Title: Lecture - Gravitational Physics, PHYS 636 [Zoom]

Speakers: Ruth Gregory

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Review of Gravitational Physics

Ruth Gregory

KING'S College LONDON

ROADMAP

Today: Review of Manifold Basics. *It is possible I will go a bit faster with slides, so please take a look at https://pirsa.org/20010045 to review the material if you need.*

Wednesday / Friday: More on Manifolds – Forms, Lie Derivative, Covariant Derivative, Curvature and Cartan.

Down the Road: Black Holes, Causal Structure, Actions & Thermodynamics

Advanced Topics: Submanifolds, Walls & Branes, Perturbation Theory, Analog Gravity.

Conventions

A trap for the unwary!

Signature $+,-,-,-$ [hep-ph] $\hbar = c = 1$ [relativist's]

Curvature:

$$R^a{}_{bcd} = 2\Gamma^a_{b[d,c]}$$

$$R_{ab} = R^c{}_{acb}$$

So the Einstein equations are:

$$R_{ab} - \frac{1}{2}Rg_{ab} = 8\pi G T_{ab}$$

Typically a, b.. are spacetime indices, but need not be a co-ordinate basis, μ, ν.. will usually be a co-ordinate basis, i, j.. are usually space indices, and A, B... submanifold indices.

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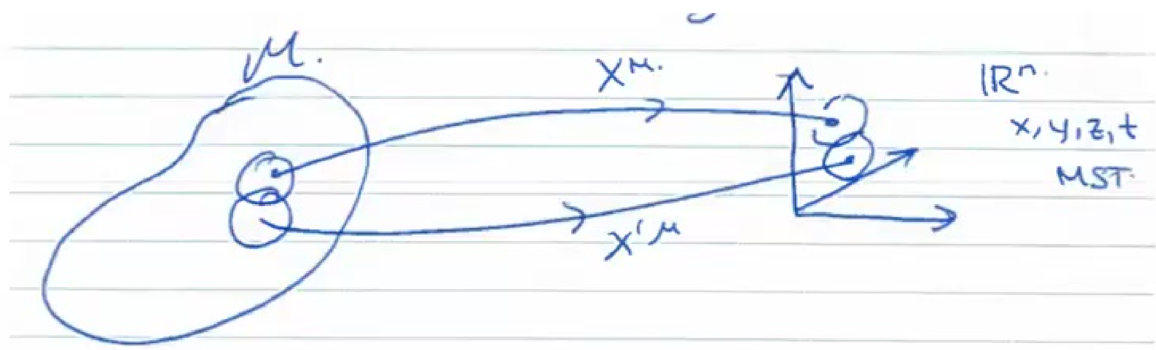
Manifolds-1

A manifold $\mathcal{M}$ (i.e. spacetime) is a set of events that looks locally like $\mathbb{R}^n$.

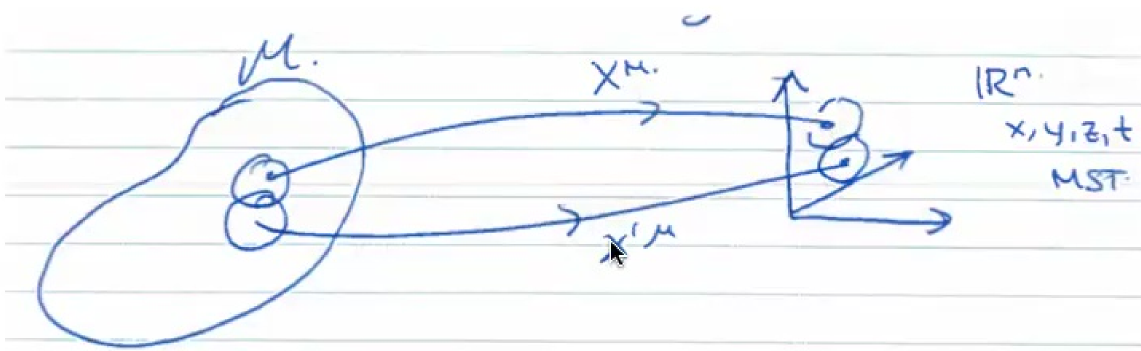


i.e. we can cover $\mathcal{M}$ with a collection of *charts* (open sets together with a map to flat $\mathbb{R}^n$).

$$\mathcal{M} = \cup\, \mathcal{U}_i$$

Manifolds-2

These maps label the points locally - co-ordinates - and where different charts overlap we ask that the transformation between the two sets of co-ordinates is infinitely differentiable.



This gives a *C^∞ manifold*, and the set of charts is called an *ATLAS*.

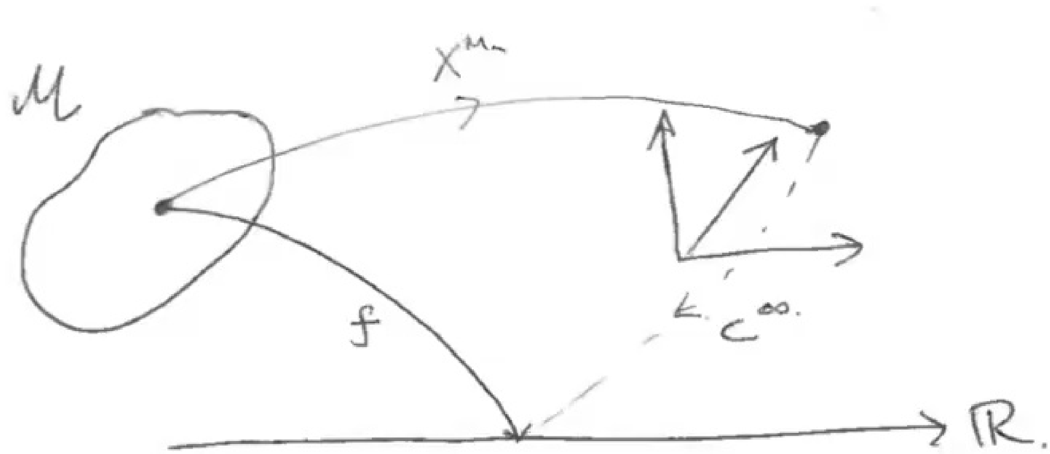
We transport structure from $\mathbb{R}^n$ to the manifold, building up our understanding of geometry.

Functions

A C^∞ function on $\mathcal{M}$ is a map

$$f : \mathcal{M} \to \mathbb{R}$$

that is locally C^∞ in all charts. The set of all C^∞ functions is denoted $C^\infty(\mathcal{M})$.

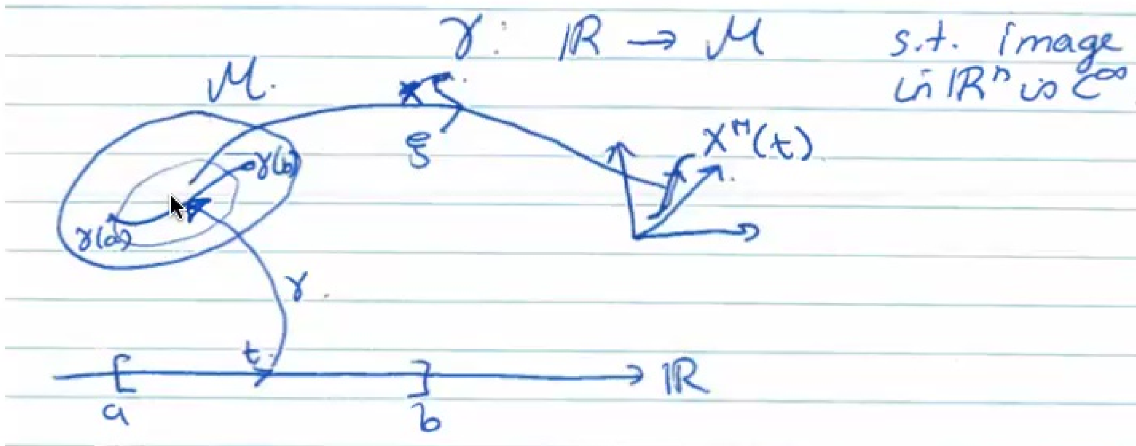


Curves

A C^∞ curve is a map

$$\gamma : \mathbb{R} \rightarrow \mathcal{M}$$

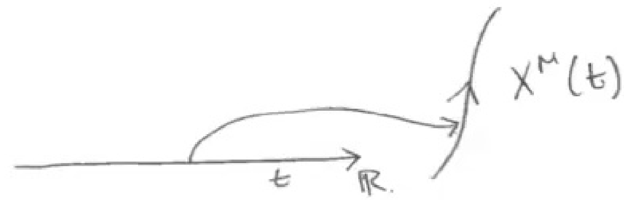
such that the image in a local chart is infinitely differentiable.



Curves

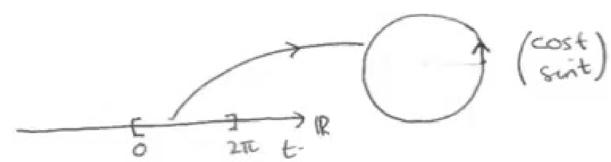
Examples are the worldline of an observer,

$$X^\mu(t) = (t, \mathbf{x}(t))$$



Or the circle:

$$S^1 : [0, 2\pi] \to \mathbb{R}^2$$
$$t \mapsto (\cos t, \sin t)$$



Note that the curve is the *path-plus-parametrisation*, so the same path traversed at a different rate is a different curve.

Tangent Vectors

A vector is defined as a linear operator interpreted as the tangent to a curve at a point P

$$\mathbf{T} : C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})$$

$$f \mapsto \frac{df}{dt} \quad \forall f \in C^\infty(\mathcal{M})$$

Here, for $f \in C^\infty(\mathcal{M})$, $f(t) = f \circ \gamma(t)$ is a real function, so is differentiable

Tangent Space

These operators form a vector space at P, the *Tangent Space* $T_P(\mathcal{M})$.

We can rescale vectors by changing the parametrisation of the curve

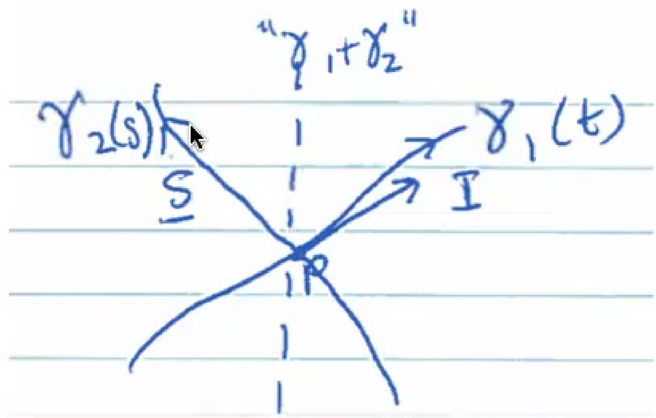
$$\gamma_\lambda(t) = \gamma(t/\lambda) \quad \text{takes} \qquad \mathbf{T} \to \lambda \mathbf{T}$$

and to add we go to a local chart and construct a "composite curve":

$$X_3^\mu = X_1^\mu + X_2^\mu$$

$$\mathbf{T} + \mathbf{S} : f \mapsto \left.\frac{df}{dt}\right|_{\gamma_1} + \left.\frac{df}{ds}\right|_{\gamma_2}$$

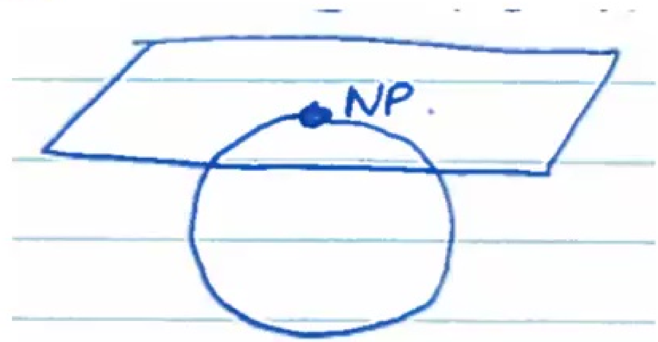
$$\forall f \in C^\infty(\mathcal{M})$$



Tangent Bundle

Note that $T_P(\mathcal{M})$ is a distinct space from $\mathcal{M}$, and is only defined at P. We can defined tangent spaces at all points of $\mathcal{M}$, and the collection of these is called the *Tangent Bundle $T(\mathcal{M})$*.

For example, the tangent plane to the North Pole of the sphere is a *plane*, $\mathbb{R}^2$, and we can directly visualise this as a plane sitting on top of the sphere.



Covectors

These are defined via maps from the tangent space to the reals (think of the dot-product).

$$\omega : T_P(\mathcal{M}) \to \mathbb{R}$$

$$\mathbf{v} \mapsto \omega(\mathbf{v}) \quad \text{or} \quad \langle \omega | \mathbf{v} \rangle$$

The set of all such ω also forms a vector space at P and is called the *cotangent* or *dual space* at P, $T^*_P(\mathcal{M})$

Bases

$T_P(\mathcal{M})$ and $T^*_P(\mathcal{M})$ are vector spaces, so we can choose a basis for each. The most common basis is the *co-ordinate basis*

$$I f = \frac{df}{dt} \quad \text{at } P$$

$$= \underbrace{\frac{dX^\mu}{dt}}_{\text{tangent to } X^\mu(t)} \underbrace{\frac{\partial f}{\partial X^\mu}}_{\text{derivs of } f \text{ in coord basis.}}$$

Here, the operators $\partial/\partial X^\mu$ are the co-ordinate basis vectors.

Components

We call the general derivatives of the coordinates of γ (the curve defining **T**) the *components* of **T**.

$$\underline{T} = \frac{dX^\mu}{dt}\frac{\partial}{\partial X^\mu} = T^\mu \frac{\partial}{\partial X^\mu}.$$

$\underline{T}$ ↑ VECTOR (geometric)

T^μ ↑ COMPONENTS (scalars)

$\frac{\partial}{\partial X^\mu}$ ↑ OPERATOR: BASIS VECTOR

Similarly, we can define the *covector co-ordinate basis* and covector components:

$$\underset{\sim}{\omega} = \omega_\mu \, dX^\mu.$$

ω_μ ↑ components.

dX^μ ↑ basis covector.

General Basis

It is often useful to use non-coordinate bases, in which case, the components of a vector/covector may not be directly related to a particular direction. A general basis can be written in terms of the coordinate basis:

$$\underline{e}_a = e_a{}^\mu \frac{\partial}{\partial x^\mu}$$

a = 1 -- n.

vier or viel bein

the most common of which is the *orthonormal* basis (which requires a metric!) when

$$\mathbf{g}(\mathbf{e}_a, \mathbf{e}_b) = \eta_{ab}$$

(geometric) (scalars)

Abstract Index Notation

Introduced by Penrose, to "legitimise" the working methods of physicists!

We often write T^μ to denote a vector, and then execute geometric operations, such as covariant derivative, on this "vector":

$$\nabla_\mu T^\nu = \partial_\mu T^\nu + \Gamma^{\ \nu}_{\mu\lambda} T^\lambda$$

However, strictly, T^μ are the *components* of the (geometric) vector **T**, and are therefore *scalars*.

The *vector* is the geometric **T**, and the correct expression is:

$$\nabla \mathbf{T} = \nabla(T^a \mathbf{e}_a) = (\nabla T^a)\mathbf{e}_a + T^a(\nabla \mathbf{e}_a)$$

Partial Derivatives   Connection

This method of using the index notation for meaning a geometric object is the *Abstract Index Notation*. Once we start to do calculations in gravity, we will revert to this, but for the first part of the course we will be building our differential geometry toolkit, so will be using geometric notation.

$$\begin{cases} \mathbf{T} & \text{The geometric object} \\ T^a & \text{Components - scalars!} \end{cases}$$

BE CAREFUL!

Co-ordinate transformations

Under a change of coordinates

$$\underbrace{\frac{\partial}{\partial X^{\mu}}}_{\text{OLD BASIS}} \rightarrow \underbrace{\frac{\partial X'^{\nu'}}{\partial X^{\mu}}}_{\text{MATRIX TRANSFM}} \underbrace{\frac{\partial}{\partial X'^{\nu'}}}_{\text{NEW BASIS}}$$

but **T** is the *same* geometric object, so is not affected by this transformation:

$$\mathbf{T} = T^{\mu}\frac{\partial}{\partial X^{\mu}} = \underbrace{T^{\mu}\frac{\partial X'^{\nu'}}{\partial X^{\mu}}}_{T'^{\nu'}}\frac{\partial}{\partial X'^{\nu'}}$$

which tells us how the components of **T** transform, and was the "old" definition of a vector – a statement of how the components changed under a coordinate transformation (NOT geometric!).

Co- and Contra-variant

Vectors and co-vector components transform in the opposite way:

$$T'^{\nu'} = \frac{\partial X'^{\nu'}}{\partial X^{\mu}} T^{\mu} \qquad \text{CONTRAVARIANT}$$

$$\omega_{\nu'} = \frac{\partial X^{\mu}}{\partial X'^{\nu'}} \omega_{\mu} \qquad \text{COVARIANT}$$

wherein we quickly see the problem of differentiating vectors:

$$\frac{\partial T^{\mu}}{\partial X^{\nu}} = \frac{\partial X'^{\nu'}}{\partial X^{\nu}} \frac{\partial}{\partial X'^{\nu'}} \left[\frac{\partial X^{\mu}}{\partial X'^{\mu'}} T'^{\mu'} \right]$$

$$= \underbrace{\frac{\partial X'^{\nu'}}{\partial X^{\nu}} \frac{\partial X^{\mu}}{\partial X'^{\mu'}} \frac{\partial T'^{\mu'}}{\partial X'^{\nu'}}}_{\checkmark} + \underbrace{\frac{\partial X'^{\nu'}}{\partial X^{\nu}} \frac{\partial^2 X^{\mu}}{\partial X'^{\nu'} \partial X'^{\mu'}} T'^{\mu'}}_{\times}$$

Lie Bracket

Consider the commutator of two vectors:

$$\underbrace{(\mathbf{uv}-\mathbf{vu})}_{\text{VECTOR}}[f] = u^\mu \frac{\partial}{\partial x^\mu}\left(v^\nu \frac{\partial f}{\partial x^\mu}\right) - v^\mu \frac{\partial}{\partial x^\mu}\left(u^\nu \frac{\partial f}{\partial x^\mu}\right)$$

$$= \left(u^\mu \frac{\partial v^\nu}{\partial x^\mu} - v^\mu \frac{\partial u^\nu}{\partial x^\mu}\right)\frac{\partial f}{\partial x^\nu} = \big(\underbrace{[\mathbf{u},\mathbf{v}]}_{\text{LIE BRACKET}}\big)^\nu \frac{\partial f}{\partial x^\nu}$$

EX: Check what happens under a coordinate transformation

$$u^\mu \frac{\partial v^\nu}{\partial x^\mu} \to u^{\mu'} \frac{\partial v^{\nu'}}{\partial x'^{\mu'}} \frac{\partial x^\nu}{\partial x'^{\nu'}} + u^{\mu'} v^{\nu'} \frac{\partial^2 x^\nu}{\partial x'^{\mu'} x'^{\nu'}}$$

and confirm that the bracket is indeed covariant.

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