Title: Lecture - Mathematical Physics, PHYS 777-

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Recap We discussed tangent space at p & dual vector spaces.

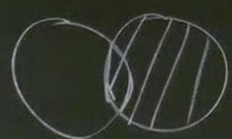
$T_pM = \mathrm{Der}_p(C^\infty(M,\mathbb{R}))$ - vector space, n-dim for $\dim M = n$

There is a basis for (U,φ), $\varphi(p) = (x^1(p), \dots, x^n(p))$

$$\Rightarrow T_pM = \left\langle \frac{\partial}{\partial x^1}\Big|_p, \dots, \frac{\partial}{\partial x^n}\Big|_p \right\rangle$$

$$\frac{\partial}{\partial x^i}\Big|_p f = \frac{\partial}{\partial x^i} f\circ\varphi^{-1}\Big|_{x^i=\varphi(p)}$$

Under the change $(V,\psi)\to(U,\varphi)$, $\psi = (y^1, \dots, y^n)$



on $U\cap V$: $\frac{\partial}{\partial x^i}\Big|_p = \frac{\partial y^j}{\partial x^i}\Big|_p \frac{\partial}{\partial y^j}\Big|_p$ → Jacobi matrix $\frac{\partial y^j}{\partial x^i}\Big|_p = \frac{\partial}{\partial x^i}(\psi\circ\varphi^{-1})\Big|_{x=\varphi(p)}$

Vectors (

$X =$

Elements

- assi

X

Vectors (as opposite to their "coordinate" representation) are invariant objects:

$$X = X^i \frac{\partial}{\partial x^i}\Big|_p = (X')^j \frac{\partial}{\partial y^j}\Big|_p$$

$$ = X^i \frac{\partial y^j}{\partial x^i}\Big|_p \frac{\partial}{\partial y^j}$$

$$\Rightarrow \boxed{(X')^j = \frac{\partial y^j}{\partial x^i}\Big|_p X^i}$$

obviously a vector space

Elements of T_pM have "global" analog: vector fields $X \in \mathrm{Vect}(M)$ — assignment of vector X_p to each point p in a smooth way.

$$X = X^i \frac{\partial}{\partial x^i} \in \mathrm{Der}(C^\infty(M,\mathbb{R})) : (Xf)(p) = X^i(p) \frac{\partial f}{\partial x^i}\Big|_p , \quad X^i \in C^\infty(U,\mathbb{R})$$

We also learned dual vector spaces. V – vector space

V^* – space of covectors

$$V^* = \mathrm{Hom}(V, \mathbb{R}), \quad V = \langle e_1, \dots, e_n \rangle, \quad V^* = \langle \varepsilon^1, \dots, \varepsilon^n \rangle$$

$$\varepsilon^i(e_j) = \delta^i_j$$

$\underset{A}{\mathrm{Hom}}(V, W)$ – space of linear maps $V \to W$

Applied to tangent spaces, we get T_p^*M.

$$\omega(p) \in T_p^*M = \mathrm{Hom}(T_pM, \mathbb{R})$$

Basis: $T_p^*M = \langle dx^1|_p, \dots, dx^n|_p \rangle \Rightarrow \omega(p) = \omega(p)_i \, dx^i|_p$

$$dx^i|_p\left(\frac{\partial}{\partial x^j}\Big|_p\right) = \delta^i_j$$

Using that $\omega(p)$ is a "geometric" object: $dx^i|_p = \frac{\partial x^i}{\partial y^j}(p)\, dy^j|_p$

This space can be "globalized" to $T^*M = \Lambda^1 T^*M = \Lambda^1 M = \Omega^1(M)$.

$\omega \overset{?}{=} \omega_i\, dx^i$, $\omega^i \in C^\infty(U, \mathbb{R})$, glued together by

Alternatively $\omega \in$ "$(\mathrm{Vect}(M))^*$" : $\omega : \mathrm{Vect}(M) \longrightarrow C^\infty(M, \mathbb{R})$

$$\omega(X)(p) = \omega_i(p) X^i(p)$$

Remark: Both $\mathrm{Vect}(M) \overset{\text{"}}{=} TM$ and T^*M have structure of module over $C^\infty(M, \mathbb{R})$:

$\forall\ v \in \mathrm{Vect}(M)$ $\quad (fv)(p) = f(p)v(p)$

$\omega \in T^*M$ $\quad : (f\omega)(p) = f(p)\omega(p)$.

$f \in C^\infty(M, \mathbb{R})$

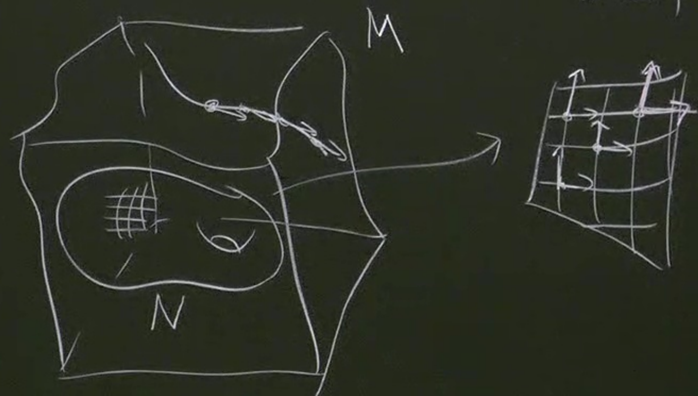
Let's define integration on manifold $I = \int_N \omega \rightsquigarrow m\text{-form.}$

To integrate we need

1) Integration cycle $N \subset M$

$\dim N = n$
$\dim M = m$

2) Integrand (what do we integrate)



... m vectors defining "local" geometry."

We want $\{m \text{ tangent vectors}\} \longrightarrow \mathbb{R}$

$\Lambda^1 M$ is an example for $m=1$

We need $\Lambda^n M$ whatever it means.

Tensor algebra

V, U, W - vector spaces

$U \otimes V = (\text{vector space with basis labled by elements of } U \times V)/\sim$

$$\sim : \begin{cases} (u+u', v) = (u,v) + (u',v) \\ (u, v+v') = (u,v) + (u,v') \\ (\alpha u, v) = \alpha(u,v) \\ (u, \alpha v) = \alpha(u,v) \end{cases} \quad \alpha \in \mathbb{R}$$

$U \otimes V$ - space of - bilinear combinations of U and V

Properties:

- $\mathbb{R} \otimes V = V \otimes \mathbb{R} = V$
- $(U \otimes V) \otimes W = U \otimes (V \otimes W)$
  $\rightsquigarrow U \otimes V \otimes W$ is well difined.
- $(U \oplus V) \otimes W = (U \otimes W) \oplus (V \otimes W)$
- If $U = \langle u_1, \dots, u_n \rangle$, $V = \langle v_1, \dots, v_m \rangle$
  $U \otimes V = \langle u_i \otimes v_j \rangle_{\substack{i=1,\dots,n \\ j=1,\dots,m}}$ $\dim(U \otimes V) = \dim U \cdot \dim V$
- $f \in Hom(U, U')$, $g \in Hom(V, V')$: $(f \otimes g)(u \otimes v) = f(u) \otimes g(v)$

Tensor algebra $T(V) = \bigoplus_{r,s=0}^{+\infty} T^r_s(V)$, $T^r_s(V) = \underbrace{V\otimes\cdots\otimes V}_{r} \otimes \underbrace{V^*\otimes\cdots}_{s}$

$$\Downarrow$$

$$T = T^{i_1\ldots i_r}_{j_1\ldots j_s}\, e_{i_1}\otimes\cdots\otimes e_{i_r}\otimes e^{j_1}\otimes\cdots$$

- Decomposable tensors:

$T^r_s(V) \ni v_1\otimes \cdots \otimes v_r \otimes u_1 \otimes \cdots \otimes u_s$, $v_i\in V$, $u_j\in V^*$

$n \quad n \quad n \quad (r+s)\cdot n$

$$\begin{pmatrix} v_1^1 \\ v_1^2 \\ \vdots \\ v_1^n \end{pmatrix} \cdots \begin{pmatrix} v_r^1 \\ \vdots \\ v_r^n \end{pmatrix} \cdots$$

- Contraction $T^r_s(V) \longrightarrow T^{r-1}_{s-1}(V)$

$$\mathbb{R} \ni u_1(v_1)$$

$$v_1\otimes\cdots\otimes v_r\otimes u_1\otimes\cdots\otimes u_s \longmapsto u_1(v_1)\, v_2\otimes\cdots\otimes v_r\otimes u_1\otimes\cdots\otimes u_s$$

$V) = \underbrace{V\otimes \cdots \otimes V}_{r} \otimes \underbrace{V^*\otimes \cdots \otimes V^*}_{s.} = V^{\otimes r}\otimes (V^*)^{\otimes s}$, $\dim T^r_s(V) = n^{r+s}$

$T^{i_1\ldots i_r}_{j_1\ldots j_s}\, e_{i_1}\otimes\cdots\otimes e_{i_r}\otimes e^{j_1}\otimes\cdots\otimes e^{j_s}$

$\in V^*$

- Evaluation: $T^r_s(V)\otimes V \longrightarrow T^r_{s-1}(V)$

$\otimes v_r\otimes u_1\otimes\cdots\otimes u_s$ $\quad (v_1\otimes\cdots\otimes v_r\otimes u_1\otimes\cdots\otimes u_s)\otimes v \mapsto v_1\otimes\cdots\otimes v_r\otimes u_2\otimes\cdots\otimes u_s \cdot u_1(v)$

Examples

- $V = T^1_0(V)$, $V^* = T^0_1(V)$
- Space of linear maps $= \mathrm{Hom}(U,V) = V \otimes U^*$, $\mathrm{Hom}(V,V) = T^1_1(V)$

  $\mathrm{ev}: (V \otimes U^*) \otimes U \to V$
- Bilinear forms $= \mathrm{Hom}(U \otimes U, \mathbb{R}) = U^* \otimes U^* = T^0_2(U)$