Title: Lecture - Mathematical Physics, PHYS 777-

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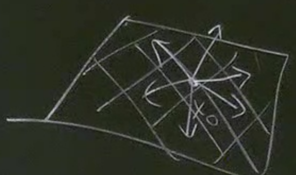
Recap on T_pM

"Naïve" definition of the space of tangent vectors to $\mathbb{R}^n$

$$T_{x_0}\mathbb{R}^n \cong \mathbb{R}^n \ni v = v^i e_i$$



Directional derivative

$$v \rightsquigarrow D_{v,x_0}$$

$$(D_{v,x_0} f)(x) = v^i \frac{\partial f}{\partial x_i}(x_0)$$

"infinitesimal changes of func"

Tangent space $T_{x_0}\mathbb{R}^n$ as space of derivations

$$X : C^\infty(\mathbb{R}^n, \mathbb{R}) \longrightarrow \mathbb{R}$$

- $X(\alpha f + \beta g) = \alpha X(f) + \beta X(g)$
- $X(fg) = f(x_0) X(g) + X(f) g(x_0)$

$\boxed{\text{Tangent space } T_pM \text{ is a space of derivations of } C^\infty(M,\mathbb{R}) \text{ at } p.}$

(g)

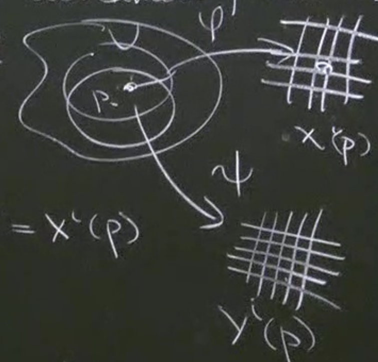
$g(x_0)$

Q) How to coordinatize "abstract" picture?

(U,φ), $U \subset X$, $\varphi(p) = (x^1(p), \dots, x^n(p))$ coordinates or coordinate functions

$\mathbb{R}^n \to \mathbb{R}$

$$X = \underset{\mathbb{R}}{v^i} \underbrace{\frac{\partial}{\partial x^i}\Big|_p}_{\text{basis derivations}}, \quad \frac{\partial}{\partial x_i}\Big|_p \cdot f = \frac{\partial f \circ \varphi^{-1}(x)}{\partial x^i}\Big|_{x^i = x^i(p)}$$



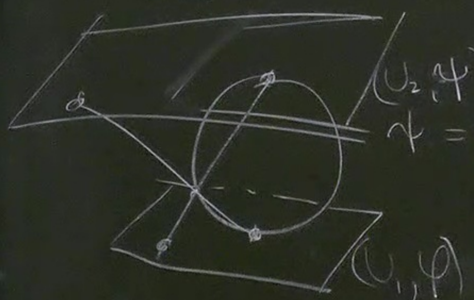
$$T_pM = \left\langle \frac{\partial}{\partial x^1}\Big|_p, \dots, \frac{\partial}{\partial x^n}\Big|_p \right\rangle$$

Other chart (V,ψ), $\psi = (y^1, \dots, y^n)$,

Jacobian matrix of a trasformation $\psi \circ \varphi^{-1}$

$$\frac{\partial}{\partial x^i}\Big|_p \cdot f = \frac{\partial\left((f \circ \psi^{-1}) \circ (\psi \circ \varphi^{-1})\right)}{\partial x^i}\Big|_p = \frac{\partial (f \circ \psi^{-1})}{\partial y^j}\Big|_{y^j = y^j(p)} \cdot \frac{\partial y^j(x)}{\partial x^i}\Big|_{x^i = x^i(p)}$$

Example S^2



$\psi = (X', Y')$

$x^2 + y^2 + z^2 = 1 \qquad \varphi = (X, Y)$

$$\psi \circ \varphi^{-1} : \begin{cases} X' = \dfrac{4X}{X^2+Y^2} \\[2ex] Y' = \dfrac{4Y}{X^2+Y^2} \end{cases}$$

$$\left.\frac{\partial}{\partial X}\right|_p = \left.\frac{\partial X'}{\partial X}\right|_p \left.\frac{\partial}{\partial X'}\right|_p + \left.\frac{\partial Y'}{\partial X}\right|_p \frac{\partial}{\partial Y'}$$

$$\left.\frac{\partial}{\partial Y}\right|_p = \left.\frac{\partial X'}{\partial Y}\right|_p \left.\frac{\partial}{\partial X'}\right|_p + \left.\frac{\partial Y'}{\partial Y}\right|_p \frac{\partial}{\partial Y'}$$

$\psi(p)$

$$\frac{\partial}{\partial X}\bigg|_p = \frac{\partial X'}{\partial X}\bigg|_p \frac{\partial}{\partial X'}\bigg|_p + \frac{\partial Y'}{\partial X}\bigg|_p \frac{\partial}{\partial Y'}\bigg|_p = 4\,\frac{Y^2 - X^2}{(Y^2+X^2)^2}\,\frac{\partial}{\partial X'}\bigg|_p - \frac{8Y^2}{(X^2+Y^2)^2}\,\frac{\partial}{\partial Y'}\bigg|_p$$

$$\frac{\partial}{\partial Y}\bigg|_p = \frac{\partial X'}{\partial Y}\bigg|_p \frac{\partial}{\partial X'}\bigg|_p + \frac{\partial Y'}{\partial Y}\bigg|_p \frac{\partial}{\partial Y'}\bigg|_p .$$

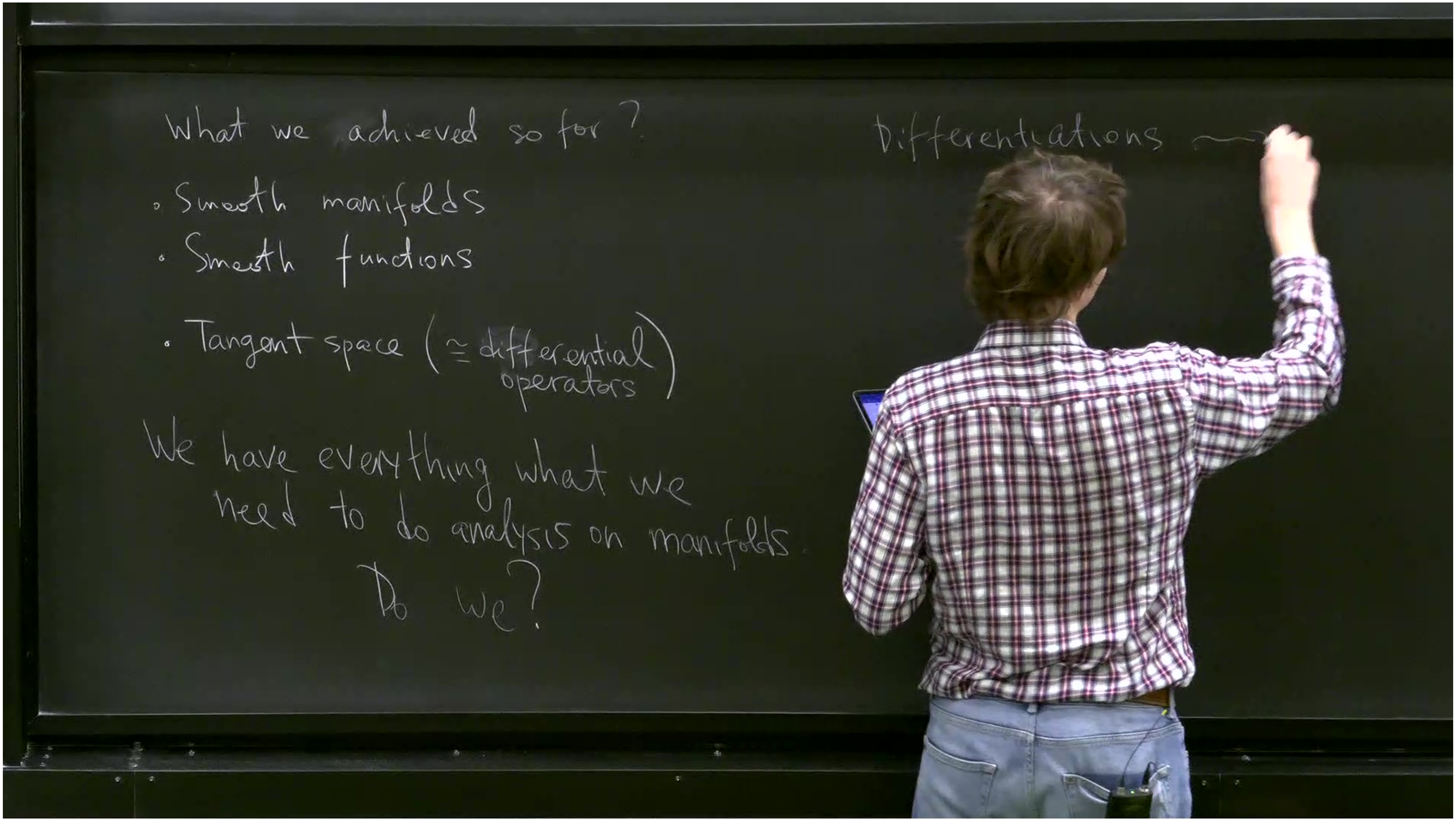
$$\Big|_p = 4\,\frac{y^2-x^2}{(y^2+x^2)^2}\,\frac{\partial}{\partial x'}\Big|_p - \frac{8y^2}{(x^2+y^2)^2}\,\frac{\partial}{\partial y'}\Big|_p = \frac{1}{4}\left((y')^2-(x')^2\right)\frac{\partial}{\partial x'}\Big|_p - \frac{1}{2}(y')^2\,\frac{\partial}{\partial y'}\Big|_p$$

$$\Big|_p = \cdots\cdots = -\frac{1}{2}(x')^2\,\frac{\partial}{\partial x'}\Big|_p + \frac{1}{4}\left((x')^2-(y')^2\right)\frac{\partial}{\partial y'}\Big|_p$$

$$\Big|_p = 4\,\frac{Y^2-X^2}{(Y^2+X^2)^2}\,\frac{\partial}{\partial X^1}\Big|_p - \frac{8Y^2}{(X^2+Y^2)^2}\,\frac{\partial}{\partial Y^1}\Big|_p = \frac{1}{4}\left((Y^1)^2-(X^1)^2\right)\frac{\partial}{\partial X^1}\Big|_p - \frac{1}{2}(Y^1)^2\,\frac{\partial}{\partial Y^1}\Big|_p$$

$$\Big|_p = \text{- - - - - - - -} \quad = -\frac{1}{2}(X^1)^2\,\frac{\partial}{\partial X^1}\Big|_p + \frac{1}{4}\left((X^1)^2-(Y^1)^2\right)\frac{\partial}{\partial Y^1}\Big|_p$$

$$X^1 = Y^1 = 0$$


What we achieved so far?
· Smooth manifolds
· Smooth functions
· Tangent space (≅ differential operators)
We have everything what we need to do analysis on manifolds.
Do we?
Differentiations ⇝

✓ Differentiations $\rightsquigarrow \frac{\partial}{\partial x^i}\Big|_p$

$\boxed{\text{Integration}}$ $\rightsquigarrow dx^i\big|_p$ - differential forms $\rightsquigarrow$ something to integrate

i.e. pair with tangent vectors

T_p^*M - cotangent space

Metric tensor (measuring length) $\rightsquigarrow g : T_pM \otimes T_pM \longrightarrow \mathbb{R}$

Curvature tensor $\rightsquigarrow R : T_pM \otimes T_pM \otimes T_pM \longrightarrow T_pM$

Differentiations of tensors $\rightsquigarrow L_X$ - Lie derivative

$\rightarrow$ tensors, i.e. multilinear combinations of T_pM and T_p^*M.

lds.

Symplectic form $\rightsquigarrow \omega : T_pM \otimes T_pM \longrightarrow \mathbb{R}$.

Poisson bivector. $\rightsquigarrow \{\,,\,\} \in T_pM \otimes T_pM \cong T_p^*M \times T_p^*M \longrightarrow \mathbb{R}$

$\{\,,\,\}_p : C^\infty \times C^\infty \longrightarrow \mathbb{R}$

vectors

otangent space

e. multilinear

tions of T_pM and T_p^*M

Connection 1-form

$A : T_pM \longrightarrow \mathfrak{g}$

$A \in T_p^*M \otimes \mathfrak{g}$

(parallel transport on M)

Fibre bundle

Co-vectors and tensors , V, U, W – vector spaces over $\mathbb{R}$.

Space of covectors (dual space) $V^* =$ space of linear functions $V \longrightarrow \mathbb{R}$

$V^* = \mathrm{Hom}(V, \mathbb{R})$ linear maps.

$\alpha, \beta \in V^*$, $u, v \in V$, $x, y \in \mathbb{R}$

- $\alpha(xu + yv) = x\,\alpha(u) + y\,\alpha(v)$
- $(x\alpha + y\beta)(u) = x\,\alpha(u) + y\,\beta(u)$

$V = \langle e_1, \dots, e_n \rangle$, $v = v^i e_i$, $v^i \in \mathbb{R}$, $e_i \in V$; $\varepsilon^i(e_j) = \delta^i_j \ \forall\, i,j \in 1, \dots, n$.

$V^* = \langle \varepsilon^1, \dots, \varepsilon^n \rangle$, $\alpha = \alpha_i \varepsilon^i$

$$\alpha(v) = \Big(\sum_i \alpha_i \varepsilon^i\Big)\Big(\sum_j v^j e_j\Big) = \sum_{i,j} \alpha_i v^j \cdot \varepsilon^i(e_j) = \sum \alpha_i v^j \delta^i_j = \sum_i \alpha_i v^i$$

• Double dual space $(V^*)^* \cong V$ (for finite dim. space V)

Inf. dim: $(V^*)^* \supset V$

$v \in V \longmapsto \phi_v \in (V^*)^* \cong \mathrm{Hom}(V^*, \mathbb{R})$

$\alpha \in V^*$: $\phi_v(\alpha) \overset{\text{def}}{=} \alpha(v)$ - it is linear, map is bijection.

• Transformation properties: $e_i \in V \rightsquigarrow A^j_i e_j = e'_i \Longleftrightarrow (\varepsilon')^i(e'_j) = \delta^i_j$

$\varepsilon^i \in V^* \rightsquigarrow (A^{-1})^i_j \varepsilon^j = \varepsilon'^i$

Rule of thumb:
v, α should be invariant

$\dots, n$