Title: Lecture - Mathematical Physics, PHYS 777-

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Subject: Mathematical physics

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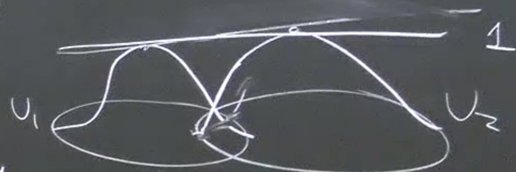
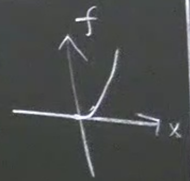
Recap We want to integrate n-form $\omega \in \Lambda^n T^*M$ over $i : N \hookrightarrow M$

- $\bigcup_{j\in I} U_j = N$ (locally finite), $R_j \in C^\infty(N, \mathbb{R})$
  - $\rightarrow R_j = 0$ outside of U_j
  - $\rightarrow 0 \leq R_j \leq 1$
  - $\rightarrow \sum_{j\in I} R_j(p) = 1 \;\; \forall \; p \in N$

  } Partition of unity



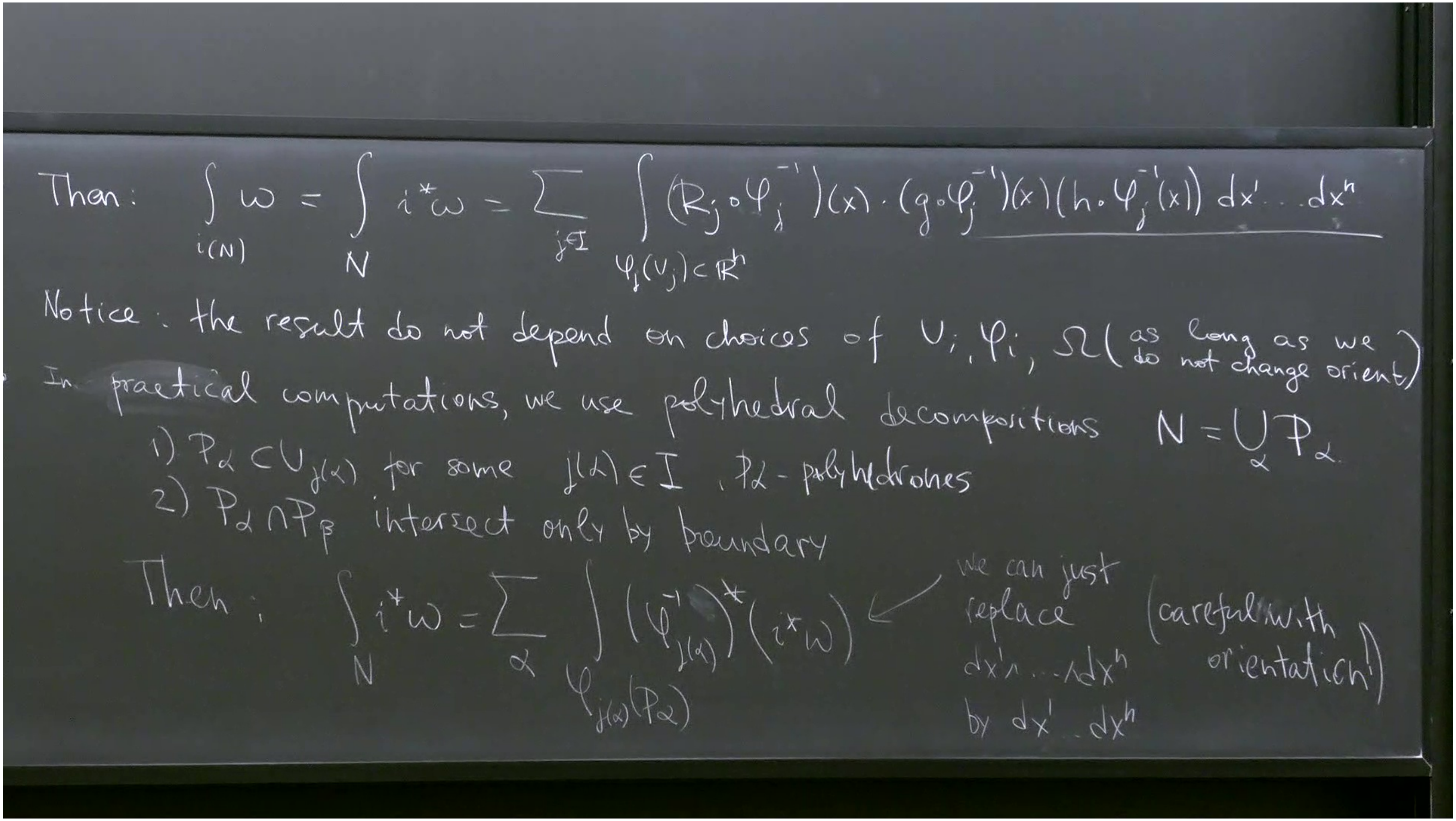
$$f = \begin{cases} e^{-\frac{1}{x}}, & x > 0 \\ 0, & x < 0 \end{cases}$$



- $\Omega = h \, dx^1 \wedge \ldots \wedge dx^n \in \Lambda^n T^*N$ - "orientation" on N, i.e. s.t. $h(x) > 0$

$\rightsquigarrow$ every other top form $\omega \in \Lambda^n T^*N$ can be written as $\omega = g \cdot \Omega$

$g \in C^\infty(N, \mathbb{R})$



Then: $\int_{i(N)} \omega = \int_N i^*\omega = \sum_{j\in I} \int_{\varphi_j(U_j)\subset\mathbb{R}^n} (R_j\circ\varphi_j^{-1})(x)\cdot(g\circ\varphi_j^{-1})(x)\underline{(h\circ\varphi_j^{-1}(x))\,dx^1\ldots dx^n}$

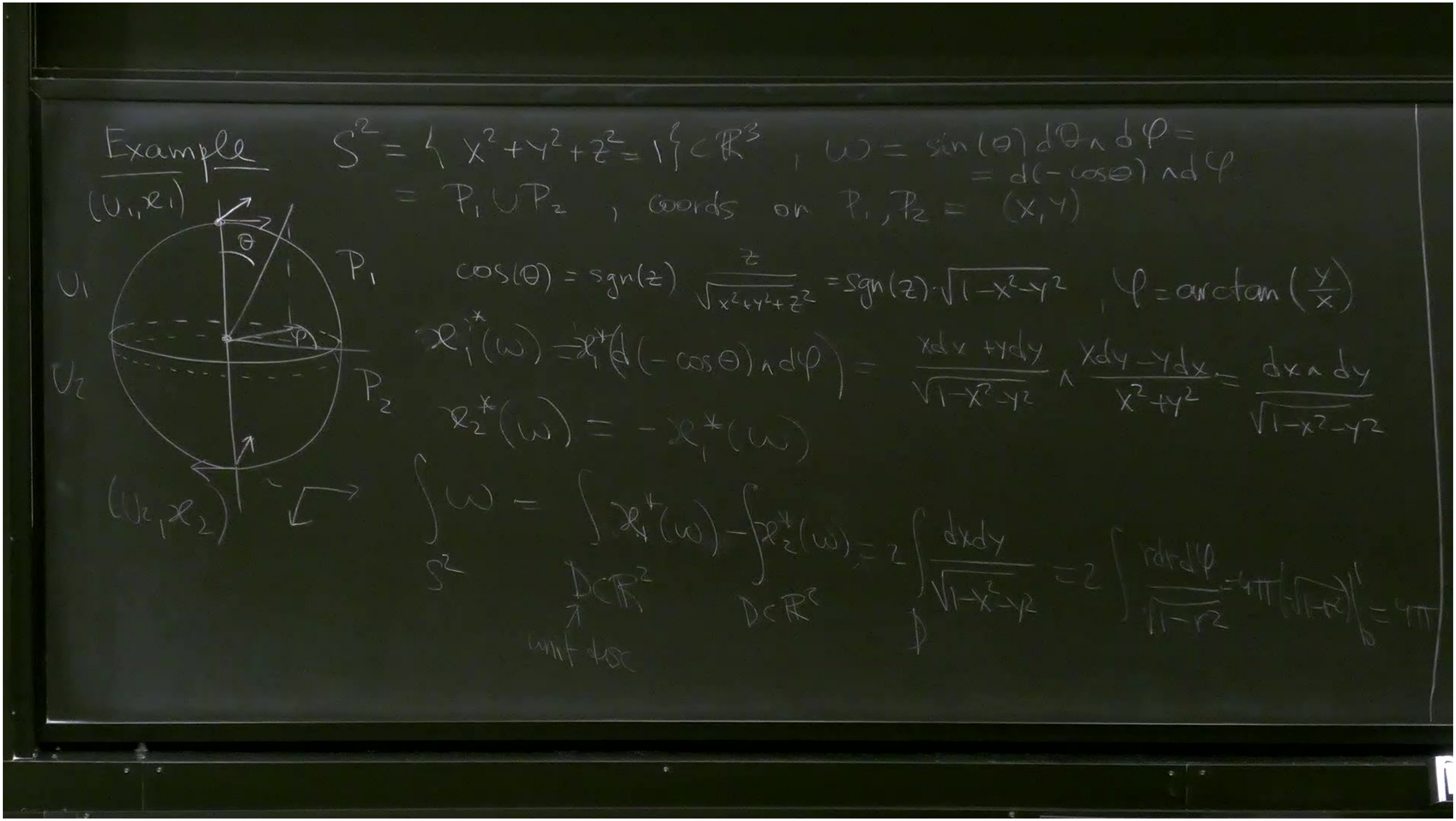
Notice: the result do not depend on choices of U_i, φ_i, Ω (as long as we do not change orient.)

In practical computations, we use polyhedral decompositions $N = \bigcup_\alpha P_\alpha$.

1) $P_\alpha \subset U_{j(\alpha)}$ for some $j(\alpha)\in I$, P_α - polyhedrones

2) $P_\alpha \cap P_\beta$ intersect only by boundary

Then: $\int_N i^*\omega = \sum_\alpha \int_{\varphi_{j(\alpha)}(P_\alpha)} \left(\varphi_{j(\alpha)}^{-1}\right)^*(i^*\omega)$ ← we can just replace $dx^1\wedge\ldots\wedge dx^n$ by $dx^1\ldots dx^n$ (careful with orientation!)



Example

$$S^2 = \{ x^2+y^2+z^2 = 1 \} \subset \mathbb{R}^3 \quad , \quad \omega = \sin(\theta)\, d\theta \wedge d\varphi = $$
$$= d(-\cos\theta) \wedge d\varphi$$

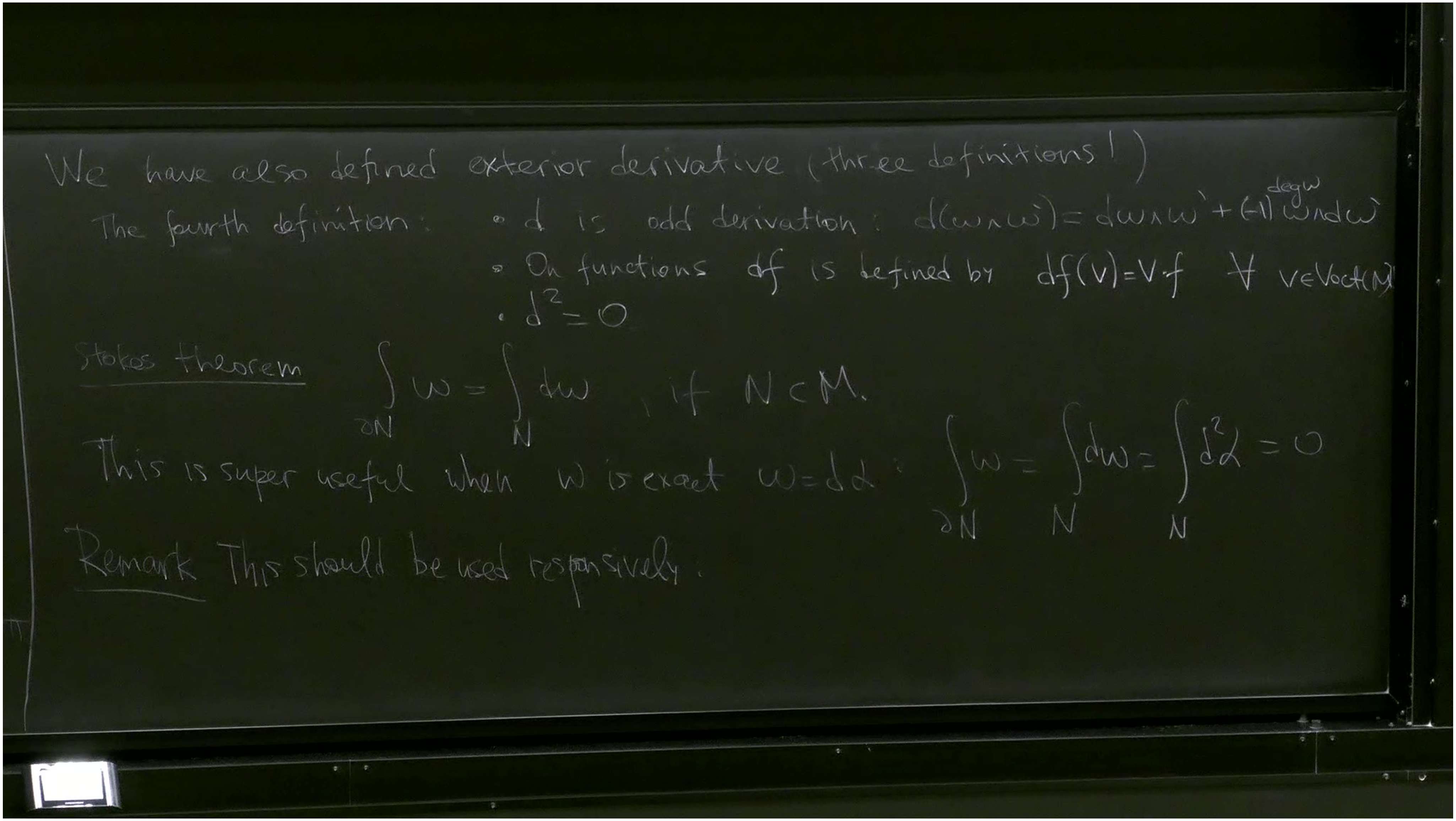
$= P_1 \cup P_2$, coords on $P_1, P_2 = (x, y)$

$$\cos(\theta) = \mathrm{sgn}(z) \frac{z}{\sqrt{x^2+y^2+z^2}} = \mathrm{sgn}(z) \cdot \sqrt{1-x^2-y^2} \quad , \quad \varphi = \arctan\left(\frac{y}{x}\right)$$

$$\chi_1^*(\omega) = \chi_1^*\big(d(-\cos\theta) \wedge d\varphi\big) = \frac{x\,dx + y\,dy}{\sqrt{1-x^2-y^2}} \wedge \frac{x\,dy - y\,dx}{x^2+y^2} = \frac{dx \wedge dy}{\sqrt{1-x^2-y^2}}$$

$$\chi_2^*(\omega) = -\chi_1^*(\omega)$$

$$\int_{S^2} \omega = \int_{\underset{\substack{\uparrow \\ \text{unit disc}}}{D \subset \mathbb{R}^2}} \chi_1^*(\omega) - \int_{D \subset \mathbb{R}^2} \chi_2^*(\omega) = 2 \int_D \frac{dx\,dy}{\sqrt{1-x^2-y^2}} = 2 \int \frac{r\,dr\,d\varphi}{\sqrt{1-r^2}} = 4\pi \left(-\sqrt{1-r^2}\right)\Big|_0^1 = 4\pi$$



We have also defined exterior derivative (three definitions!)

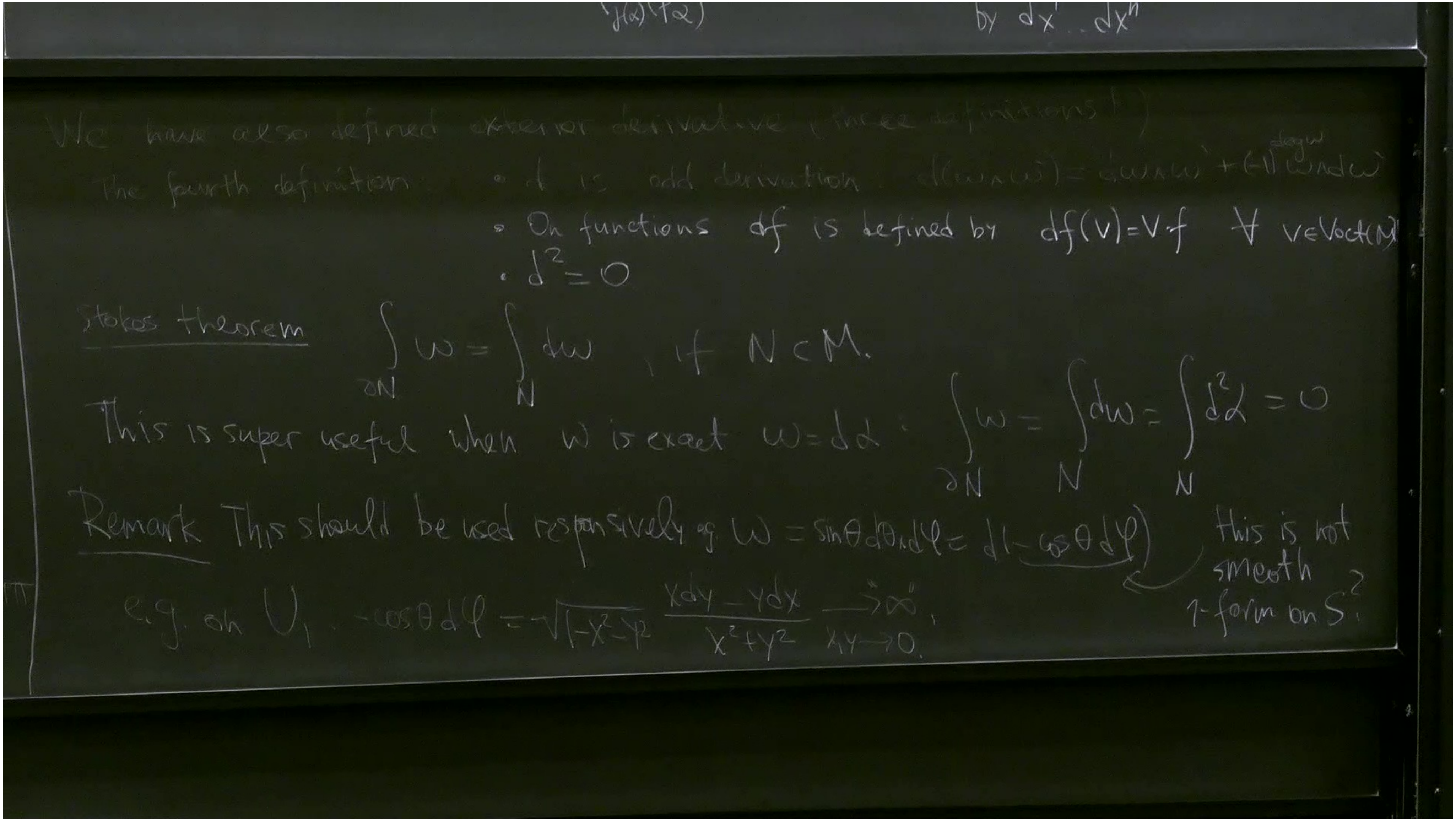
The fourth definition:

- d is odd derivation: $d(\omega\wedge\omega') = d\omega\wedge\omega' + (-1)^{\deg\omega}\,\omega\wedge d\omega'$
- On functions df is defined by $df(V)=V\cdot f$ $\forall\, V\in \mathrm{Vect}(M)$
- $d^2=0$

Stokes theorem $\int_{\partial N}\omega = \int_N d\omega$, if $N\subset M$.

This is super useful when ω is exact $\omega = d\alpha$: $\int_{\partial N}\omega = \int_N d\omega = \int_N d^2\alpha = 0$

Remark This should be used responsively.



$j(\alpha)(\tau\alpha)$ by $dx^1 \ldots dx^n$

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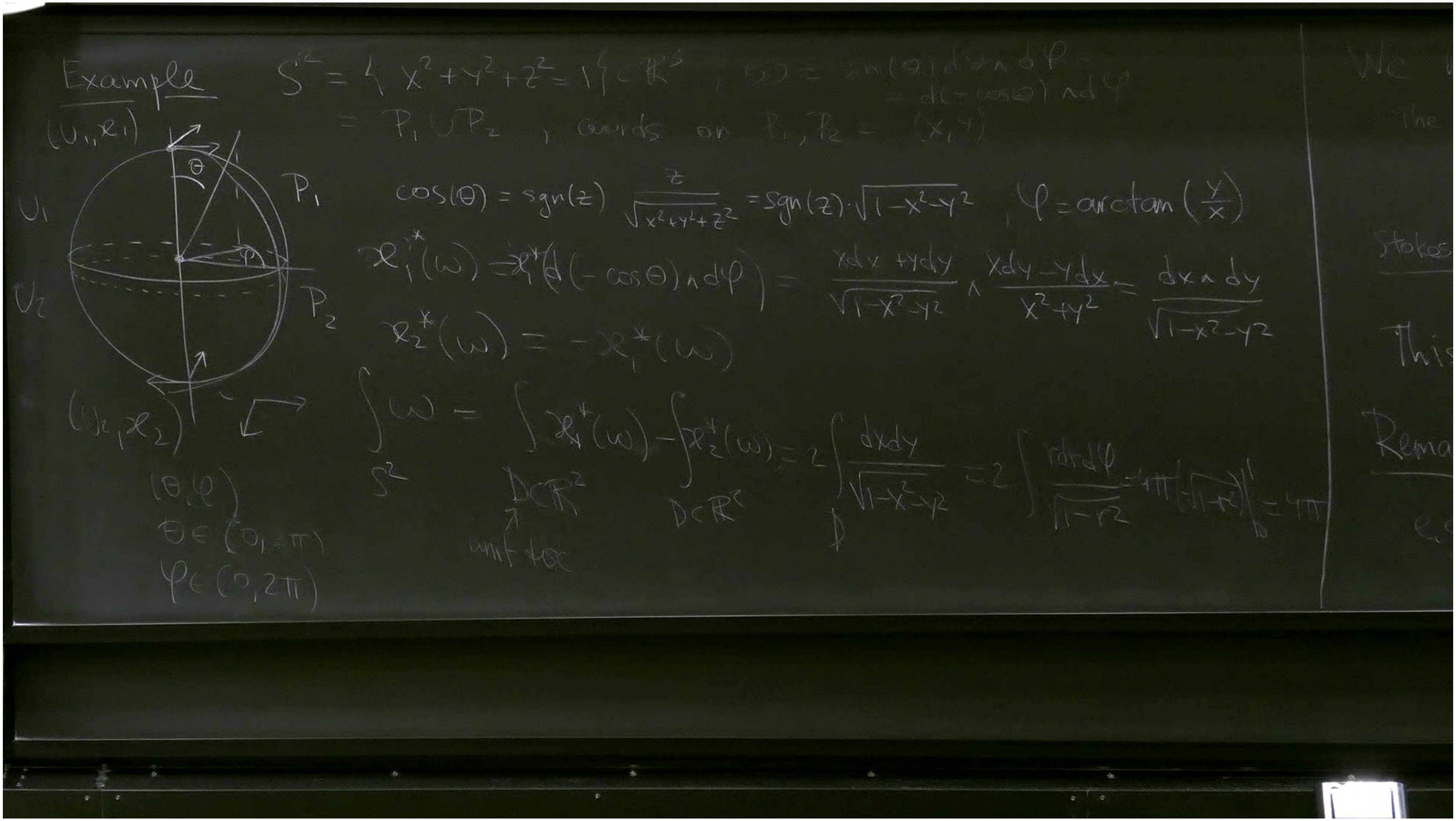
Remark This should be used responsively eg $\omega = \sin\theta\, d\theta \wedge d\varphi = d(\underline{-\cos\theta\, d\varphi})$ This is not smooth 1-form on S^2!

e.g. on U_1 $\displaystyle -\cos\theta\, d\varphi = -\sqrt{1-x^2-y^2}\ \frac{x\,dy - y\,dx}{x^2+y^2} \xrightarrow[x,y \to 0]{} \text{"}\infty\text{"}$.



Example

$S^2 = \{x^2+y^2+z^2=1\} \subset \mathbb{R}^3$, $\omega = \sin(\theta)\, d\theta \wedge d\varphi = d(-\cos\theta) \wedge d\varphi$

$= P_1 \cup P_2$, coords on $P_1, P_2 = (x, y)$

(U_1, x_1)

θ

φ

P_1

P_2

U_1

U_2

(U_2, x_2)

$$\cos(\theta) = \mathrm{sgn}(z) \frac{z}{\sqrt{x^2+y^2+z^2}} = \mathrm{sgn}(z) \cdot \sqrt{1-x^2-y^2}, \quad \varphi = \arctan\left(\frac{y}{x}\right)$$

$$x_1^*(\omega) = x_1^*\left(d(-\cos\theta) \wedge d\varphi\right) = \frac{x\,dx + y\,dy}{\sqrt{1-x^2-y^2}} \wedge \frac{x\,dy - y\,dx}{x^2+y^2} = \frac{dx \wedge dy}{\sqrt{1-x^2-y^2}}$$

$$x_2^*(\omega) = -x_1^*(\omega)$$

$$\int_{S^2} \omega = \int_{D \subset \mathbb{R}^2} x_1^*(\omega) - \int_{D \subset \mathbb{R}^2} x_2^*(\omega) = 2 \int_D \frac{dx\,dy}{\sqrt{1-x^2-y^2}} = 2 \int \frac{r\,dr\,d\varphi}{\sqrt{1-r^2}} = 4\pi \left(-\sqrt{1-r^2}\right)\Big|_0^1 = 4\pi$$

unit disc

(θ, φ)

$\theta \in (0, \pi)$

$\varphi \in (0, 2\pi)$

We

The

Stokes

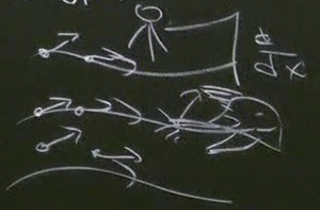
This

Rema

e.

Lie derivatives

(fisher's derivative)



$X \in \mathrm{Vect}(M)$ ~ defines a "flow" on M:

$X = X^\mu \frac{\partial}{\partial x^\mu}$ in $(U, \varphi = (x^i)) \Rightarrow \frac{dx^\mu}{dt} = X^\mu(x)$

- $\frac{d\sigma^\mu(t, x_0)}{dt} = X^\mu(\sigma(t, x_0))$, $\sigma^\mu(0, x_0) = (x_0)^\mu$

It defines a smooth map $\sigma : \mathbb{R} \times M \to M$ - flow.

These σ they form a group (for compact M) $\sigma_t = \sigma(t, \cdot) : M \to M$.

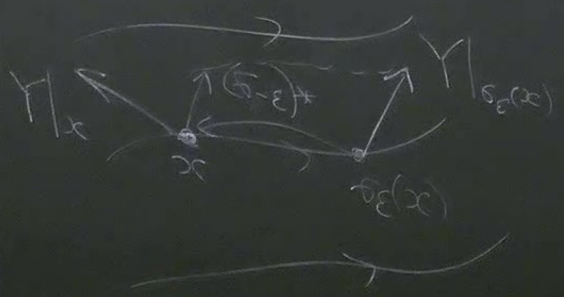
- $\sigma_t(\sigma_s(x)) = \sigma_{t+x}(x) \Rightarrow \sigma_t \circ \sigma_s = \sigma_{t+s}$
- $\sigma_0 = id$
- $\sigma_{-t} = (\sigma_t)^{-1}$

Example $X = x\frac{\partial}{\partial y} + y\frac{\partial}{\partial x}$ on $\mathbb{R}^2$: $\begin{pmatrix} \sigma_t^1(x_0) \\ \sigma_t^2(x_0) \end{pmatrix} = \begin{pmatrix} \mathrm{ch}\,t & \mathrm{sh}\,t \\ \mathrm{sh}\,t & \mathrm{ch}\,t \end{pmatrix} \begin{pmatrix} x_0^1 \\ x_0^2 \end{pmatrix}$ $\sigma_t : \sigma_{t+s} = \sigma_t\, \sigma_s$

Lie derivative $\mathcal{L}_X$ "defines how do tensors change under flow X"

Let $X, Y \in \mathrm{Vect}(M)$: $\mathcal{L}_X Y\big|_x = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\Big[(\sigma_{-\varepsilon})_* Y\big|_{\sigma_\varepsilon(x)} - Y\big|_x\Big] =$

$= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\Big[Y\big|_{\sigma_\varepsilon(x)} - (\sigma_\varepsilon)_* Y\big|_x \Big]$



$${}^{\mu}\left(x^{\nu}+\varepsilon X^{\nu}(x)\right)\left(\phi_{-\varepsilon}\right)_{*}\left(\frac{\partial}{\partial(x+\varepsilon X)^{\mu}}\right) \simeq \left[Y^{\mu}+\varepsilon X^{\nu}\partial_{\nu}Y^{\mu}\right]\cdot\left(\delta_{\mu}^{\lambda}-\varepsilon\partial_{\mu}X^{\lambda}\right)\frac{\partial}{\partial x^{\lambda}}+\dots$$

$${}^{\nu}-\underbrace{Y^{\mu}\partial_{\mu}X^{\nu}}_{} = \left([X,Y]\right)^{\nu} \quad , \quad \left([X,Y]\right)f = X(Yf)-Y(Xf)$$

$$x)-\omega|_{x}\Big] = \left(X^{\nu}\partial_{\nu}\omega_{\mu}+\partial_{\mu}X^{\nu}\omega_{\nu}\right)dx^{\mu}$$