Title: Lecture - Gravitational Physics, PHYS 636

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Subject: Cosmology, Strong Gravity

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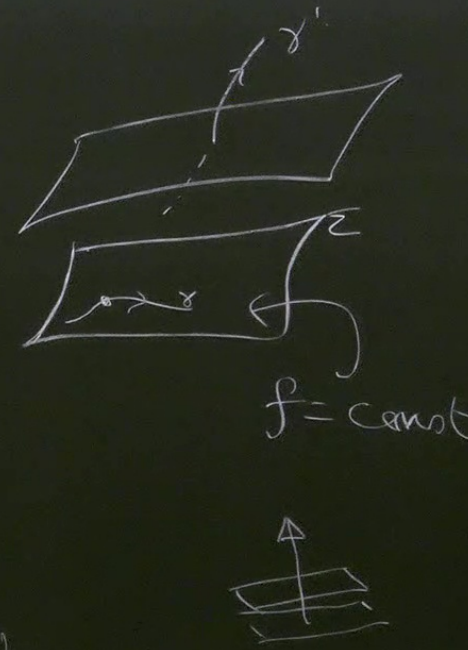
L3 Forms & Duals

Consider $f \in C^\infty(\mathcal{M})$

If $\gamma \subset \Sigma$ then $\dot{\gamma} f = 0$

as f is const on Σ, hence γ



Define a covector, $\underset{\sim}{d}f$

$$\langle \underset{\sim}{d}f \,|\, \underline{T} \rangle = \underline{T}f \qquad \forall\, \underline{T} \in T_p(\mathcal{M})$$

any f

$$\underset{\sim}{d} : C^\infty(\mathcal{M}) \longrightarrow T^*(\mathcal{M})$$

(Think "grad")

In components $\underset{\sim}{d}f = \dfrac{\partial f}{\partial x^\mu}\, \underset{\sim}{d}x^\mu$

A <u>p-form</u> is a totally antisymmetric rank p covariant tensor. Construct similarly to tensors.

$$\underline{A} \wedge \underline{B} = \underline{A} \otimes \underline{B} - \underline{B} \otimes \underline{A} \quad \leftarrow \text{2-form.}$$

covectors / 1-forms

The wedge product is anti-symmetric.

In components: $\left(\underline{A}^{(p)} \wedge \underline{B}^{(q)}\right)_{a_1 \dots a_{p+q}} = \frac{(p+q)!}{p!\, q!} A_{[a_1 \dots a_p} B_{a_{p+1} \dots a_{p+q}]}$

where (eg) $\underline{A}^{(p)} = \frac{1}{p!} A_{a_1 \dots a_p} dx^{a_1} \wedge \dots dx^{a_p}$

$\underline{A}^{(p)} \wedge \underline{B}^{(q)} = (-)^{pq}\, \underline{B} \wedge \underline{A}$

covectors / 1-forms

There are only a finite number of forms, $p \leq n$, we denote the bundle of p-forms as $\Lambda^p(M)$.

The n-form is unique up to a factor

where (eg) $\underline{A}^{(p)} = \frac{1}{p!} A_{a_1 \dots a_p} dx^{a_1} \wedge \dots dx^{a_p}$

$\underline{A}^{(p)} \wedge \underline{B}^{(q)} = (-)^{pq} \, \underline{B} \wedge \underline{A}$

$\epsilon_{a_1 \dots a_n} = \pm 1$ (sign of permutation)

$\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = +1$

$\epsilon_{132} = \epsilon_{213} = \epsilon_{321} = -1$

This ϵ is actually a tensor density

$$\underline{\epsilon} = \frac{1}{n!} \epsilon_{\mu_1 \dots \mu_n} dx^{\mu_1} \wedge \dots dx^{\mu_n}$$

$$= \frac{1}{n!} \underbrace{\det\left(\frac{\partial x}{\partial x'}\right)}_{} \epsilon_{\mu'_1 \dots \mu'_n} dx'^{\mu'_1} \wedge \dots dx'^{\mu'_n}$$

In 4D have 4 lin. indep 1-forms,

6 2-forms, 4 3 forms & 1 4-form

? Can we relate 1 and 3-forms?

In 3D 3 1-forms } relate via
3 2-forms. } X product.

Recall in 3D.

$$(\underline{a} \times \underline{b})_i = \varepsilon_{ijk}\, a_j\, b_k$$

With a metric, note that

$$\det g \longrightarrow \det g' = \det\left(g_{\mu\nu} \frac{\partial x^{\mu}}{\partial x'^{\mu'}} \frac{\partial x^{\nu}}{\partial x'^{\nu'}} \right)$$

$$= \det g \cdot \det\left(\frac{\partial x}{\partial x'}\right)^2$$

Recall in 3D.

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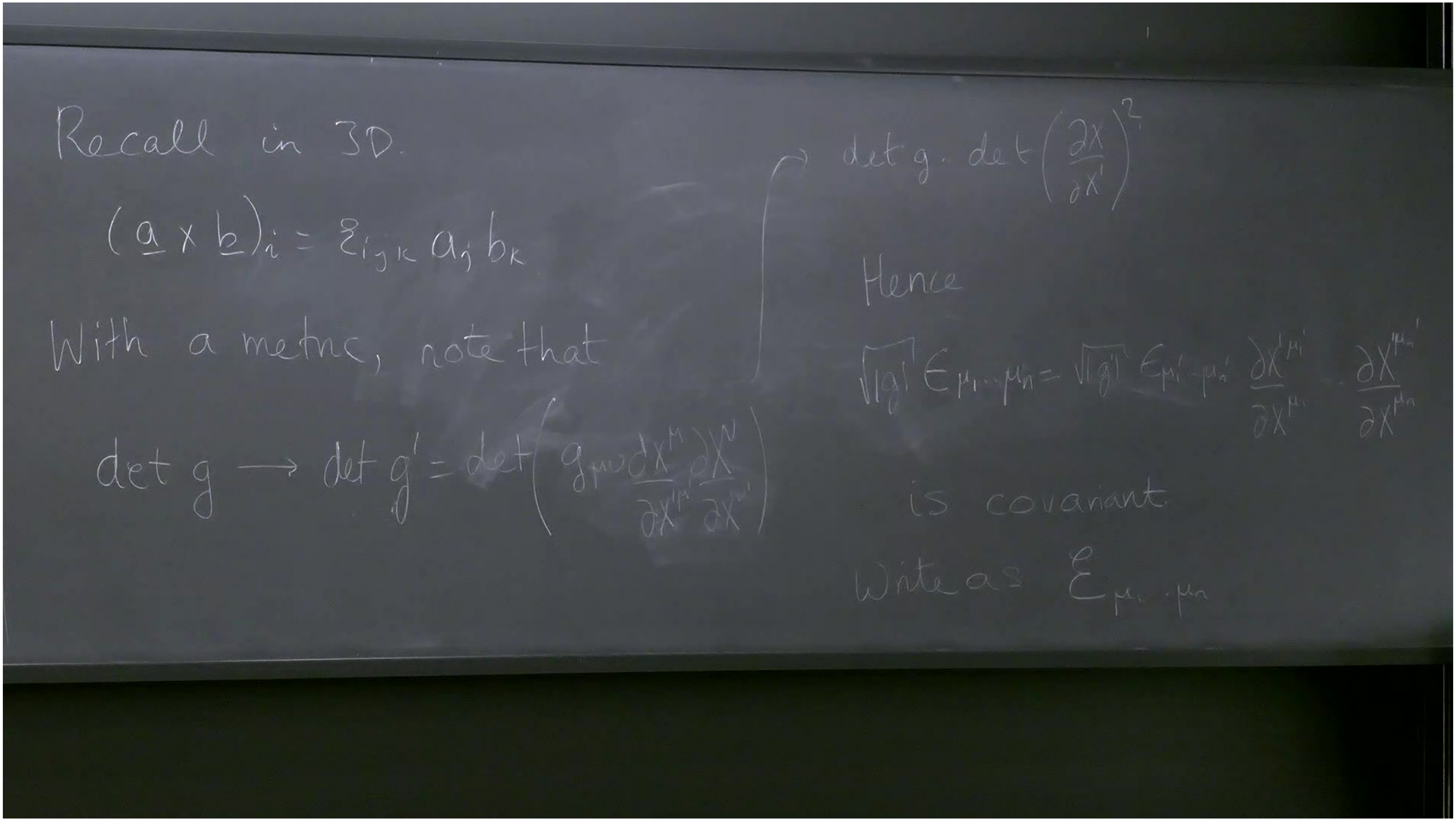
Hence

$$\sqrt{|g|}\,\epsilon_{\mu_1 \ldots \mu_n} = \sqrt{|g'|}\,\epsilon_{\mu_1' \ldots \mu_n'} \frac{\partial x'^{\mu_1'}}{\partial x^{\mu_1}} \cdots \frac{\partial x'^{\mu_n'}}{\partial x^{\mu_n}}$$



Recall in 3D.
$(\underline{a} \times \underline{b})_i = \varepsilon_{ijk} a_j b_k$
With a metric, note that
$\det g \longrightarrow \det g' = \det\left(g_{\mu\nu} \frac{\partial x^\mu}{\partial x'^{\mu'}} \frac{\partial x^\nu}{\partial x'^{\nu'}} \right)$
$\det g \cdot \det\left(\frac{\partial x}{\partial x'}\right)^2$
Hence
$\sqrt{|g|}\, \epsilon_{\mu_1 \dots \mu_n} = \sqrt{|g'|}\, \epsilon_{\mu_1' \dots \mu_n'} \frac{\partial x'^{\mu_1'}}{\partial x^{\mu_1}} \dots \frac{\partial x'^{\mu_n'}}{\partial x^{\mu_n}}$
is covariant.
Write as $\mathcal{E}_{\mu_1 \dots \mu_n}$

We can then define a map

$$* \quad \Lambda^p \to \Lambda^{n-p}$$

$$\underset{\sim}{A} \longmapsto * \underset{\sim}{A}$$

via $$(*A)_{a_1 \ldots a_{n-p}} = \frac{1}{p!} \varepsilon_{a_1 \ldots a_{n-p}}{}^{b_1 \ldots b_p} A_{b_1 \ldots b_p}$$

eg $\mathbb{R}^3$

$dx^1 \leftrightarrow dx^2 \wedge dx^3$

This is the Hodge dual

Can now extend $\underline{d}$

to forms

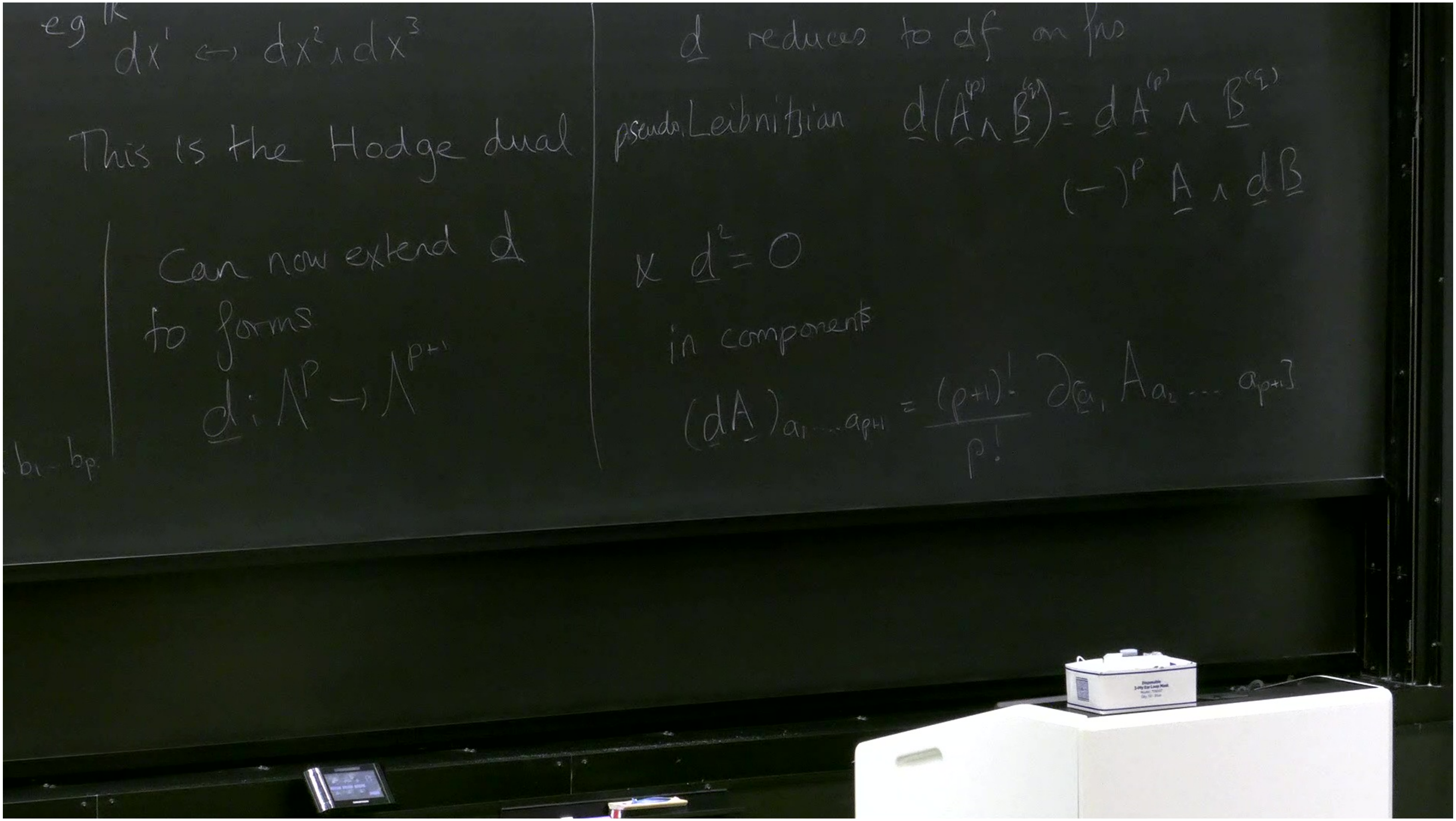
$\underline{d} : \Lambda^p \to \Lambda^{p+1}$

$b_1 \dots b_p$

$\underline{d}$ reduces to $\underline{d}f$ on fns

pseudo Leibnitzian $\underline{d}(\underline{A}^{(p)} \wedge \underline{B}^{(q)}) = \underline{d}\,\underline{A}^{(p)} \wedge \underline{B}^{(q)}$

$(-)^p\, \underline{A} \wedge \underline{d}\,\underline{B}$


eg $\mathbb{R}^3$ $dx^1 \leftrightarrow dx^2 \wedge dx^3$
This is the Hodge dual
Can now extend $\underline{d}$ to forms
$\underline{d} : \Lambda^p \to \Lambda^{p+1}$
$b_1 .. b_p$.
$\underline{d}$ reduces to df on fns
pseudo Leibnitzian $\underline{d}(\underline{A}^{(p)} \wedge \underline{B}^{(q)}) = \underline{d}\,\underline{A}^{(p)} \wedge \underline{B}^{(q)}$
$(-)^p\, \underline{A} \wedge \underline{d}\,\underline{B}$
x $\underline{d}^2 = 0$
in components
$(d\underline{A})_{a_1 \dots a_{p+1}} = \frac{(p+1)!}{p!}\, \partial_{[a_1} A_{a_2 \dots a_{p+1}]}$

e.g. Electromagnetism

A_μ 1-form gauge field

$\underline{A} = A_\mu dx^\mu$

$\underline{F} = \underline{d}\,\underline{A}$

$$F_{\mu\nu} = \frac{2!}{1!} \partial_{[\mu} A_{\nu]} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

If we add $\underline{d} f$ to $\underline{A}$

$$\underline{d}(\underline{A} + \underline{d} f) = \underline{d}\,\underline{A} + \cancel{\underline{d}^2} f$$
$$= \underline{F}$$

covectors / 1-forms

e.g. $B_{\mu\nu}$ RR 2-form

A_μ couples to point charges

$B_{\mu\nu}$ " " line charges

$$\underline{H} = \underline{d}\underline{B} \leftrightarrow \partial_\mu B_{\nu\lambda} + \partial_\nu B_{\lambda\mu} + \partial_\lambda B_{\mu\nu}.$$

$\underline{A} \wedge \underline{B} = (-)^{pq} \underline{B} \wedge \underline{A}$

The Hodge dual gives rise to a dual derivation.

$\delta : \Lambda^p \to \Lambda^{p-1} \qquad \delta = * d *$

$A_{a_1 \ldots a_p} \to (-)^p \nabla^{a_1} A_{a_1 \ldots a_p}$

Free Maxwell: $dF = 0$ (identity). $\delta F = 0$

$A_{a_1 \ldots a_p}$

$\underline{H} = \underline{d}\underline{B} \leftrightarrow \partial_\mu B_{\nu\lambda} + \partial_\nu B_{\lambda\mu} + \partial_\lambda B_{\mu\nu}.$

Free Maxwell : $dF = 0$ (iden

Useful identity

$$\langle \underline{d\omega} \mid \underline{u}, \underline{v} \rangle = \underline{u}\big(\langle \underline{\omega} \mid \underline{v} \rangle\big) - \underline{v}\big(\langle \underline{\omega} \mid \underline{u} \rangle\big) - \langle \underline{\omega} \mid [\underline{u}, \underline{v}] \rangle$$