Title: Lecture - Gravitational Physics, PHYS 636

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Collection/Series: Gravitational Physics (Elective), PHYS 636, January 6 - February 5, 2025

Subject: Cosmology, Strong Gravity

Date: January 08, 2025 - 9:00 AM

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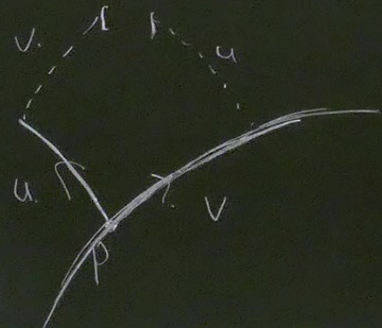
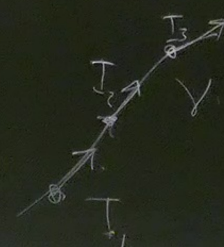
L2 Lie Derivative

Recall $([\underline{u},\underline{v}])^\mu = U^\nu V^\mu{}_{,\nu} - V^\nu U^\mu{}_{,\nu}$



Integral curves. γ is the integral curve of $\underline{T}$ if at all pts on γ the tangent to γ is $\underline{T}$



In a

In a local chart, can express γ as a Taylor series.

$$X^\mu_\gamma = \underbrace{X_0^\mu + \varepsilon T^\mu}_{Y^\mu} + \frac{1}{2}\varepsilon^2 T^\nu T^\mu{}_{,\nu} + \cdots$$

vector field generates a coord transfm.

Concept of derivative:

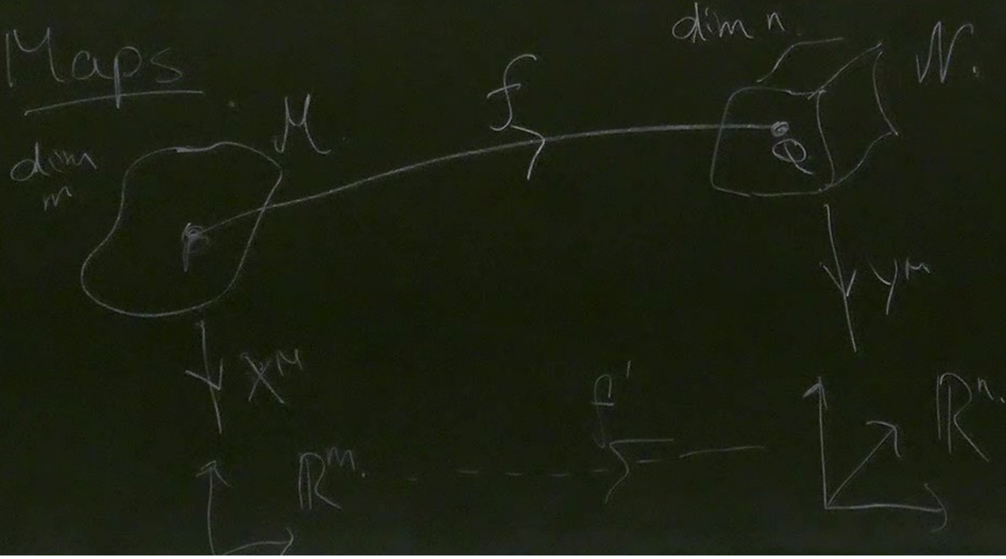
$$\frac{dQ}{dt} = \lim_{\delta t \to 0} \frac{Q(t+\delta t) - Q(t)}{\delta t}$$

Challenges: what is δt? or $Q(t+\delta t)$?

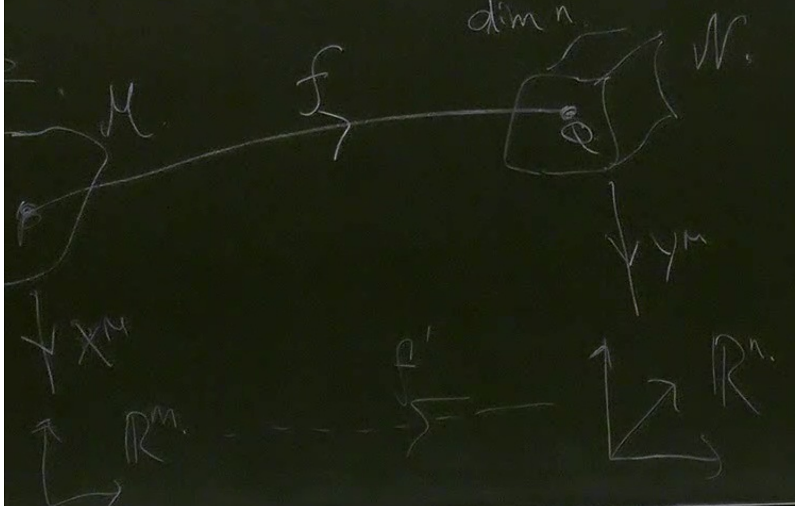
If "$t+\delta t$" and "t" are not the same point on manifold, they live in different spaces.

The integral curves give a way of "linking" nearby pts.

Maps

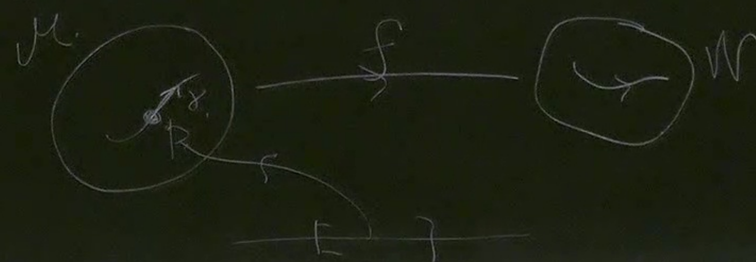


...tegral curves give a
...f "linking" nearby pts.



If $m=n$, f is 1-1 and onto, say M & N are the same manifold & f is a diffeomorphism.

<u>Push Forward</u> $f_* \; T_p(M) \to T_{\varphi}(N)$



If "$t+\delta t$" and "t" are not the same point on manifold, they live in different spaces.

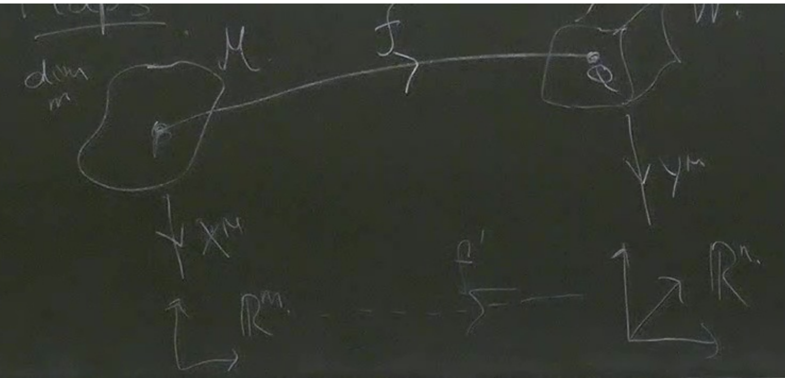
$\downarrow x^\mu$

$\mathbb{R}^m$

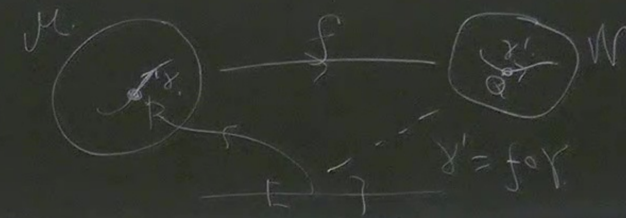
γ' is the image of γ under f, and has its own tangent vector in $T_Q(W)$.

Pull Back:

erent Spaces.



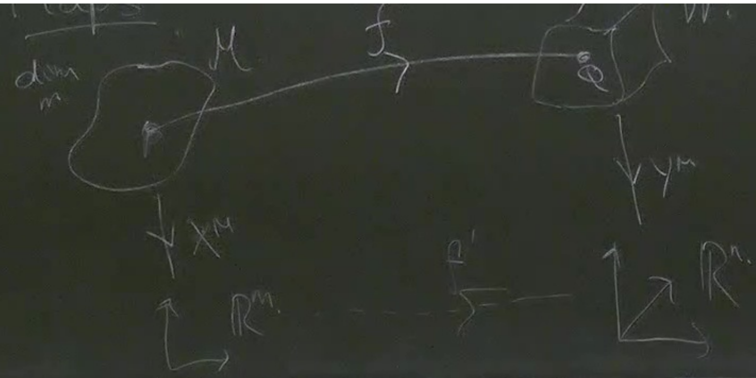
Push Forward $f_* \; T_p(M) \to T_q(N)$



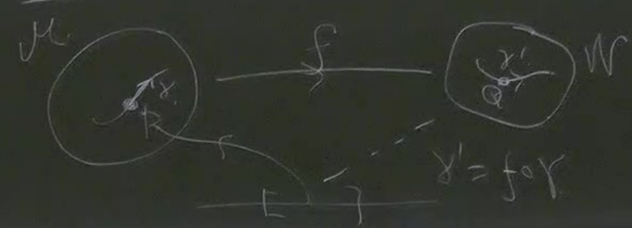
Pull Back: $f^*: T_q^*(N) \longrightarrow T_p^*(M)$ s.t. $\underbrace{\langle f^*(\underline{\omega}) \mid I \rangle}_{\text{in } M} = \underbrace{\langle \underline{\omega} \mid f_*(I) \rangle}_{\text{in } N}$

$$\begin{tikzcd} T_p(M) \arrow[r, "f_*"] \arrow[dr, "f^*(\underline{\omega})"'] & T_q(N) \arrow[d, "\underline{\omega} \in T_q^*(N)"] \\ & \mathbb{R} \end{tikzcd}$$

ferent spaces



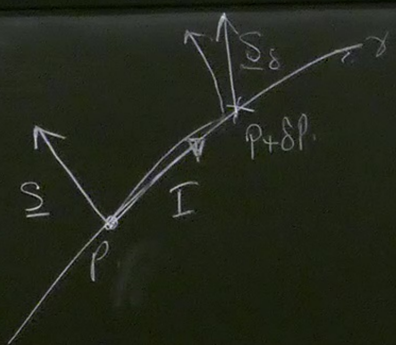
Push Forward $f_*: T_p(\mathcal{M}) \to T_q(\mathcal{N})$



Pull Back: $f^*: T^*_q(\mathcal{N}) \to T^*_p(\mathcal{M})$ s.t. $\underbrace{\langle f^*(\underline{\omega}) \mid \underline{I} \rangle}_{\text{in } \mathcal{M}} = \underbrace{\langle \underline{\omega} \mid f_*(\underline{I}) \rangle}_{\text{in } \mathcal{N}}$

$$\begin{tikzcd} T_p(\mathcal{M}) \arrow[r, "f_*"] \arrow[dr, "f^*(\underline{\omega})"'] & T_q(\mathcal{N}) \arrow[d, "\underline{\omega} \in T^*_q(\mathcal{N})"] \\ & \mathbb{R} \end{tikzcd}$$

For f a diffeo, the push forward + pull backs work in both directions and allow us to "slide" our geometric objects along integral curves



Actual value of $\underline{S}$ at $P+\delta P$
(in a local chart) is

$$S_\delta^\mu(P+\delta P) = S^\mu(P) + \epsilon T^\nu \partial_\nu S^\mu$$

To compare to the original value, must push forward from P

$$X^\mu \to X^\mu + \epsilon T^\mu = Y^\mu$$

$$\underline{S} = S^\mu \frac{\partial}{\partial X^\mu} = S^{\mu'} \frac{\partial}{\partial Y^\mu}$$

o the original

sh forward from P.

$+\epsilon T^{\mu} = y^{\mu}$

$S^{\mu'} \dfrac{\partial}{\partial y^{\mu'}}$

$$S^{\mu'} = \frac{\partial y^{\mu'}}{\partial x^{\mu}} S^{\mu} = S^{\mu'} + \epsilon S^{\mu} T^{\mu'}{}_{,\mu}$$

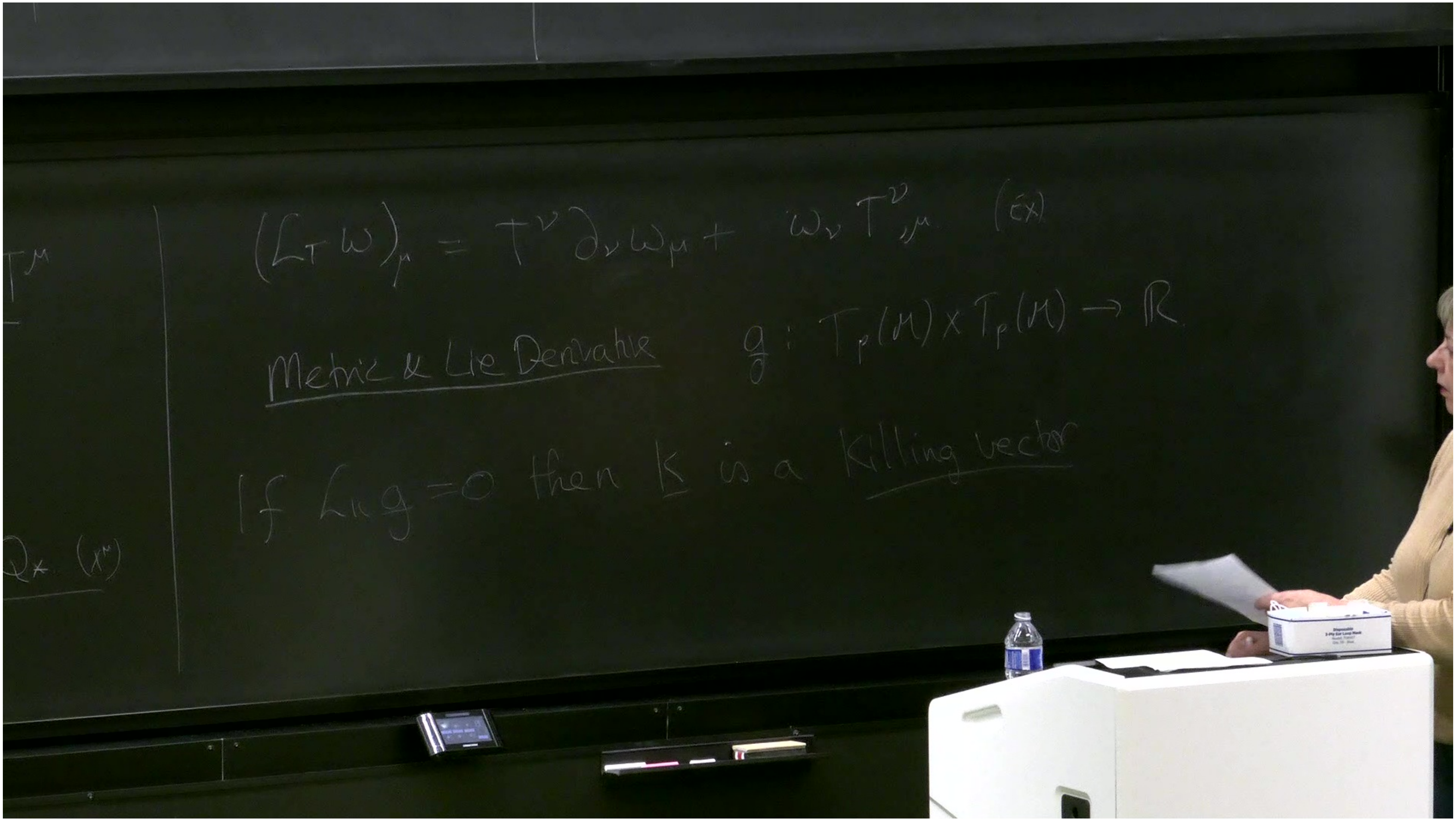
$$\left(\frac{\partial y^{\mu'}}{\partial x^{\mu}} = \delta^{\mu'}_{\mu} + \epsilon T^{\mu'}{}_{,\mu} \right)$$

$$= \frac{\epsilon T^{\nu}\partial_{\nu} S^{\mu} - \epsilon S^{\nu}\partial_{\nu} T^{\mu}}{\epsilon}$$

$$= [\underline{T}, \underline{S}]^{\mu}$$

Lie Derivative : $\lim_{\epsilon \to 0} \frac{Q(x^{\mu}+\epsilon T^{\mu}) - Q_{*}(x^{\mu})}{\epsilon}$

$(\mathcal{L}_T \omega)_{\mu} = T$


$(\mathcal{L}_T \omega)_\mu = T^\nu \partial_\nu \omega_\mu + \omega_\nu T^\nu_{,\mu}$ (ex)
Metric & Lie Derivative
$g : T_p(M) \times T_p(M) \to \mathbb{R}$
If $\mathcal{L}_k g = 0$ then k is a Killing vector

$$= [\underline{T}, \underline{S}]^\mu$$

Lie Derivative : $\lim_{\epsilon \to 0} \frac{Q(x^\mu + \epsilon T^\mu) - Q_*(x^\mu)}{\epsilon}$

If $\mathcal{L}_k g = 0$ then $\underline{k}$ is a Killing vector

k represents a symmetry of the geometry

e.g. S^2: $ds^2 = d\theta^2 + \sin^2\theta \, d\varphi^2$

$$\underline{k} = \frac{\partial}{\partial \varphi} \qquad k^\mu = (0,1) \qquad k^\mu{}_{,\nu} = 0$$

$\frac{\partial}{\partial \varphi}(g_{\mu\nu}) = 0$ hence

$$\mathcal{L}_k g_{\mu\nu} = k^\sigma \partial_\sigma g_{\mu\nu} + g_{\sigma\nu} K^\sigma{}_{,\mu} + g_{\sigma\mu} K^\sigma{}_{,\nu}$$
$$\equiv 0.$$

$$= [\underline{T}, \underline{S}]^\mu$$

Lie Derivative: $\lim_{\epsilon \to 0} \dfrac{Q(x^\mu + \epsilon T^\mu) - Q_*(x^\mu)}{\epsilon}$

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$k^\mu{}_{,\nu} = 0$

$\dfrac{\partial}{\partial \phi}(g_{\mu\nu}) = 0$ hence

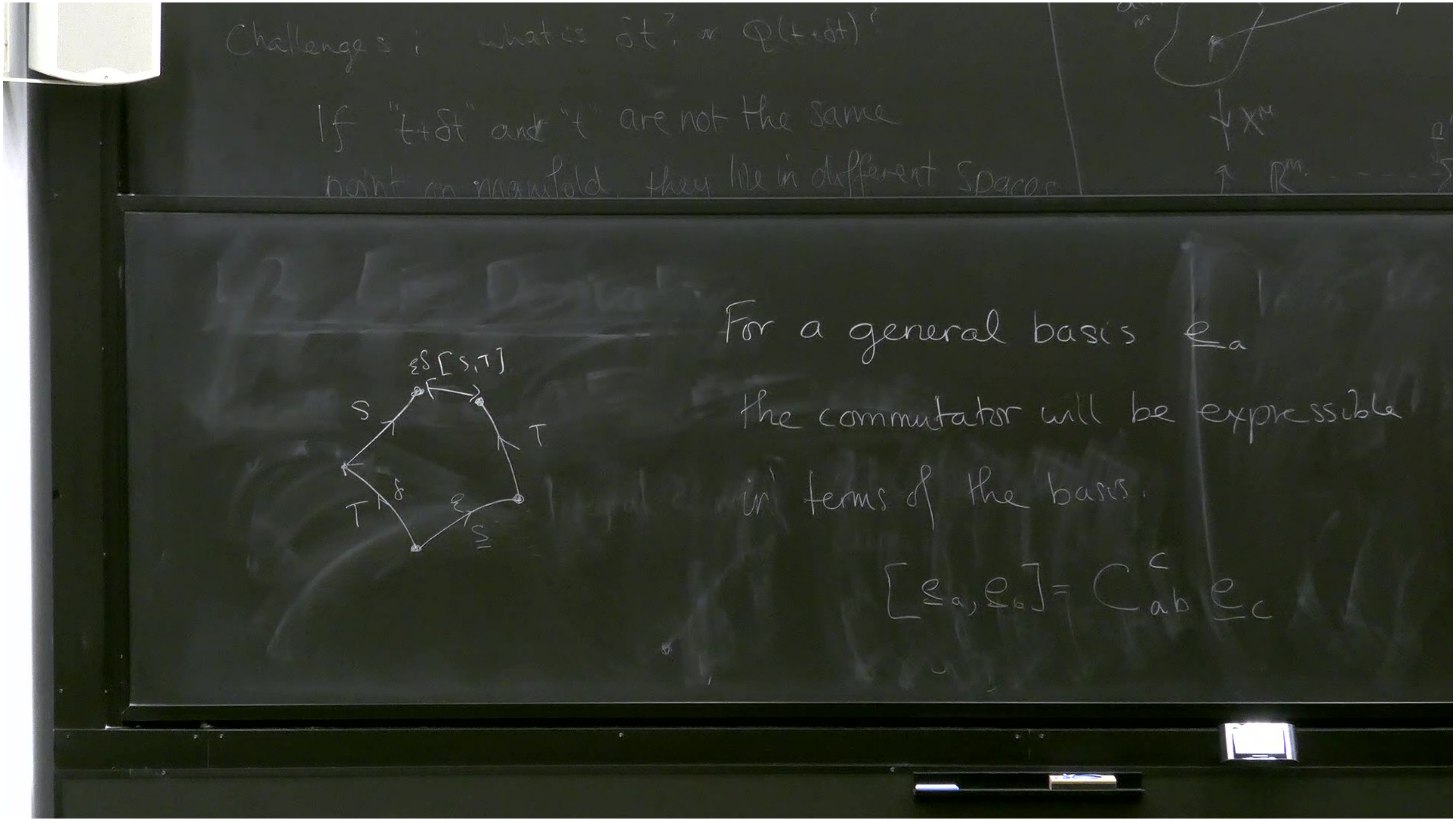
$$\mathcal{L}_k g_{\mu\nu} = k^\sigma \partial_\sigma g_{\mu\nu} + g_{\sigma\nu} k^\sigma{}_{,\mu} + g_{\sigma\mu} k^\sigma{}_{,\nu}$$

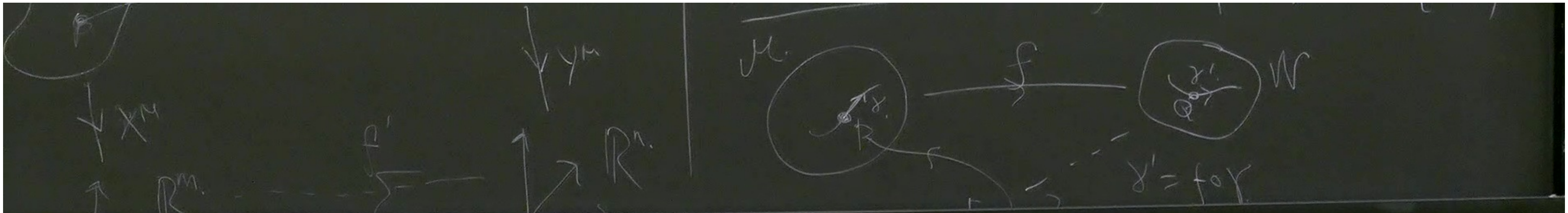
$$\equiv 0.$$

Also 2 more Killing vectors

that form an SO(3) algebra.

$$[\underline{\xi}_i, \underline{\xi}_j] = \epsilon_{ijk} \underline{\xi}_k$$


Challenges: what is δt? or Q(t+δt)?
If "t+δt" and "t" are not the same
point on manifold they live in different spaces
For a general basis $\underline{e}_a$
the commutator will be expressible
in terms of the basis.
$[\underline{e}_a, \underline{e}_b] = C^c_{ab}\, \underline{e}_c$
εδ[S,T]
S
T
δ
ε
$\underline{S}$



$\underline{e}_a$

be expressible

${}_{ab}^{\ c}\ \underline{e}_c$

$C_{ab}^{\ \ c}$ are called structure constants