Title: Lecture - Mathematical Physics, PHYS 777-

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Recap Last time $\rightsquigarrow$ tensor algebra $T^r_s(V) = \underbrace{V\otimes\ldots\otimes V}_{r} \otimes \underbrace{V^*\otimes\ldots\otimes V^*}_{s}$

Decomposable tensors in $T^r_0(V)$ $\Leftrightarrow$ $v_1\otimes\ldots\otimes v_r$, $v_i \in V$

$V\otimes V \ni \overset{t}{t^{ij}} e_i\otimes e_j$, $t_{ij}\in\mathbb{R}$

$n\begin{pmatrix} & \overset{n}{t_{ij}} & \end{pmatrix}$

$v_1\otimes\ldots\otimes v_r + v_1'\otimes\ldots\otimes v_r'$ most likely not decomposable

t is decomposable $\Leftrightarrow$ $\mathrm{rk}\, t = 1$

$\mathrm{rk}\, t = 1 \Rightarrow t^{ij} = u^i v^j$

$\Rightarrow t = u^i v^j e_i\otimes e_j = u\otimes v$

t is decomp. $\Rightarrow t = u\otimes v \Rightarrow t^{ij} = u^i v^j$

In quantum many body: $H = \underbrace{\mathbb{C}^2\otimes\ldots\otimes\mathbb{C}^2}_{N}$, N - big

$\mathbb{C}^{2^n} \leftarrow \underbrace{\mathbb{C}^2\otimes\ldots\otimes\mathbb{C}^2}_{n}\otimes\underbrace{\mathbb{C}^2\otimes\ldots\otimes\mathbb{C}^2}_{N-n} \rightarrow \mathbb{C}^{2^{N-n}}$

Exterior (Grassmann) algebra (& symmetric algebra) $\Lambda^r V,\ S^r V \subset T^r_0 V$

Antisymmetrizer : $A: T^r_0 V \to T^r_0 V$, $A(v_1, \dots, v_r) = \frac{1}{r!} \sum_{\sigma \in S_r} \mathrm{sgn}(\sigma)\, v_{\sigma(1)} \otimes \dots \otimes v_{\sigma(r)}$

Symmetrizer : $S: T^r_0 V \to T^r_0 V$, $S(v_1, \dots, v_r) = \frac{1}{r!} \sum_{\sigma \in S_r} v_{\sigma(1)} \otimes \dots \otimes v_{\sigma(r)}$

They are projectors: $A^2 = A$, $S^2 = S$

Ex $r=2$ $A(v_1, v_2) = \frac{1}{2}(v_1 \otimes v_2 - v_2 \otimes v_1)$, $S = \frac{1}{2}(v_1 \otimes v_2 + v_2 \otimes v_1)$

$\Lambda^r V \overset{def}{=} A(T^r_0(V))$, $S^r V \overset{def}{=} S(T^r_0(V))$ - antisym and sym. tensors.

$\Lambda^* V = \bigoplus_{r=0}^{n} \Lambda^r V$, $S^* V = \bigoplus_{r=0}^{+\infty} S^r V$

$\Lambda^{n+1} V = 0$

Q] What is the multiplication $\Lambda^\bullet V \otimes \Lambda^\bullet V \longrightarrow \Lambda^\bullet V$

Not a $\otimes$: $\underset{\Lambda^1 V \cong V}{e_i} \otimes e_j \notin \Lambda^2 V$.

It's is a wedge product : $\underset{\Lambda^r V}{t} = t^{i_1 \dots i_r} e_{i_1} \otimes \dots \otimes e_{i_r}$, $\underset{\Lambda^s V}{s} = s^{i_1 \dots i_s} e_{i_1} \otimes \dots \otimes e_{i_s}$

fully antisymmetric i.e.

$t^{i_{\sigma(1)}, \dots, i_{\sigma(r)}} = \mathrm{sgn}(\sigma)\, t^{i_1 \dots i_r}$

$$t \wedge s := \frac{1}{(r+s)!} \sum_{\sigma \in S_{r+s}} \mathrm{sgn}(\sigma)\, t^{i_{\sigma(1)} \dots i_{\sigma(r)}} s^{i_{\sigma(r+1)} \dots i_{\sigma(r+s)}}$$

$$(t \wedge s)^{i_1 \dots i_{r+s}} = t^{[i_1 \dots i_r} s^{i_{r+1} \dots i_{r+s}]}$$

Notice that it is well-defined $\Lambda^\circ V \times \Lambda^\circ V \to \Lambda^\circ V$.

$$t \wedge 1 = \frac{1}{r!} \sum_{\sigma \in S_r} \mathrm{sgn}(\sigma)\, t^{\sigma(i_1) \ldots \sigma(i_r)} e_{i_1} \otimes \ldots \otimes e_{i_r} = \cancel{\frac{1}{r!} \sum_{\sigma \in S_r} (\mathrm{sgn}\,\sigma)^2}\, t^{i_1 \ldots i_r} e_{i_1} \otimes \ldots \otimes e_{i_r} = t$$

Defining properties:

- $\wedge$ is a bilinear operation.
- $X \in V \Rightarrow X \wedge X = 0$ $\quad \left((Y+Z) \wedge (Y+Z) = 0 \Rightarrow Y \wedge Z = -Z \wedge Y \right)$
- $\wedge$ is associative. $(X \wedge Y) \wedge Z = X \wedge (Y \wedge Z)$.

The whole $\Lambda^\circ V$ can be generated from V this way.

The basis in $\Lambda^r \mathbb{R}^n = \langle e_{i_1} \wedge \ldots \wedge e_{i_r} \mid i_1 < \ldots < i_r \rangle$

$\Rightarrow \dim \Lambda^r \mathbb{R}^n = \binom{n}{r} \Rightarrow \Lambda^\circ \mathbb{R}^n = 2^n$

Example $\Lambda^0\mathbb{R}^2$, $\mathbb{R}^2 = \langle e_1, e_2 \rangle$. $\Lambda^0\mathbb{R}^2 = \mathbb{R}$, $\Lambda^1\mathbb{R}^2 = \mathbb{R}^2$, $\Lambda^2\mathbb{R}^2 = \langle e_1 \wedge e_2 \rangle$

$$e_1 \wedge (e_1 \wedge e_2) = (e_1 \wedge e_1) \wedge e_2 = 0, \quad e_2 \wedge (e_1 \wedge e_2) = $$
$$= e_1 \wedge e_2 \wedge e_2 = 0$$

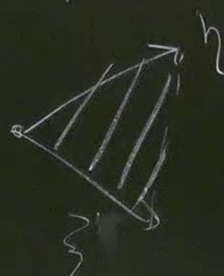
Example $\Lambda^2(\mathbb{R}^2)^*$ – space of bilinear forms on $\mathbb{R}^2$

2-forms

$$\Lambda^2(\mathbb{R}^2)^* = \langle \varepsilon^1 \wedge \varepsilon^2 \rangle$$

$\xi_1, \xi_2 \in \mathbb{R}^2$

$$(\varepsilon^1 \wedge \varepsilon^2)(\xi, \eta) = \frac{1}{2}(\varepsilon^1 \otimes \varepsilon^2 - \varepsilon^2 \otimes \varepsilon^1)(\xi, \eta)$$
$$= \frac{1}{2}(\xi^1 \eta^2 - \xi^2 \eta^1) = \frac{1}{2} \begin{vmatrix} \xi^1 & \eta^1 \\ \xi^2 & \eta^2 \end{vmatrix}$$



basis in $\Lambda^r \mathbb{R}^n = \langle e_{i_1}\wedge\dots\wedge e_{i_r} \mid i_1<\dots<i_r\rangle$

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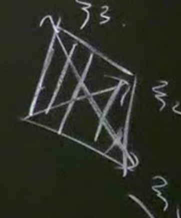
Example $\Lambda^n(\mathbb{R}^n)^* = \langle \varepsilon^1\wedge\dots\wedge\varepsilon^n\rangle$

" Ω – standard volume form

$$\Omega(\xi_1,\dots,\xi_n) = \frac{1}{n!}\sum_{\sigma\in S_n} \mathrm{sgn}(\sigma)\,\xi_1^{\sigma(1)}\cdots\xi_n^{\sigma(n)} = \frac{1}{n!}\det(\xi^i_j)$$

⇝ geometrically this is the volume of simplex $\xi_1,\dots,\xi_n$



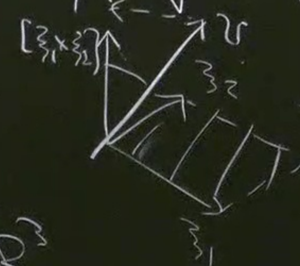
Example $\omega\in\Lambda^2(\mathbb{R}^3)^*$ (in general $\Lambda^\bullet V^*$ – these are "forms" on V)

$$\omega(\xi_1,\xi_2) = (v,[\xi_1\times\xi_2])$$

scalar product.

vector product.

Some vector in $\mathbb{R}^3$



– geometrically this is a flux of v through $\langle\xi_1,\xi_2\rangle$

Pullback of forms on vector space

$f : U \longrightarrow V$, $\omega \in \Lambda^k V$

We can define $f^* \omega \in \Lambda^k U$

$$(f^*\omega)(\xi_1, \dots, \xi_k) = \omega(f\xi_1, \dots, f\xi_k), \quad \xi_i \in U.$$

- This is a linear map on a space of forms.
- $f^*(\omega \wedge \omega') = (f^*\omega) \wedge (f^*\omega')$
- If $U \xrightarrow{f} V \xrightarrow{g} W \Rightarrow (g \circ f)^* = f^* \circ g^* : \Lambda^\bullet W \longrightarrow \Lambda^\bullet V$

- k-forms on M give a space of k-linear, skew-sym. maps $(\mathrm{Vect})^{\otimes k} \longrightarrow C^\infty(M, \mathbb{R})$

$$\omega(X_1, \dots, X_k)(p) = \sum_{i_1 < \dots < i_k} \omega_{i_1 \dots i_k}(\varphi(p)) X_1^{i_1}(\varphi(p)) \dots X_k^{i_k}(\varphi(p)), \quad X_i = X_i^j \frac{\partial}{\partial x^j}$$

Let's "globalize" this

- $\Lambda^k T_p^* M$ - space of k-forms on $T_p M$.
- Space $\Lambda^k T^* M \cong \Lambda^k M \rightsquigarrow$ smooth assigment of $\Lambda^k T_p^* M$ to each p

in a basis $\omega = \sum_{i_1 < \dots < i_k} \omega_{i_1 \dots i_k} \, dx^{i_1} \wedge \dots \wedge dx^{i_k}$ $\qquad dx^i\big|_p \left(\frac{\partial}{\partial x^j}\Big|_p \right) = \delta^i_j$

basis 1-forms

- Under change of basis:

$$\hat{\omega}_{i_1 \dots i_k} = \omega_{j_1 \dots j_k} \frac{\partial x^{j_1}}{\partial y^{i_1}} \cdots \frac{\partial x^{j_k}}{\partial y^{i_k}}$$

form in chart (U, φ)

$\varphi = (x^1, \dots, x^n)$

in chart (V, ψ)

$\psi = (y^1, \dots, y^n)$

. $N \subset M$ - integration cycle. $f : N \longrightarrow M$. $\int f^*(\omega) \in \Lambda^n N$ - top form, we

. $\omega \in \Lambda^n M$

$f : N \longrightarrow M.$ $\int f^*(\omega) \in \Lambda^n N$ - top form, we can integrate this.