Title: Lecture - Mathematical Physics, PHYS 777-

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Recap • $\omega \in \Lambda^k T^*M$ – k-form. In chart (U, φ): $\omega|_p : T_pM \otimes \dots \otimes T_pM \to \mathbb{R}$.

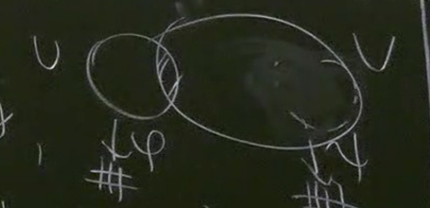
$$\omega = \sum_{i_1 < \dots < i_k} \omega_{i_1 \dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k} = \frac{1}{k!} \sum \omega_{i_1 \dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k}$$

$\omega_{i_1 \dots i_k} \in C^\infty(U, \mathbb{R})$

$$t \wedge s := \frac{1}{(r+s)!} \sum_{\sigma \in S_{r+s}} \operatorname{sgn}(\sigma)\, t_{i_{\sigma(1)} \dots i_{\sigma(r)}} s_{i_{\sigma(r+1)} \dots i_{\sigma(r+s)}} dx^{i_1} \otimes \dots \otimes dx^{i_{r+s}}$$

$$(t \wedge s)^{i_1 \dots i_{r+s}} = t^{[i_1 \dots i_r} s^{i_{r+1} \dots i_{r+s}]}$$

Under transitions to other charts: $dy^i = \left(\frac{\partial y^i}{\partial x^j}\right) dx^j$,



$\Rightarrow$ to keep ω to be "invariant" $y^i(x) = \psi^i \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$

$$\omega'_{j_1 \dots j_k} = \frac{\partial x^{i_1}}{\partial y^{j_1}} \dots \frac{\partial x^{i_k}}{\partial y^{j_k}} \omega_{i_1 \dots i_k}$$

Rema

Remarks: $\omega_{i_1 \ldots i_k}$ is not a function $M \to \mathbb{R}$, only $U \to \mathbb{R}$.
this is because it is a local section of bundle.

• We essentially use that $\frac{\partial x}{\partial y}$ is non-degenerate

Pushforwards & pullbacks: $N \hookrightarrow M$

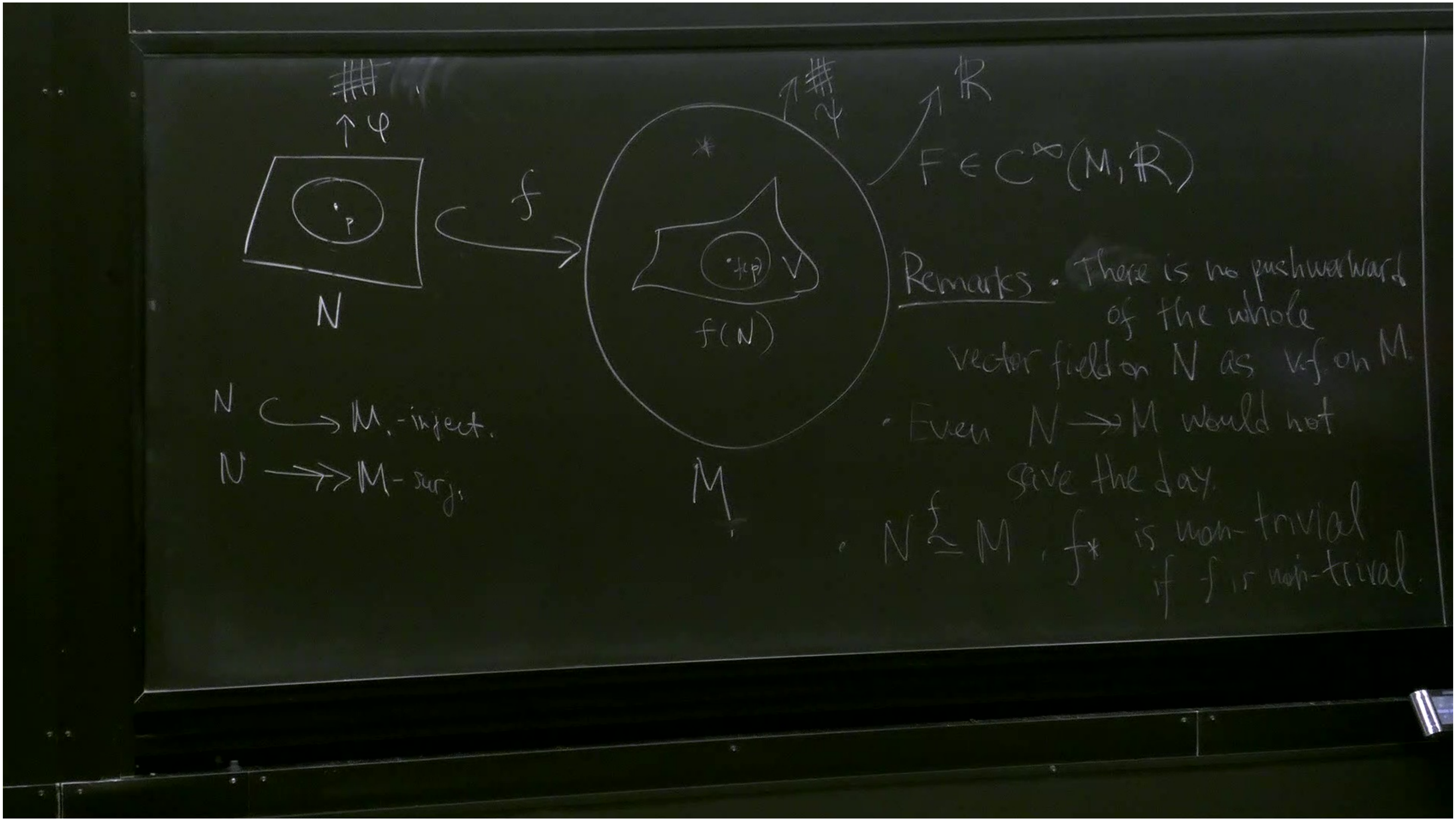
We pushforward tangent vectors to N at p to $f(p)$ on M, f_*

$$X_p \in T_pN = \mathrm{Der}_p\left(C^\infty(N,\mathbb{R})\right) \xrightarrow{f_*} f_*(X_p) \in T_{f(p)}M = \mathrm{Der}\left(C^\infty(M,\mathbb{R})\right)$$

$$(f_* X_p)(F) \overset{\text{def}}{=} X_p(F \circ f)$$

, in coordinates $\frac{\partial}{\partial x^i}(F \circ f \circ \varphi^{-1}) = \frac{\partial}{\partial x^i}(F \circ \psi \circ \psi^{-1} \circ \varphi^{-1})$

$$\Rightarrow \boxed{f_*\left(\frac{\partial}{\partial x^i}\Big|_p\right) = \frac{\partial y^j}{\partial x^i}\Big|_p \frac{\partial}{\partial y^j}\Big|_{f(p)}, \quad \frac{\partial y^j}{\partial x^i}\Big|_p = \frac{\partial(\psi^j \circ f \circ \varphi^{-1})}{\partial x^i}\Big|_{\varphi(p)}}$$



↑ φ
ψ
ℝ
F ∈ C^∞(M, ℝ)
f
p
f(p)
V
N
f(N)
M
Remarks • There is no pushforward of the whole vector field on N as v.f. on M.
• Even N ↠ M would not save the day.
• N ≅ M, f* is non-trivial if f is non-trivial.
N ↪ M - inject.
N ↠ M - surj.

ward

on M.

l

rival.

Pullbacks $\omega \in \Lambda^k T^*M \Rightarrow f^*(\omega) \in \Lambda^k T^*N$

$$(f^*\omega)\big|_p(\xi_1, \dots, \xi_k) = \omega\big|_{f(p)}(f_*\xi_1, \dots, f_*\xi_k), \quad \xi_i \in T_pN$$

In coordinates: $f^*\left(dy^i\big|_{f(p)}\right) = \frac{\partial y^i}{\partial x^j}\Big|_p dx^j\Big|_p$

And this (in contrast to $f_*(X_p)$) can be globalized)

To $f^*: \Lambda^k T^*M \to \Lambda^k T^*N$: $f^*(\omega)(p) = \frac{1}{k!}\omega_{i_1 \dots i_k}(f(p)) \frac{\partial y^{i_1}}{\partial x^{j_1}} \cdots \frac{\partial y^{i_k}}{\partial x^{j_k}} dx^{j_1} \wedge \dots \wedge$

Obviously. f^* is linear and respect wedge product.

Remark. There is no naive pullbacks of vectors.

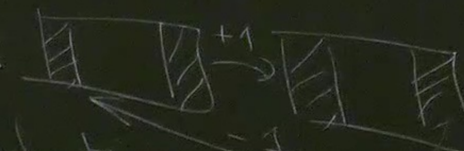
- All these things are better in the theory of sheafs.

Integration of forms

· M is orientable if (U_i,φ_i) s.t. $\forall\, U\cap V \neq \emptyset$ $\det\left(\frac{\partial X}{\partial Y}\right)(p) > 0 \quad \forall p\in U\cap V$

Otherwise it is non-orientable.

Möbius strip :



· For orientable M one can pick orientation

– top form $\Omega \in \Lambda^n T^*M$ s.t. in charts

$$\Omega = h\, dx^1\wedge\dots\wedge dx^n \quad \text{with } h>0$$

for all points of charts.

Other orientation is given by $h\to -h$

Remark Every other $\omega \in \Lambda^n T^*M$ can be presented as $\omega = g \cdot \Omega$, $g \in C^\infty(M, \mathbb{R})$

- For every locally finite open cover $\{U_i\}$, the partition of unity is set $R_i \in C^\infty(M, \mathbb{R})$, s.t.
  - R_i are vanishing outside U_i
  - $0 < R_i < 1$
  - $\forall p \in M: \sum_i R_i(p) = 1$

Other orientation is given by $h \to -h$ for all points of charts.

Integral of top form: $\omega \in \Lambda^n T^*M$, $(U_i \varphi_i)$ $\omega = g\,\Omega$

$$\int_M \omega \stackrel{\text{def}}{=} \sum_i \int_{\varphi_i(U_i) \subset \mathbb{R}^n} \underbrace{(R \circ \varphi_i^{-1})(x)}\,\underbrace{(g \circ \varphi_i^{-1})(x)}\,\underbrace{(h \circ \varphi_i^{-1})(x)\, dx^1 \ldots dx^n}$$

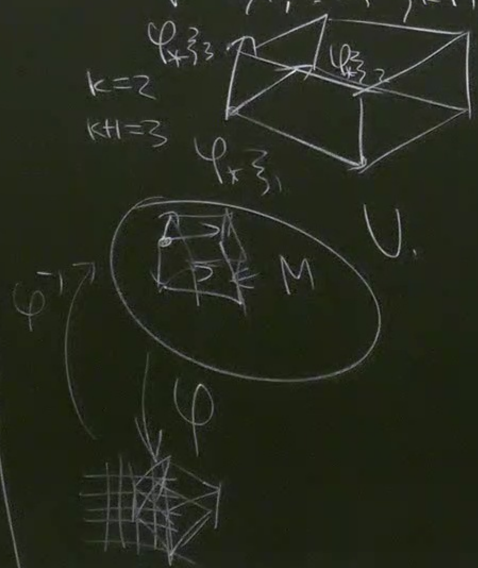
Other forms: $N \xhookrightarrow{f} M$, $\dim N = k$, $\omega \in \Lambda^k T^*M$

$$\Rightarrow \int_{f(N)} \omega \stackrel{\text{def}}{=} \int_N f^*(\omega)$$

Exterior derivative

$\omega \in \Lambda^k T^*M$, $d\omega \in \Lambda^{k+1} T^*M$

$\forall\, \xi_1, \dots, \xi_{k+1} \in T_pM$: $(d\omega)(\xi_1, \dots, \xi_{k+1}) = \lim_{\varepsilon \to 0} \left(\frac{1}{\varepsilon^k} \int_{\varphi^{-1}(\partial \Pi(\varepsilon \varphi_* \xi))} \omega \right)$

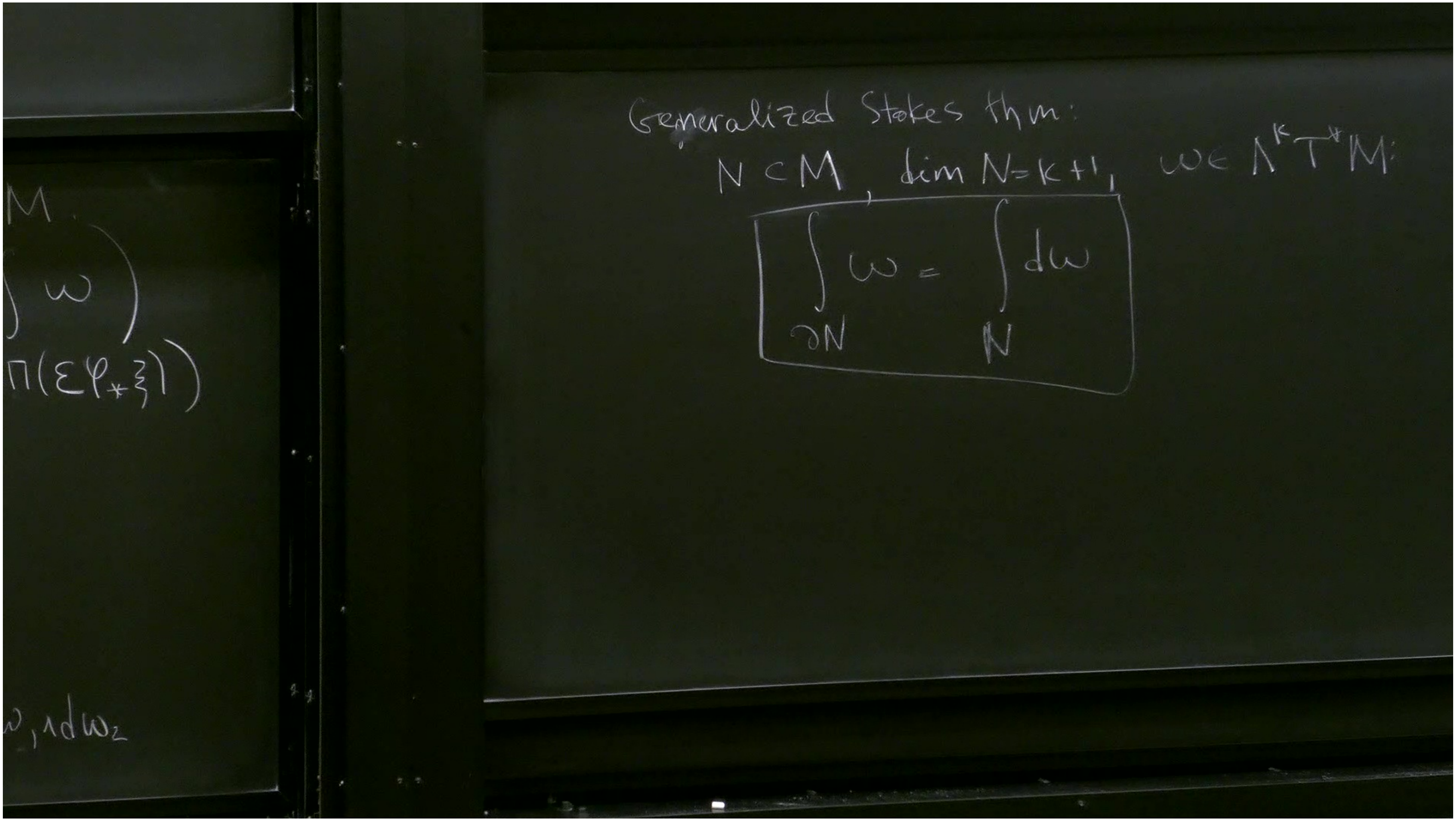


$= \Pi(\varepsilon \varphi_* \xi)$

In coordinates

$$d\omega = \frac{\partial \omega_{i_2 \dots i_{k+1}}}{\partial x^{i_1}}\, dx^{i_1} \wedge \dots \wedge dx^{i_{k+1}}$$

- $d\omega$ is well defined.
- d is linear, $d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^{\deg \omega_1} \omega_1 \wedge d\omega_2$


Generalized Stokes thm:
$N \subset M$, $\dim N = k+1$, $\omega \in \Lambda^k T^*M$:
$\boxed{\int_{\partial N} \omega = \int_N d\omega}$