}$

Proof: $\underline{d}(g_{ab}) = \underline{\nabla}(g_{ab})$

$= \underline{\nabla}(\langle g \mid \underline{e}_a\, \underline{e}_b \rangle)$

$= \langle g \mid \nabla \underline{e}_a, \underline{e}_b \rangle + \langle g \mid \underline{e}_a\, \nabla \underline{e}_b \rangle$

$= \langle g \mid \underset{\sim}{\theta}^{c}{}_{a}\, \underline{e}_c, \underline{e}_b \rangle + \langle g \mid \underline{e}_a\, \underset{\sim}{\theta}^{c}{}_{b}\, \underline{e}_c \rangle$

ith contractions
n fns.

In GR take a metric torsion-free connection.

Defn The connection 1-forms or spin connection are:

1-form $\underset{\sim}{\theta}^{a}{}_{c} = \Gamma^{a}{}_{bc}\, \underset{\sim}{\omega}^{b}$

labels

So that

$$\underset{\sim}{\nabla} \underset{\sim}{e}_b = \underset{\sim}{\theta}^{c}{}_{b}\, \underset{\sim}{e}_c$$

Note $\underset{\sim}{d}\, g_{ab} = \underset{\sim}{\theta}_{ab} + \underset{\sim}{\theta}_{ba}$

Proof:
$$
\begin{aligned}
\underset{\sim}{d}(g_{ab}) &= \underset{\sim}{\nabla}(g_{ab}) \\
&= \underset{\sim}{\nabla}(\langle g \,|\, \underset{\sim}{e}_a\, \underset{\sim}{e}_b \rangle) \\
&= \langle g \,|\, \underset{\sim}{\nabla} \underset{\sim}{e}_a, \underset{\sim}{e}_b \rangle + \langle g \,|\, \underset{\sim}{e}_a\, \underset{\sim}{\nabla} \underset{\sim}{e}_b \rangle \\
&= \langle g \,|\, \underset{\sim}{\theta}^{c}{}_{a}\, \underset{\sim}{e}_c, \underset{\sim}{e}_b \rangle + \langle g \,|\, \underset{\sim}{e}_a\, \underset{\sim}{\theta}^{c}{}_{b}\, \underset{\sim}{e}_c \rangle \\
&= g_{cb}\, \underset{\sim}{\theta}^{c}{}_{a} + g_{ac}\, \underset{\sim}{\theta}^{c}{}_{b}
\end{aligned}
$$

where $C^a{}_{bc} = \langle \underline{\omega}^a | [\underline{e}_b, \underline{e}_c] \rangle$

So that

$\underline{\nabla}\, \underline{e}_b =$

Cartan's 1st Structural Eqn

Torsion 2-form: $\underset{\sim}{T}^a = \frac{1}{2} T^a{}_{bc}\, \underline{\omega}^b \wedge \underline{\omega}^c$

$$\underset{\sim}{T}^a = \frac{1}{2}\left(\Gamma^a{}_{bc} - \Gamma^a{}_{cb} - C^a{}_{bc} \right) \underline{\omega}^b \wedge \underline{\omega}^c$$

$$= \underset{\sim}{\Theta}^a{}_c \wedge \underline{\omega}^c$$

labels

so that

$$\nabla e_b = \theta^c{}_b \, e_c$$

$$= \langle g | \nabla e_a, e_b \rangle + \langle g | e_a, \nabla e_b \rangle$$

$$= \langle g | \theta^c{}_a e_c, e_b \rangle + \langle g | e_a, \theta^c{}_b e_c \rangle$$

$$= g_{cb}\, \theta^c{}_a + g_{ac}\, \theta^c{}_b$$

But $\langle d\omega^a | e_b, e_c \rangle = e_b(\underbrace{\cancel{\langle \omega^a | e_c \rangle}}_{\delta^a_c}) - e_c(\underbrace{\cancel{\langle \omega^a | e_b \rangle}}_{\delta^a_b}) - \langle \omega^a | [e_b, e_c] \rangle$

Hence

$$\boxed{T^a = \theta^a{}_c \wedge \omega^c + d\omega^a}$$

When torsion vanishes

$$d\omega^a = -\theta^a{}_b \wedge \omega^b$$

Cartan's 2^{nd} Structural Eqn

Recall curvature is the commutator of the connection:

$$\underset{\sim}{R}(\underline{u},\underline{v})\underline{W} = \nabla_{\underline{u}}\nabla_{\underline{v}}\underline{W} - \nabla_{\underline{v}}\nabla_{\underline{u}}\underline{W} - \nabla_{[\underline{u},\underline{v}]}\underline{W}$$

$$R^a{}_{bcd} = \partial_c\Gamma^a{}_{bd} - \partial_d\Gamma^a{}_{bc} + \Gamma^a{}_{ce}\Gamma^e{}_{db} - \Gamma^a{}_{de}\Gamma^e{}_{cb} - C^e{}_{cd}\Gamma^a{}_{eb}$$

2^{nd} Cartan relates curvature to exterior deriv of spin connection

Define curvature 2-form $\underline{R}^a{}_b = \frac{1}{2} R^a{}_{bcd}\, \underline{\omega}^c \wedge \underline{\omega}^d$

Then $\boxed{\underline{R}^a{}_b = \underline{d}\,\underline{\theta}^a{}_b + \underline{\theta}^a{}_c \wedge \underline{\theta}^c{}_b}$

$d\Gamma^a{}_{eb}$

Example $ds^2 = A^2(r)\,dt^2 - B^2(r)\,dr^2 - r^2[d\theta^2 + \sin^2\theta\, d\varphi^2]$

Define o/n basis.

$\underset{\sim}{\omega}^t = A\,dt$

$\underset{\sim}{\omega}^r = B\,dr$

$\underset{\sim}{\omega}^\theta = r\,d\theta$

$\underset{\sim}{\omega}^\varphi = r\sin\theta\, d\varphi$

$\eta_{ab}\, \underset{\sim}{\omega}^a \otimes \underset{\sim}{\omega}^b$

$\Downarrow$

$\underset{\sim}{\theta}_{ab} = -\underset{\sim}{\theta}_{ba}$

Take $\underset{\sim}{d}$:

$\underset{\sim}{d}\,\underset{\sim}{\omega}^t = A'\,\underset{\sim}{dr} \wedge \underset{\sim}{dt}$

$= -\dfrac{A'}{AB}\, \underset{\sim}{\omega}^t \wedge \underset{\sim}{\omega}^r$

Example $ds^2 = A^2(r)\,dt^2 - B^2(r)\,dr^2 - r^2\left[d\theta^2 + \sin^2\theta\, d\varphi^2\right]$

Define o/n basis.

$\underset{\sim}{\omega}^t = A\,\underset{\sim}{d}t$

$\underset{\sim}{\omega}^r = B\,\underset{\sim}{d}r$

$\underset{\sim}{\omega}^\theta = r\,\underset{\sim}{d}\theta$

$\underset{\sim}{\omega}^\varphi = r\sin\theta\,\underset{\sim}{d}\varphi$

$\eta_{ab}\,\underset{\sim}{\omega}^a \otimes \underset{\sim}{\omega}^b$

$\Downarrow$

$\underset{\sim}{\theta}_{ab} = -\underset{\sim}{\theta}_{ba}$

Take $\underset{\sim}{d}$:

$\underset{\sim}{d}\,\underset{\sim}{\omega}^t = A'\,\underset{\sim}{d}r \wedge \underset{\sim}{d}t$

$= -\dfrac{A'}{AB}\,\underset{\sim}{\omega}^t \wedge \underset{\sim}{\omega}^r$

$\underset{\sim}{d}\,\underset{\sim}{\omega}^\theta = \dfrac{-1}{Br}\,\underset{\sim}{\omega}^\theta \wedge \underset{\sim}{\omega}^r$

$\underset{\sim}{d}\,\underset{\sim}{\omega}^\varphi =$

$$d\omega^\varphi = \sin\theta\, dr\wedge d\varphi + r\cos\theta\, d\theta\wedge d\varphi$$

$$= -\frac{1}{Br}\,\omega^\varphi\wedge\omega^r - \frac{\cot\theta}{r}\,\omega^\varphi\wedge\omega^\theta$$

Read off: $\Theta^t{}_r = \dfrac{A'}{AB}\,\omega^t$ $\quad \Theta^t{}_{\theta,\varphi} = 0$

$\Theta^\theta{}_r = \dfrac{1}{Br}\,\omega^\theta$ $\quad \Theta^\varphi{}_\theta = \dfrac{\cot\theta}{r}\,\omega^\varphi$

$\Theta^\varphi{}_r = \dfrac{1}{Br}\,\omega^\varphi$ all others zero.

$$R^t{}_r = d\Theta^t{}_r = \left(\frac{A'}{B}\right)'\frac{\omega^r\wedge\omega^t}{AB}$$

$$\underline{d}\,\underline{\omega}^\varphi = \sin\theta\,\underline{d}r\wedge\underline{d}\varphi + r\cos\theta\,\underline{d}\theta\wedge\underline{d}\varphi$$

$$= -\frac{1}{Br}\,\underline{\omega}^\varphi\wedge\underline{\omega}^r - \frac{\cot\theta}{r}\,\underline{\omega}^\varphi\wedge\underline{\omega}^\theta$$

Read off: $\underset{\sim}{\Theta}{}^t{}_r = \frac{A'}{AB}\,\underset{\sim}{\omega}{}^t$, $\quad \underset{\sim}{\Theta}{}^t{}_{\theta,\varphi} = 0$

$$\underset{\sim}{\Theta}{}^\theta{}_r = \frac{1}{Br}\,\underset{\sim}{\omega}{}^\theta \qquad \underset{\sim}{\Theta}{}^\varphi{}_\theta = \frac{\cot\theta}{r}\,\underline{\omega}^\varphi$$

$$\underset{\sim}{\Theta}{}^\varphi{}_r = \frac{1}{Br}\,\underline{\omega}^\varphi$$

all others zero.

$$\underline{R}^t{}_r = \underline{d}\,\underline{\Theta}^t{}_r = \left(\frac{A'}{B}\right)'\frac{\underset{\sim}{\omega}{}^r\wedge\underline{\omega}^t}{AB}$$

$$\underset{\sim}{R}{}^t{}_\theta = \underset{\sim}{\Theta}{}^t{}_r\wedge\underset{\sim}{\Theta}{}^r{}_\theta$$

$$= \frac{A'}{AB}\,\underline{\omega}^t\wedge\frac{-1}{Br}\,\underline{\omega}^\theta$$

Simly

$$\underline{R}^t{}_\varphi = -\frac{A'}{AB^2r}\,\underline{\omega}^t\wedge\underline{\omega}^\varphi$$

$$\underline{\omega}^\varphi = r\sin\theta\,\underline{d}\varphi$$

$$\underline{d}\underline{\omega}^\theta = \frac{-1}{Br}\,\underline{\omega}^\theta \wedge \underline{\omega}^r$$

$$\leftarrow \tfrac{1}{2} R^\theta{}_{rab}\,\underline{\omega}^a \wedge \underline{\omega}^b$$

$$\underline{R}^\theta{}_r = \underline{d}\underline{\Theta}^\theta{}_r = -\frac{B'}{B^2}\,\frac{\underline{\omega}^r \wedge \underline{\omega}^\theta}{Br}$$

$$\underline{R}^\varphi{}_r = \underline{d}\underline{\Theta}^\varphi{}_r + \underline{\Theta}^\varphi{}_\theta \wedge \underline{\Theta}^\theta{}_r$$

$$= \underline{d}\left(\frac{\sin\theta\, d\varphi}{B}\right) + \frac{\cot\theta}{r}\,\underline{\omega}^\varphi \wedge \frac{1}{Br}\underline{\omega}^\theta$$

$$= -\frac{B'}{B^2}\,\frac{\underline{\omega}^r \wedge \underline{\omega}^\varphi}{rB}$$

$$\underline{R}^\varphi{}_\theta = \underline{d}\,\underline{\Theta}^\varphi{}_\theta + \underline{\Theta}^\varphi{}_r \wedge \underline{\Theta}^r{}_\theta$$

$$= -\sin\theta\,\underline{d}\theta \wedge \underline{d}\varphi + \frac{1}{Br}\,\underline{\omega}^\varphi \wedge$$

$$= -\frac{1}{r^2}\left(1 - \frac{1}{B^2}\right)\underline{\omega}^\theta \wedge \underline{\omega}$$

Read off

$$R^{\hat t}{}_{\hat r \hat t \hat r} = -\left(\frac{A'}{B}\right)' \Big/ AB \qquad R^{\hat t}{}_{\hat\theta}$$

$$\underset{\sim}{\omega}^\theta = \frac{-1}{Br}\,\underset{\sim}{\omega}^\theta \wedge \underset{\sim}{\omega}^r$$

$$\underset{\sim}{\theta}^{\varphi}{}_r = \frac{1}{Br}\,\underset{\sim}{\omega}^{\varphi}$$

all others zero.

$$\underset{\sim}{R}^t{}_\varphi = \frac{-A}{AB^2 r}\,\underset{\sim}{\omega}^t \wedge \underset{\sim}{\omega}$$

$$R^{\hat\theta}{}_{\hat r\hat\theta\hat r} = \frac{B'}{rB^3} = R^{\hat\varphi}{}_{\hat r\hat\varphi\hat r}$$

$$\underset{\sim}{R}^{\varphi}{}_\theta = \underset{\sim}{d}\,\underset{\sim}{\theta}^{\varphi}{}_\theta + \underset{\sim}{\theta}^{\varphi}{}_r \wedge \underset{\sim}{\theta}^r{}_\theta$$

$$= -\sin\theta\, \underset{\sim}{d}\theta \wedge \underset{\sim}{d}\varphi + \frac{1}{Br}\,\underset{\sim}{\omega}^{\varphi} \wedge \frac{-1}{Br}\,\underset{\sim}{\omega}^{\theta}$$

$$= -\frac{1}{r^2}\left(1 - \frac{1}{B^2}\right)\underset{\sim}{\omega}^{\theta} \wedge \underset{\sim}{\omega}^{\varphi}$$

$$R^{\hat\varphi}{}_{\hat\theta\hat\varphi\hat\theta} = \frac{1}{r^2} - \frac{1}{r^2B^2}$$

Read off

$$R^{\hat t}{}_{\hat r\hat t\hat r} = -\left(\frac{A'}{B}\right)'\Big/AB \qquad R^{\hat t}{}_{\hat\theta\hat t\hat\theta} = -\frac{A'}{AB^2 r} = R^{\hat t}{}_{\hat\varphi\hat t\hat\varphi}$$

Hats emphasise this is o/n basis

e.g. $$\underset{\sim}{Q} = \underbrace{Q^{\hat a}{}_{\hat b}}_{\text{o/n cpts.}} \underset{\sim}{\omega}^{\hat b} \underset{\sim}{e}_{\hat a} = \underbrace{Q^{\mu}{}_{\nu}}_{\text{co-ord cpts.}} dx^{\nu} \frac{\partial}{\partial x^{\mu}}$$

$$\underset{\sim}{\omega}^{\hat b} = \boxed{\omega^{b}{}_{\nu}}\, dx^{\nu}$$

$$\underset{\sim}{e}_{\hat a} = \boxed{e_{\hat a}{}^{\mu}}\, \frac{\partial}{\partial x^{\mu}}$$

Here (eg)

$$\omega^{\hat t}{}_{t} = A$$

$$e^{t}{}_{\hat t} = 1/A$$

$$R^{\hat{t}}{}_{\hat{t}} = R^t{}_t = \frac{1}{B^2}\left[\frac{A''}{A} - \frac{A'B'}{AB} + 2\frac{A'}{Ar}\right]$$

$$R^\theta{}_\theta = R^\varphi{}_\varphi = \frac{1}{B^2}\left[\frac{A'}{Ar} + \frac{1}{r^2} - \frac{B'}{rB}\right] - \frac{1}{r^2}$$

$$R^r{}_r = \frac{1}{B^2}\left(\frac{A''}{A} - \frac{B'A'}{AB} - 2\frac{B'}{Br}\right)$$

So get

$$G^t{}_t = \frac{1}{r^2} - \frac{1}{B^2r^2} + \frac{2B'}{B^3r}$$

$$= \frac{1}{r^2} - \frac{1}{r^2}\left(\frac{r}{B^2}\right)' = 8\pi G T^0_0$$

$$\Rightarrow B^{-2} = 1 - \frac{2G}{r}\underbrace{\int 4\pi r^2 T^0_0\, dr}_{M(r)}$$

$$R^t{}_t - R^r{}_r = \frac{2}{B^2}\left(\frac{A'}{A} + \frac{B'}{B}\right)$$

In vacuum $\quad A^2 = 1/B^2 = 1 - \frac{2GM}{r}.$