Title: Lecture - Mathematical Physics, PHYS 777-

Speakers: Mykola Semenyakin

Collection/Series: Mathematical Physics (Core), PHYS 777-, January 6 - February 5, 2025

Subject: Mathematical physics

Date: January 28, 2025 - 9:00 AM

URL: https://pirsa.org/25010010

Recap Fiber bundle: it is a tuple

$$\begin{tikzcd} F \arrow[r] & E \\ & B \arrow[from=u, "\pi"] \end{tikzcd}$$

fiber: F

total space: E

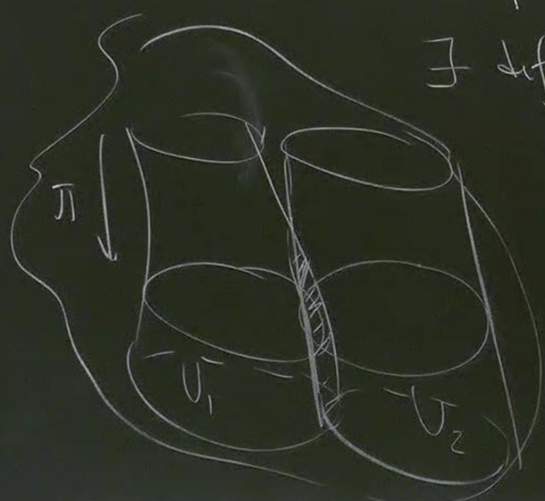
projection map: π

base: B

π - continuous surjection,
statisfying local triviality condition.

$\forall\, x \in B$ $\exists$ open neiborhoud of x $U \subseteq B$ such that

$\exists$ diffeomorphism $\varphi : \pi^{-1}(U) \longrightarrow U \times F$ st.



$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\varphi"] \arrow[d, "\pi"'] & U \times F \arrow[dl, "proj"] \\ U & \end{tikzcd}$$

$$proj_1 \circ \varphi = \pi$$

space
tion
p
e

- Set of all (U_i, φ_i) is called local trivialisation
- Fiber over every point $\pi^{-1}(x) = F_x = F$ is the same
- Local trivialisation might serve as an alternative way to define fiber bundle

transition functions: $t_{ij} = \varphi_i \circ \varphi_j^{-1} : U_i \cap U_j \times F \longrightarrow U_i \cap U_j \times F$

$$(x, \xi) \longmapsto (x, t_{ij}(x, \xi))$$

$\rightsquigarrow t_{ii}(x, \xi) = \xi \quad \forall x \in U_i$

$\rightsquigarrow t_{ij}(x, t_{ji}(x, \xi)) = \xi \quad \forall x \in U_i \cap U_j$

$\rightsquigarrow t_{ij}(x, t_{jk}(x, \xi)) = t_{ik}(x, \xi) \quad \forall x \in U_i \cap U_j \cap U_k$

$\leftarrow$ cocycle condition.

Alternatively, we can take this as a definition

• Section of a bundle: map $S : B \longrightarrow E$ s.t. $\pi \circ S = id$

i.e it acts $x \longmapsto S(x) \in F_x$

We will denote the set of all sections of E over B by $\Gamma(E) = \Gamma(B,E)$

• If $F_1 \to E_1 \xrightarrow{\pi_1} B_1$, $F_2 \to E_2 \xrightarrow{\pi_2} B_2$, a bundle map is a pair:

$f : E_1 \longrightarrow E_2$

$g : B_1 \longrightarrow B_2$ s.t.

$$\begin{tikzcd} E_1 \arrow[r, "f"] \arrow[d, "\pi_1"] & E_2 \arrow[d, "\pi_2"] \\ B_1 \arrow[r, "g"] & B_2 \end{tikzcd}$$

$\pi_2 \circ f = g \circ \pi_1$

• if $B_1 = B_2$

$\Downarrow$

$g = id$

$$\begin{tikzcd} E_1 \arrow[rr, "f"] \arrow[dr, "\pi_1"'] & & E_2 \arrow[dl, "\pi_2"] \\ & B & \end{tikzcd}$$

$\pi_1 = \pi_2 \circ f$

Ex

Extended set of examples

- Trivial bundle $E = B \times F$, sections $\Gamma(B, B\times F) = C^\infty(B, F)$

  e.g. cylinder $E = S^1 \times [0,1]$, $\Gamma(E) = C^\infty(S^1, [0,1])$

  $E = S^1 \times \mathbb{R}$, $\Gamma(E) = C^\infty(S^1, \mathbb{R})$

- Möbius strip M: $[0,1] \to M \to S^1$, $\mathbb{R} \to M' \to S^1$

$\Gamma(M) = \{ f \in C^\infty([0,2\pi], [0,1]) ; f(2\pi) = 1 - f(0) \}$

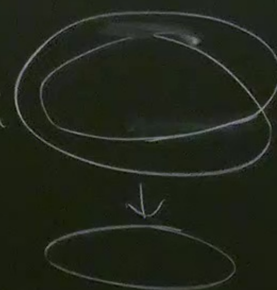
$\Gamma(M') = \{ f \in C^\infty([0,2\pi], \mathbb{R}), f(2\pi) = -f(0) \}$

Notice that in the cases of $S^1 \times \mathbb{R}$ & M' the fiber $F = \mathbb{R}$ is a vector space

Covers – the fiber is a discrete set (collection of disjoint points)
often in these, $\Gamma(B, E) = \emptyset$

$$S^1 = \{|z| = 1 \subset \mathbb{C}\} \xrightarrow{w = z^2} S^1 = \{|w| = 1 \subset \mathbb{C}\}$$

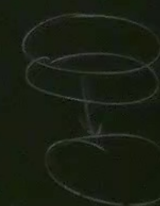
⇝ Discrete version of Möbius strip



$$\mathbb{Z}/2\mathbb{Z} \to S^1 \xrightarrow[2:1]{\pi} S^1$$

$$\Gamma(S^1) = \emptyset$$

$$\mathbb{Z}/2 \to S^1 \cup S^1 \to S^1$$



⇝ Universal covers.
such covers that E is contractible

$$\mathbb{Z}^2 \to \mathbb{R}^2 \to T^2$$

Universal property: for any other cover E over the same B

$\exists!$ map of bundles $E_{univ} \to E$

Vector bundles - when the fiber is a vector space, and transition maps respect this

everything is similar to the general case, except of extra

- $\pi^{-1}(x) = \mathbb{R}^k = F$ as a vector space
- $\varphi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ is such that,

  $\varphi|_{\pi^{-1}(x)} : \pi^{-1}(x) \to \{x\} \times \mathbb{R}^k$ is a linear isomorphism of vect spaces $\forall x$

k is called rank of E

$k=1$ is called line bundle

- In a local trivialization one has $t_{ij} = \varphi_i \circ \varphi_j^{-1}$.

$$U_i \cap U_j \times \mathbb{R}^k \to U_i \cap U_j \times \mathbb{R}^k$$

$$(x, v) \mapsto (x, \underbrace{g_{ij}(x) v}_{t_{ij}(x,v)})$$

$$g_{ij} : U_i \cap U_j \to GL(k)$$

Here $GL(k) = GL_k = GL(\mathbb{R}^k)$ is called a general linear group.

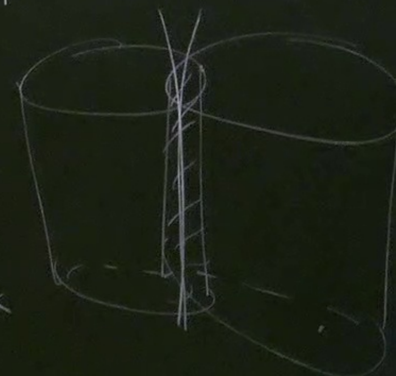
and it is defined by $GL(k) = \{ M \in Mat_{k\times k}(\mathbb{R}) \mid \det M \neq 0 \}$

These g_{ij} satisfy:

$\rightsquigarrow g_{ii}(x) = id \quad \forall x \in U_i$

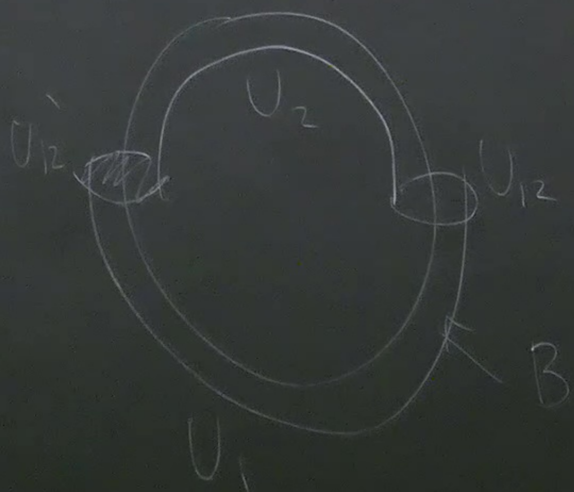
$\rightsquigarrow g_{ij}(x) \circ g_{ji}(x) = id \quad \forall x \in U_i \cap U_j$

$\rightsquigarrow g_{ij}(x) \cdot g_{jk}(x) \cdot g_{ki}(x) = id \quad \forall x \in U_i \cap U_j \cap U_k$



Remark: The group $GL(k)$ is called structural group. In principle, if there is extra structure on fibers (scalar product, skew-symmetric form) $GL(k)$ can be reduced to $G \subset GL(k)$ (e.g. $O(k)$, $Sp(k)$)

$g_{ij} : U_i \cap U_j \to GL(k)$



$\mathbb{Z}/2\mathbb{Z} = O(1) \subset GL(1) = \mathbb{R}^*$

$\{+1, -1\}$

$g_{12}|_{U_{12}'} = +1 \in O(1)$

$g_{12}|_{U_{12}} = -1$

$GL(k)$ can be reduced to $G \subset GL(k)$ (e.g. $O(k)$, $Sp(k)$)

Tangent and cotangent bundles: $(U, \varphi = (x^1, \ldots, x^n))$

$(V, \psi = (y^1, \ldots, y^n))$

$$X = X^i \frac{\partial}{\partial x^i} = (X^i)' \frac{\partial}{\partial y^i} \Rightarrow \underbrace{(X^j)'(x)}_{= S_V(x)} = \underbrace{\frac{\partial y^j}{\partial x^i}(x)}_{= g_{VU}(x)} \underbrace{X^i(x)}_{= S_U(x)}$$

$$\begin{array}{ccc} \mathbb{R}^n & \longrightarrow & TB \\ & & \downarrow \\ & & B \end{array}$$

$$\boxed{\Gamma(B, TB) = \mathrm{Vect}(B)}$$

Remarks

- Space of sections of the vector bundle is a vector space.
- $\Gamma(B, E)$ is a module over $C^\infty(B, \mathbb{R})$