Title: Lecture - Mathematical Physics, PHYS 777-

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Collection/Series: Mathematical Physics (Core), PHYS 777-, January 6 - February 5, 2025

Subject: Mathematical physics

Date: January 29, 2025 - 11:30 AM

URL: https://pirsa.org/25010011

Recap The vector bundle. $F = \mathbb{R}^k \longrightarrow E \xrightarrow{\pi} B$, π is smooth surj

that satisf. local triviality condition: $\forall x \in B\ \exists\ U_{\ni x} \subset B$ and diffeo $\varphi : \pi^{-1}(U) \to U \times \mathbb{R}^k$

s.t. • $\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\varphi} & U \times \mathbb{R}^k \\ \pi \downarrow & \swarrow_{\text{proj}_1} & \\ U & & \end{array}$ • $v \longmapsto \varphi^{-1}(x, v)$ is an isomorphism of the vector spaces $\mathbb{R}^k \cong F_x = \pi^{-1}(x)$

• The local triv. $\bigcup_i U_i = B$, $\varphi_i : \pi^{-1}(U_i) \to U_i \times F$.

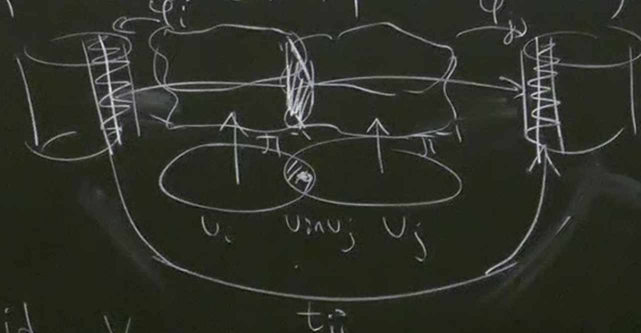
The transition func: $t_{ij} \doteq \varphi_i \circ \varphi_j^{-1} : U_i \cap U_j \times \mathbb{R}^k \to U_i \cap U_j \times \mathbb{R}^k$

$(x, v) \longmapsto (x, g_{ij}(x) \cdot v)$

$) \to U \times \mathbb{R}^k$

were $g_{ij} : U_i \cap U_j \to GL(k)$ - general linear group.

$GL(k) = \{ Mat_{k \times k}(\mathbb{R}) \mid \det M \neq 0 \}$ "isomorphisms of vector spaces"



$\rightsquigarrow g_{ii}(x) = id \quad \forall x \in U_i$

$\rightsquigarrow g_{ij}(x) \cdot g_{ji}(x) = id \quad \forall x \in U_i \cap U_j$

$\rightsquigarrow g_{ij}(x) g_{jk}(x) g_{ki}(x) = id \quad \forall x \in U_i \cap U_j \cap U_k$

cocycle condition.

g_{ij} - gluing cocycles

$) \to U \times \mathbb{R}^k$

$)$

were $g_{ij} : U_i \cap U_j \to GL(k)$; general linear group

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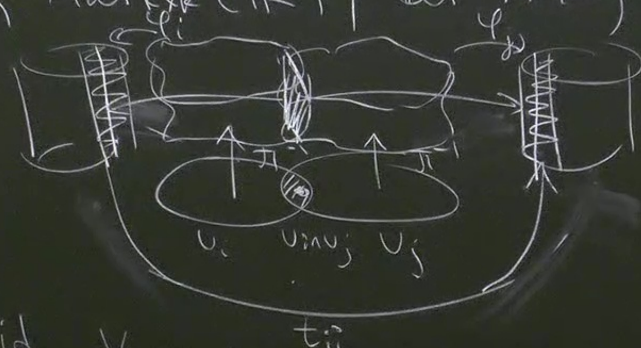


⇝ $g_{ii}(x) = id \quad \forall x \in U_i$

⇝ $g_{ij}(x) \cdot g_{ji}(x) = id \quad \forall x \in U_i \cap U_j$ $\qquad g_{ij}$ – gluing cocycles

⇝ $g_{ij}(x) g_{jk}(x) g_{ki}(x) = id \quad \forall x \in U_i \cap U_j \cap U_k$ ✓ cocycle condition.

· <u>Reconstruction teorem</u> One can reconstruct (smooth) vector bundle from its gluing cocycle.

- The sections of vector bundle $\Gamma(B,E) = \Gamma(E) \ni s: B \longrightarrow E$

s.t. $(\pi \circ s)(x) = x \quad \forall x \in B$

$s(x) \in F_x = \pi^{-1}(x)$

$\rightsquigarrow$ $\Gamma(B,E)$ is a vector space (over $\mathbb{R}$), $\vec{0}$ is given by zero section $s(x) \equiv 0$ $\forall x \in B$.

$\rightsquigarrow$ $\Gamma(B,E)$ has a multiplication by the smooth functions from $C^\infty(B,\mathbb{R})$

$$s \in \Gamma(B,E),\ f \in C^\infty(B,\mathbb{R}) \Rightarrow (f \cdot s)(x) = f(x)\, s(x)$$

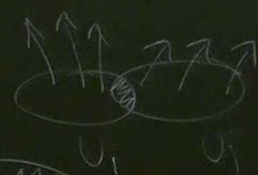
- The global sections can be glued from the local ones using g_{ij}.

Let $\{U_i\}, \{g_{ij}\}$; Local sections $s_i \in \Gamma(U_i, \underset{\substack{\text{ISO} \\ U_i \times F}}{\pi^{-1}(U_i)})$

Local sections $s_i \in \Gamma(U_i, \pi^{-1}(U_i))$ glue to the global one $\iff$ $s_i(x) = g_{ij}(x)\, s_j(x)$ $\forall\, i,j$ s.t. $U_i \cap U_j \neq \emptyset$, $\forall\, x \in U_i \cap U_j$

Remark We implicitly identify here $\pi^{-1}(U_i)$ with $U_i \times \mathbb{R}^k$, $s_i(x) \in \mathbb{R}^k$ with $\mathrm{proj}_2(\varphi_i \circ s_i)$

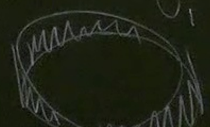
We will also do this in a following

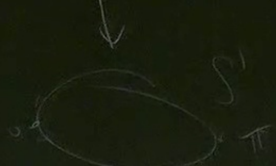


Example $R \to M$

$\downarrow$

B





s.t. $f(2\pi) = -f(0)$

$U_1 = (0, 2\pi)$

$U_2 = (-\varepsilon, \varepsilon)$

$U_1 = (0, 2\pi)$

$U_2 = (\pi, 3\pi)$

$s_i : U_i \longrightarrow \mathbb{R}$

$g_{12}\big|_{(0,\pi)} = 1$ $s_1\big|_{(0,\pi)} = s_2\big|_{(0,\pi)}$

$g_{12}\big|_{(\pi,2\pi)} = -1$ $s_1\big|_{(\pi,2\pi)} = -s_2\big|_{(\pi,2\pi)}$

∘ **Change of trivialization**

The gluing cocycle is not unique for a given vector bundle.

Let E has two trivializations (U_i, φ_i), (U_i, φ'_i)

$$\begin{tikzcd} U \times F \arrow[drr, "\mathrm{proj}_1"'] & & \pi^{-1}(U) \arrow[ll, "\varphi'_U"'] \arrow[rr, "\varphi_U"] \arrow[d, "\pi"] & & U \times F \arrow[dll, "\mathrm{proj}_1"] \\ & & U_i & & \end{tikzcd}$$

$\forall\ U$ there is a map $\varphi_U \circ (\varphi'_U)^{-1} : U \times \mathbb{R}^k \longrightarrow U \times \mathbb{R}^k$.

$$(x, v) \longmapsto (x, h_U(x) \cdot v)$$

$$h_U : U \longrightarrow GL(k)$$

$\varphi'_U \circ (\varphi_U)^{-1} : U \times \mathbb{R}^k \longrightarrow U \times \mathbb{R}^k$

$$(x, v) \longmapsto (x, h_U(x)^{-1} v)$$

It is straightforward to check: (*) $\boxed{g'_{ij} = (h_i)^{-1} g_{ij} \cdot h_j}$ over $U_i \cap U_j$

- such transformations are called gauge transformations.

Such (*) cocycles are called equivalent.

Th Two vector bundles over same base are isomorphic $\Longleftrightarrow$ The corresponding gluing cocycles are equivalent.

Remark Even though condition for gluing cocycles on $U_i \cap U_j$ & gauge transf. look similar, do not confuse them!

Remark The local sections transform as: $s_U(x) = h_U(x) \cdot s'_U(x)$.

More examples

- Tangent bundle: $\begin{array}{c} \mathbb{R}^n \longrightarrow TM \\ \downarrow \\ M \end{array}$ $\quad \Gamma(M, TM) = \mathrm{Vect}(M)$

$$s_U(x) = X_U^i(x)\frac{\partial}{\partial x^i} = (X_U^1, \dots, X_U^n) \in \mathbb{R}^n = F$$

$$s_V(y) = X_V^i(y)\frac{\partial}{\partial y^i} = (X_V^1, \dots, X_V^n) \in \mathbb{R}^n = F$$

$$\Rightarrow X_V^i = \frac{\partial y^i}{\partial x^j} X_U^j$$

For the gauge transform $x \to x'$:

$$(X')^i = \frac{\partial (x')^i}{\partial x^j} X^j$$

$$\left.\begin{aligned} s_U(x) &= \omega_{U,i}(x)\, dx^i \\ s_V(y) &= \omega_{V,i}(y)\, dy^i \end{aligned}\right\} \Rightarrow \omega_{V,i} = \underbrace{\frac{\partial x^j}{\partial y^i}}_{g_{VU}} \omega_{U,j}$$

$\Rightarrow$ the gluing cocycles are inverses of each other

$\Rightarrow$ In most of non-trivial cases $TM \not\cong T^*M$.

Remark: Čech cohomology construct invariants of bundles looking for cocycles.

We also have other bundles on M i.e. $\Lambda^k T^* M$, $\Omega^k(M) = \Gamma(M, \Lambda^k T^* M)$

$$\omega_{V, i_1 \ldots i_k} = \frac{\partial x^{j_1}}{\partial y^{i_1}} \cdots \frac{\partial x^{j_k}}{\partial y^{i_k}} \omega_{U, j_1 \ldots j_k}.$$

Obs Many bundles can be constructed using the same gluing cocycles!

Q) Can we forget about vector spaces, and talk only about gluing cocycles? Yes, we can! This is called principal bundles!

M)

- Lie group G

$\rightsquigarrow$ G is a smooth manifold

$\rightsquigarrow$ G is a group:

- There is $m: G \times G \to G$ which is assoc., smooth
- $\exists\, e \in G$: $e \cdot g = g \cdot e = g \ \forall\, g \in G$
- $\exists\, (\cdot)^{-1}: G \to G$ (smooth) $(g)^{-1} \cdot g = g \cdot g^{-1} = e$

Examples

- $GL(k) = \{ M \in \mathrm{Mat}_{k\times k}(\mathbb{R}) \mid \det M \neq 0 \}$
- $SO(k) = \{ M \in \mathrm{Mat}_{k\times k}(\mathbb{R}) \mid M^T M = \mathrm{id} \quad \text{or} \quad M^{-1} = M^T \}$
- $SO(2) = U(1) = \{ z \in \mathbb{C} \mid |z|^2 = 1 \}$

  $z^{-1} = z^*$
- Unipoten matrices $\mathbb{U}_n = \{ M \in \mathrm{Mat}_{n\times n}(\mathbb{R}) \mid M = \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \}$

  $\dim \mathbb{U}_n = \frac{n(n-1)}{2}$

  $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix}$