Title: Lecture - Mathematical Physics, PHYS 777-

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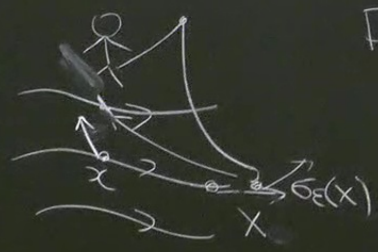
Recap Last time we defined Lie derivatives $\mathcal{L}_X$ using flows.

Flow (associated with $X \in \mathrm{Vect}(M)$).



$$\sigma : \underset{\ni t}{\mathbb{R}} \times M \longrightarrow M \qquad (\sigma^\mu = \varphi^\mu \circ \sigma)$$

$$X = X^\mu \frac{\partial}{\partial x^\mu}$$

$$\begin{cases} \dfrac{d\,\sigma^\mu(t, x_0)}{dt} = X^\mu(\sigma(t, x_0)) \\ \sigma(0, x_0) = x_0 \end{cases}$$

In other terms $\sigma^\mu(t, x_0)$ at fixed x_0 - curve tangent to X at every pt.

We will use notation $\sigma_t = \sigma(t, \cdot) : M \longrightarrow M$

The

The action of Lie derivative: $\mathcal{L}_X : T^{\otimes p} \otimes (T^*)^{\otimes q} M \to T^{\otimes p} \otimes (T^*)^{q} M$.

(p,q)

$(0,0)$: $\mathcal{L}_X f\big|_x = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left(f\big|_{\sigma_\varepsilon(x)} - f\big|_x \right) = \lim_{\varepsilon \to 0} \frac{f(x^\mu + \varepsilon X^\mu) - f(x^\mu)}{\varepsilon} = X^\mu \frac{\partial f}{\partial x^\mu}\Big|_x = (Xf)(x)$

$(1,0)$: $\mathcal{L}_X Y\big|_x = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left((\sigma_{-\varepsilon})_* Y\big|_{\sigma_\varepsilon(x)} - Y\big|_x \right) = \left(X^\mu \partial_\mu Y^\nu - Y^\mu \partial_\mu X^\nu \right) \frac{\partial}{\partial x^\nu}\Big|_x$

$1 + \varepsilon M$

$\frac{1}{1+\varepsilon M} \sim 1 - \varepsilon M$

$(0,1)$ $\mathcal{L}_X \omega\big|_x = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left((\sigma_\varepsilon)^* \omega\big|_{\sigma_\varepsilon(x)} - \omega\big|_x \right) = \left(X^\mu \partial_\mu \omega_\nu + \partial_\nu X^\mu \omega_\mu \right) dx^\nu\big|_x$

In general, for (p,q) tensors:

$$\mathcal{L}_X(Y_1\otimes\dots\otimes Y_p\otimes\omega_1\otimes\dots\otimes\omega_q)\big|_x =$$

$$= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\Big((\sigma_{-\varepsilon})_* Y_1\big|_{\sigma_\varepsilon(x)}\otimes(\sigma_{-\varepsilon})_* Y_2\big|_{\sigma_\varepsilon(x)}\otimes\dots\otimes(\sigma_\varepsilon)^*\omega_1\big|_{\sigma_\varepsilon(x)}\otimes\dots\otimes(\sigma_\varepsilon^*)\omega_q\big|_{\sigma_\varepsilon(x)} - Y_1\big|_x\otimes\dots\otimes Y_p\big|_x\otimes\omega_1\big|_x\otimes\dots\otimes\omega_q\big|_x\Big)\Big|_x =$$

$$= \sum_{i=1}^{p} Y_1\otimes\dots\otimes\mathcal{L}_X Y_i\otimes\dots\otimes\omega_1\otimes\dots\otimes\omega_q\big|_x + \sum_{j=1}^{q} Y_1\otimes\dots\otimes Y_1\otimes\omega_1\otimes\dots\otimes\mathcal{L}_X\omega_j\otimes\dots\big|_x$$

$\mathcal{L}_X$ is a derivation on tensors.

Inste

Instead of computing in coords., one can often use Cartan calculus.

- $\boxed{\mathcal{L}_X Y = [X,Y]}$, where $[X,Y]f = X(Yf) - Y(Xf)$

$[\ ,\ ]$ turns $\mathrm{Vect}(M)$ to Lie algebra.

→ $[\ ,\ ]: \mathrm{Vect}(M) \otimes \mathrm{Vect}(M) \to \mathrm{Vect}(M)$

→ $[X,Y] = -[Y,X]$

→ Satisfies Jacobi identity

$$[X[Y,Z]] + [Y[Z,X]] + [Z[X,Y]] = 0$$

- $\mathcal{L}_{fX} Y = f \cdot [X,Y] - Y(f) \cdot X$
- $\mathcal{L}_X (fY) = f[X,Y] + X(f) \cdot Y$

<u>Diff. forms</u> $i_V \; \Lambda^k T^*M \to \Lambda^{k-1} T^*M$ - interior derivative
$V \in \mathrm{Vect}(M)$ substitution of vect. fields.

$$(i_V\omega)(U_1,\dots,U_{k-1}) = \omega(V, U_1,\dots,U_{k-1})$$

(for convention $dx^{i_1}\wedge\dots\wedge dx^{i_k}(\partial_{i_1},\dots,\partial_{i_k}) = 1$)

Alternatively, there are defining prop:

→ i_V is <u>odd derivation</u> $i_V(\omega_1\wedge\omega_2) = i_V\omega_1\wedge\omega_2 + (-1)^{\deg\omega_1}\omega_1\wedge(i_V\omega_2)$

→ $\boxed{\{i_V, i_V\} = 0}$ $(\Rightarrow (i_V)^2 = 0)$, $[i_V, f] = 0$

→ On 1-forms: $i_V\omega = \omega(V)$

Cartan magic formula $\boxed{\mathcal{L}_V\omega = \{i_V, d\}\omega}$ $\mathcal{L}_V = \{i_V, d\}$

1-form

$\cdot \{i_V, d\}\omega = i_V(d\omega_\mu \wedge dx^\mu) + d(\omega_\mu\, i_V dx^\mu) = i_V\left(\frac{\partial\omega_\mu}{\partial x^\nu} dx^\nu \wedge dx^\mu\right) + d(\omega_\mu V^\mu) =$

$= \frac{\partial\omega_\mu}{\partial x^\nu} V^\nu dx^\mu - \cancel{\frac{\partial\omega_\mu}{\partial x^\nu}} V^\mu dx^\nu + \cancel{\frac{\partial\omega_\mu}{\partial x^\nu} V^\mu} dx^\nu + \omega_\mu \frac{\partial V^\mu}{\partial x^\nu} dx^\nu =$

$= \left(\frac{\partial\omega_\mu}{\partial x^\nu} V^\nu + \omega_\nu \frac{\partial V^\nu}{\partial x^\mu}\right) dx^\mu = \mathcal{L}_V\omega$

$\cdot \{i_V, d\} f = i_V df + \cancel{d(i_V f)} = \underset{df(V)}{V(f)} = \mathcal{L}_V f$

We can check that $\{i_v, \omega\}$ is an <u>even</u> derivation on forms.

$$\{i_v, d\}(\omega_1 \wedge \omega_2) = (\text{8 terms, 4 are cancelling}) = \{i_v, d\}\omega_1 \wedge \omega_2 + \omega_1 \wedge \{i_v, d\}\omega_2$$

$\Rightarrow$ Cartan magic formula (just decompose $\omega^{(k)} = \omega_1^{(1)} \wedge \dots \wedge \omega_k^{(1)}$)

$$\{i_v, d\}\, \omega_1 \wedge \dots \wedge \omega_k = \sum_{i=1}^{k} \omega_1 \wedge \dots \wedge \{i_v, d\}\omega_i \wedge \dots \wedge \omega_k =$$

$$= \sum_{i=1}^{k} \omega_1 \wedge \dots \wedge \mathcal{L}_v \omega_i \wedge \dots \wedge \omega_k = \mathcal{L}_v(\omega_1 \wedge \dots \wedge \omega_k)$$

There are more formulas of this kind:

- $[d, \mathcal{L}_v] = 0$
- $[\mathcal{L}_u, \mathcal{L}_v] = \mathcal{L}_{[u,v]}$
- $[\mathcal{L}_u, i_v] = i_{[u,v]}$

Remark:

- $[\text{even}, \text{even}] = \text{even}$
- $[\text{even}, \text{odd}] = \text{odd}$
- $\{\text{odd}, \text{odd}\} = \text{even}$

Example (classical mechanics on phase space)

Phase space: manifold M with $\{\,,\,\}: C^\infty(M,\mathbb{R}) \otimes C^\infty(M,\mathbb{R}) \longrightarrow C^\infty(M,\mathbb{R})$

- $\{f,g\} = -\{g,f\}$
- $\{f_1\{f_2,f_3\}\} + \{f_2\{f_3,f_1\}\} + \{f_3\{f_1,f_2\}\} = 0$
- $\{f\cdot g,h\} = f\{g,h\} + g\{f,h\}$

$\Rightarrow$ defines Lie alg str. on $C^\infty(M,\mathbb{R})$

Geometrically, it is represented by Poisson bivector $\Pi \in \Lambda^2 TM$

$$\Pi = \frac{1}{2}\Pi^{ij}\frac{\partial}{\partial x^i}\wedge\frac{\partial}{\partial x^j}$$

(polyvectors).

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Equations of motion: $\dfrac{df(x(t))}{dt} = \{H,f\}$.

$C^\infty(M,\mathbb{R})$

$\rightsquigarrow$ flow under vector field

$X_H = \{H, \bullet\}$ — Hamiltonian vector field

We can dualize this picture, under assumption:

$$\nexists\ C \in C^\infty(M,\mathbb{R}) : \{C, \cdot\} = 0 \quad (\text{Casimir functions})$$

Under this assump we can "invert" $\Pi \in \Lambda^2 TM \simeq \mathrm{Hom}(T^*M, TM)$

inverse is called $\Pi \circ \omega = \omega \circ \Pi = id$

symplectic form $\omega \in \mathrm{Hom}(TM, T^*M) \simeq \Lambda^2 T^*M$

Condition that Π satisfies Jacobi identity $\Rightarrow d\omega = 0$ $\quad \partial_i \omega_{jk}\, dx^i \wedge dx^j \wedge dx^k = 0$

Hamiltonian vector fields: $\boxed{dH = -i_{X_H}\omega = -\omega(X_H, \cdot)}$

$$\{H, \cdot\} = \omega^{-1}(dH) = -X_H$$