Title: Lecture - Mathematical Physics, PHYS 777-

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Recap Vector bundle: $\mathbb{R}^k \to E \xrightarrow{\pi} B$ with local trivialization (U_i, φ_i), $\varphi_i : \pi^{-1}(U_i) \xrightarrow{\sim} U_i \times \mathbb{R}^k$

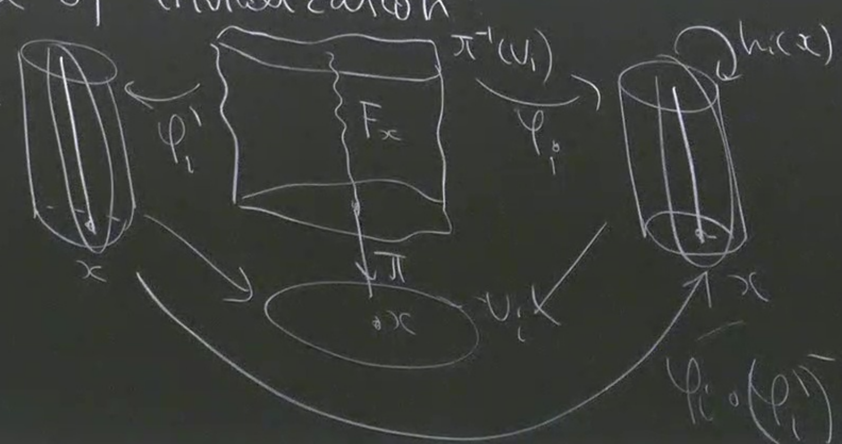
$\rightsquigarrow t_{ij} = \varphi_i^{-1} \circ \varphi_j : U_i \cap U_j \times \mathbb{R}^k \to U_i \cap U_j \times \mathbb{R}^k$

$(x, v) \mapsto (x, g_{ij}(x) v)$

The functions $g_{ij} : U_i \cap U_j \to GL(k)$ are called gluing cocycle $\rightsquigarrow$ can be used to define a bundle

- The gauge transformation is a change of trivialization

$$\begin{tikzcd} U_i \times \mathbb{R}^k \arrow[dr, "proj_1"'] & \pi^{-1}(U_i) \arrow[l, "\varphi_i'"'] \arrow[r, "\varphi_i"] \arrow[d, "\pi"] & U_i \times \mathbb{R}^k \arrow[dl, "proj_1"] \\ & U_i & \end{tikzcd}$$



$\varphi_i \circ (\varphi_i')^{-1} : U_i \times \mathbb{R}^k \longrightarrow U_i \times \mathbb{R}^k \quad , (x, v) \longmapsto (x, h_i(x) v)$ $\quad h_i : U_i \longrightarrow GL(k)$ gauge transform.

The gluing cocycle changes as $g_{ij}' = h_i^{-1} \circ g_{ij} \circ h_j$.

- The local sections $\Gamma(U_i, E)$ are $s_i : U_i \longrightarrow E$ s.t. $\pi \circ s_i = id$.

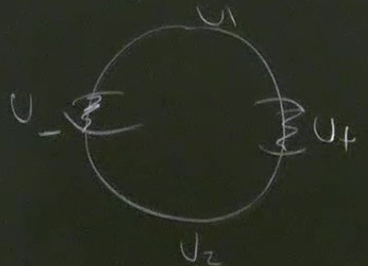
Remark We often (always) abuse notations by identifying

$\tilde{s}_i : U_i \longrightarrow \mathbb{R}^k$ defined by $(\varphi_i \circ s_i)(x) = (x, \tilde{s}_i(x))$ with s_i itself.

- The local sections glue to global ones $\Longleftrightarrow$ $\forall i,j : U_i \cap U_j \neq \emptyset : s_i(x) = g_{ij}(x) s_j(x)$ $\forall x \in U_i \cap U_j$

- Under the gauge transform : $s_i(x) = h_i(x)\, s_i'(x)$

Example Classify all the line bundles on $S^1 = U_1 \cup U_2$

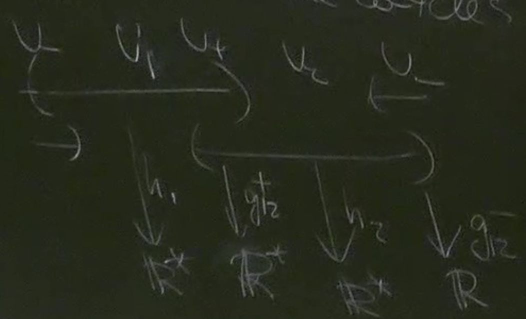


$U_1 \cap U_2 = U_+ \cup U_-$, the trivialization is defined by

$$g_{12}^+ : U_+ \to \mathbb{R}^* , \quad g_{12}^- : U_- \longrightarrow \mathbb{R}^* = GL(1)$$

The gauge transform: $h_i : U_i \to \mathbb{R}^*$

The cocycles change as: $g'^{\pm}_{12} = h_1^{-1}\big|_{U_\pm} \cdot g_{12}^{\pm} \cdot h_2\big|_{U_\pm}$



We set $h_2 = 1$. It is always possible to find h_1:

$$\begin{cases} h_1\big|_{U_+} = g_{12}^+ \\ h_1\big|_{U_-} = \mathrm{sgn}(g_{12}^+)\,|g_{12}^-| \end{cases} \Rightarrow \begin{array}{l} g'^{+}_{12} = 1 \\ g'^{-}_{12} = \mathrm{sgn}(g_{12}^+)\cdot \mathrm{sgn}(g_{12}^-) \end{array} \nearrow \text{2 bundles.}$$

Today - Principal bundles: distill gluing cocycle from the vector bundle.

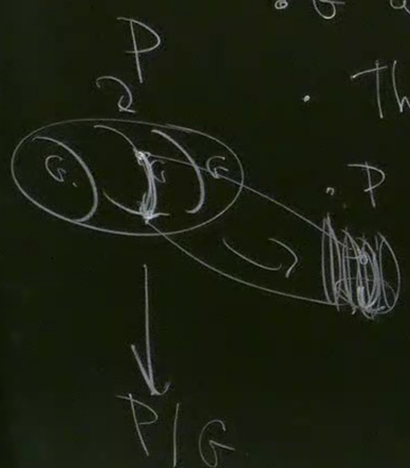
- Lie group: $\rightsquigarrow$ smooth manifold G
  $\rightsquigarrow$ group, $m: G\times G \to G$, $(\cdot)^{-1}: G \to G$ which are smooth. $e \in G$.
- Examples: matrix groups $GL(k)$, $(S)O(k)$, $Sp(k)$, $U(k)$, $U(k)$
- The left action of G on X - smooth m., is a smooth $L: G\times X \to X$
  $(g,x) \mapsto L_g x = g x$
  s.t. • $L_{gh} x = L_g L_h x = g.h.x$ $\forall x \in X$, $g,h \in G$ $g.h.x$
  • $L_e x = x$
- The right action $R: G\times X \to X$, $(g,x) \mapsto R_g x = x.g$
  • $R_{gh} x = R_h \cdot R_g x = x \cdot g \cdot h$
  • $R_e x = x$

• The G-action on X is free if $g \cdot x = x$ for some $x \Rightarrow g = e$

$\rightsquigarrow$ The action of every non-trivial element moves every point.

__Remark__ Orbits of free action $G \cdot x \simeq G$ (as smooth manifolds)

• __Principal bundle__ P, B – smooth manifolds, G – Lie grp. s.t.



∘ G acts freely on P from the right, $(u,g) \in P \times G \mapsto ug \equiv R_g u$.

· The factor space $P/G \cong B$, the projection $\pi: P \to P/G$ is smooth

∘ P is locally trivial: $\forall x\ \exists U,\ \varphi: \pi^{-1}(U) \to U \times G$

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\varphi"] \arrow[d, "\pi"'] & U \times G \arrow[dl, "\mathrm{proj}_1"] \\ U & \end{tikzcd}$$

s.t. $\varphi(ug) = \varphi(u) \cdot g$

$\forall g \in G,\ u \in \pi^{-1}(U)$

$$\begin{tikzcd} G \arrow[r] & P \arrow[d] \\ & B = P/G \end{tikzcd}$$

- The local trivialization is defined almost identical to the vector bundles

$$\varphi_i \circ \varphi_j^{-1} : U_i \cap U_j \times G \to U_i \cap U_j \times G, \quad (x, h) \longmapsto (x, h \cdot g_{ij}(x))$$

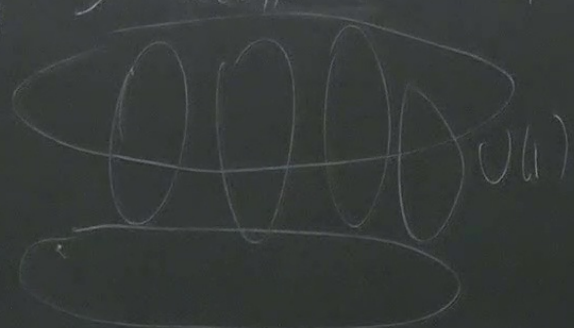
where $g_{ij} : U_i \cap U_j \to G$ satisfy all the cocycle axioms

Example Trivial G-bundle $P = B \times G$, $R_g(x,h) = (x, h\,g)$ $\quad x \in B$, $g, h \in G$

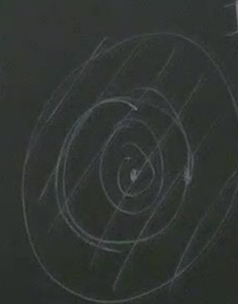
$$P/G = \{(x,g) \mid x \in B,\ g \in G\} / ((x,g) \sim (x, gh)\ \forall h \in G)$$

$(x,g) \sim (x, g g^{-1}) = (x, e)$

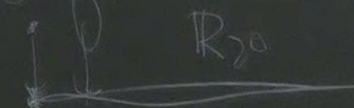
$P = S^1 \times U(1)$, $G = U(1)$, $B = S^1$



Ex $\mathbb{C} \circlearrowleft U(1)$



P, $G = U(1)$



<u>Example</u> Hopf fibration $P = S^3 = \{ |z_0|^2 + |z_1|^2 = 1,\ z_0, z_1 \in \mathbb{C} \}$

$G = \underset{\shortparallel}{U(1)}$, $B = P/G = S^2$

$R_{e^{i\alpha}}(z_0, z_1) = (z_0 e^{i\alpha}, z_1 e^{i\alpha})$

$\pi(z_0, z_1) = z_0/z_1 \in \mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$

<u>Claim</u> Every principal bundle $G \to P \xrightarrow{\pi} P/G$ is trivial $\iff$ it has global section s

$\boxed{\Rightarrow}$ $P = B \times G$, $s(x) = e$

$\boxed{\Leftarrow}$

$$\begin{tikzcd} \left\{ (x,g) \,\middle|\, \begin{matrix} x \in B \\ g \in G \end{matrix} \right\} \arrow[r, "\varphi"] \arrow[dr, "\mathrm{proj}_1"'] & P \arrow[d, "\pi"] \\ & B \end{tikzcd}$$

$\varphi(x,g) = s(x) \cdot g \in P$

- φ is injective. $s(x) g_1 = s(x) g_2 \Rightarrow g_1 = g_2$
- φ is surjective.
- $\mathrm{proj}_1 = \pi \circ \varphi$
- $\varphi((x, g \cdot h)) = \varphi((x,g)) \cdot h$