Title: Lecture - Mathematical Physics, PHYS 777-

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Cartan calculus ⟶ Stokes theorem

Operations $d, \iota_v, \mathcal{L}_v$

Integral over bdry = int over interior

Integration of forms

Use partition of 1 in charts.

k - forms

Skew - symmetric tensors

1 - form

(smooth) assignment of a covector to every pt.

Big recap: What we

Tangent space T_xM

Differentiations of the smooth funct. at the point x

Vector fields

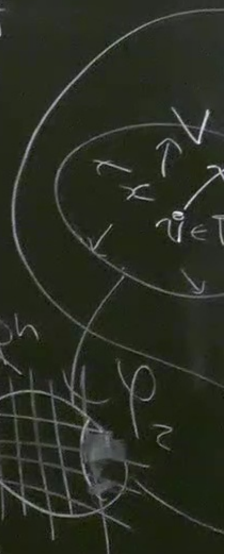
Smooth assignment of a vector to every pt. $\mathbb{R}^n$

Cotangent space T^*_xM

Dual space to T_xM

Tensors & tensor fields

Multilinear comb. of vect. and covect.



tor.

charts.

pt.

or fields

comb.

and covect.

Big recap: What we have learned so far?

General a

Tangent space T_xM

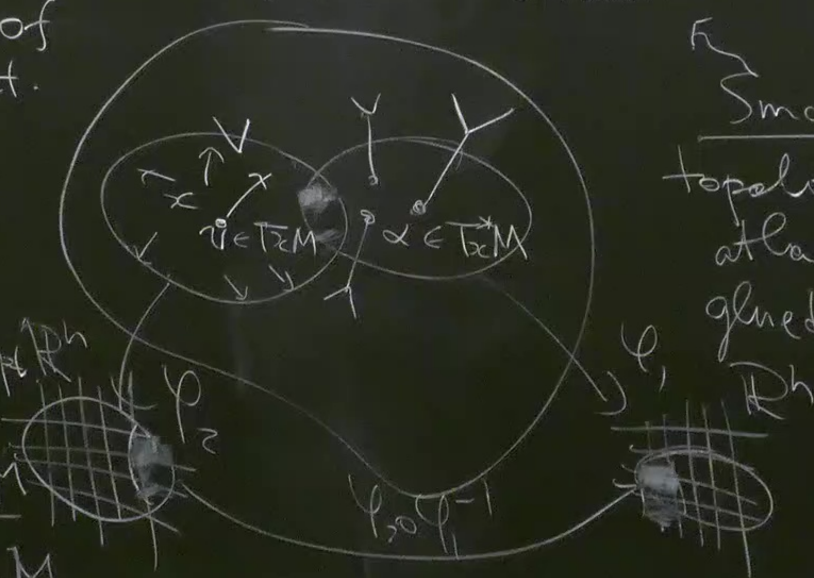
Differentiations of the smooth funct. at the point x

Vector fields

Smooth assignment of a vector to every pt.

Cotangent space T_x^*M

Dual space to T_xM



Smooth functions

Smooth in every chart.

Smooth manifold

topological space with atlas of charts in $\mathbb{R}^n$ glued together by C^∞ funct.

?

tions

very chart.

ooth manifold

ogical space with
s of charts in $\mathbb{R}^n$
d together by C^∞ funct.

General attitude it is often convenient look not only to spaces, but rather to maps between spaces.

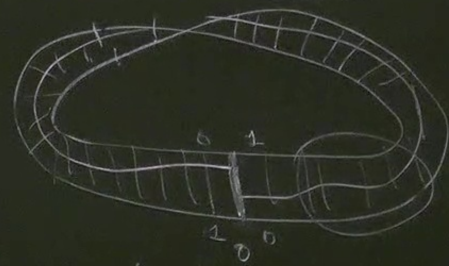
Approaches
- Local to global: Define in charts
  Check consistency
- Global to local: Give "geometric" defin.
  make it cord in charts

Application
- calculus on manifold
- classical mechanics

Bundles on manifolds

Generalize a notion of the "space, which functions are acting to"

Some picture | Möbius strip M vs cylinder $S^1 \times [0,1]$



Both can be used to construct space of functions $f: S^1 \longrightarrow [0,1]$

$\rightsquigarrow$ For cylinder – it is fine! $f(2\pi) = f(0)$

$\rightsquigarrow$ For Möbius to have a smooth funct. $f(2\pi) = -f(0)$

Dictionary

Möbius strip here $\rightsquigarrow$ total space E of the bundle

Cylinder $\rightsquigarrow$ trivial bundle

Interval $[0,1]$ $\rightsquigarrow$ fibre F of the bundle.

Circle S^1 $\rightsquigarrow$ base B of the bundle.

"Functions $f: S^1 \to [0,1]$" $\rightsquigarrow$ sections s of the bundle.

"twist $[0,1] \to [0,1]$" $\rightsquigarrow$ gluing cocycle.
transition function. f_{ij}, g_{ij}

Def Fiber bundle (E, B, π, F)

- E, B, F – topological spaces
  - E – total space
  - B – base
  - F – fibre.

$$F \hookrightarrow E \xrightarrow{\pi} B$$

$$\begin{tikzcd} F \arrow[r] & E \arrow[d, "\pi"] \\ & B \end{tikzcd}$$

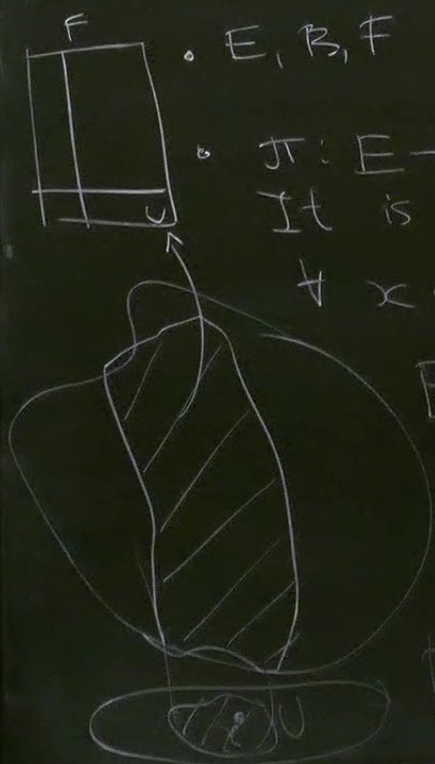
- $\pi : E \to B$ continuous surjection, called projection map

It is required to satisfy local triviality condition

$\forall\ x \in B \ \exists\ U$ – open $\subseteq B$ of x such that

$\exists$ exist homeomorphism $\varphi : \pi^{-1}(U) \longrightarrow U \times F$

(invertible, cont. func.)



E

π

B

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\varphi"] \arrow[d, "\pi"] & U \times F \arrow[dl, "\text{proj. 1.}"] \\ U & \end{tikzcd}$$

– should be commutative

Remarks

- Note that $\pi^{-1}(p) = F \quad \forall\, p \in B$.
- Set of all (U_i, φ_i) are called local trivialization.
- Local trivialization can be used to patchwork fibre bundle.

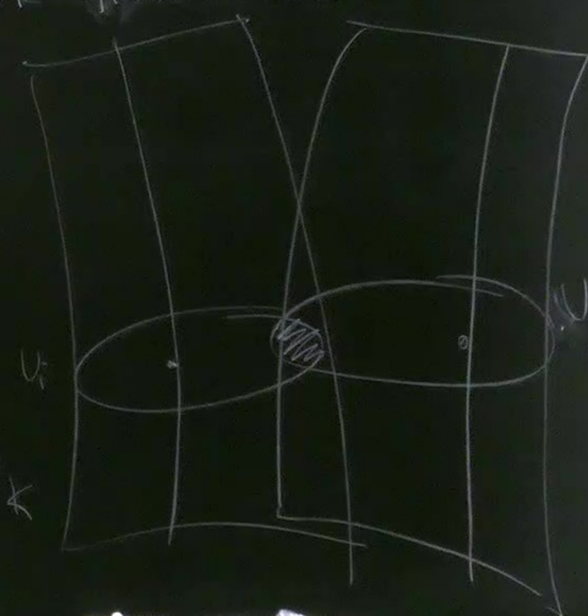
$$\varphi_{ij} = \varphi_i \circ \varphi_j^{-1} : (U_i \cap U_j) \times F \longrightarrow (U_i \cap U_j) \times F$$

$$(x, \xi) \longmapsto (x, t_{ij}(x, \xi))$$

transition function

- $t_{ii}(x, \xi) = \xi$
- $t_{ij}(x, t_{ji}(x, \xi)) = \xi$
- $t_{ik}(x, \xi) = t_{ij}(x, t_{jk}(x, \xi))$ on $U_i \cap U_j \cap U_k$

cocycle condition



- Section of fiber bundle: map $s: B \to E$ s.t. $(\pi \circ s)(x) = x \quad \forall x \in B$

Sometimes, it is impossible to have global sections

In these cases we still have local sections $s_i : U_i \to \pi^{-1}(U_i)$

- Smooth fibre bundle – all the same, but smooth

Example Trivial bundles. Sections: functions $B \to F$

$$E = F \times B$$

$$\begin{tikzcd} {[0,1]} \arrow[r] & {[0,1] \times S^1} \arrow[d] \\ & S^1 \end{tikzcd} \qquad \begin{tikzcd} S^1 \arrow[r] & S^1 \times S^1 \arrow[d] \\ & S^1 \end{tikzcd}$$

• Möbius strip: $[0,1] \to M$
$$\begin{array}{c} M \\ \downarrow \\ S^1 \end{array}$$

Sections: functions $f : [0,2\pi] \to [0,1]$

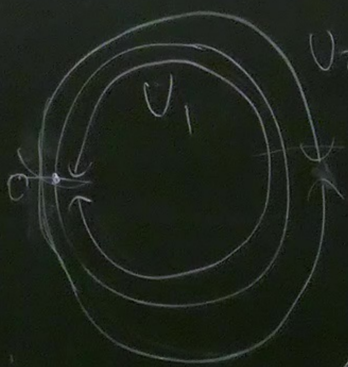
$f(2\pi) = -f(0)$

$U_1 = (0, 2\pi)$

$U_2 = (\pi, 3\pi)$

$\pi^{-1}(U_2)$

In charts:



the transition map: $t(x,\xi) = \xi$ if $x \in (0,\pi)$

$t(x,\xi) = 1-\xi$ if $x \in (\pi, 2\pi)$

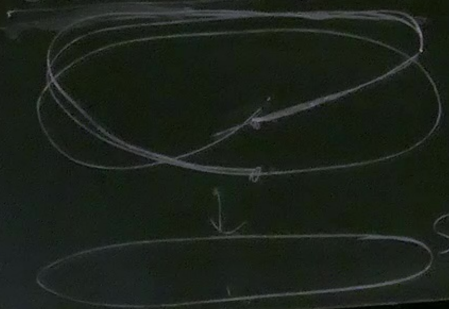
$$\begin{array}{c} \{0,1\} \to E \\ \downarrow \\ S^1 \end{array}$$

It has no global sections

There are local sections (2) and they are constant.

$\boxed{\text{Such bundles with discrete fibre are called covers.}}$

Close relative



S^1

$2 \to 1$

S^1