Title: Lecture - Mathematical Physics, PHYS 777-

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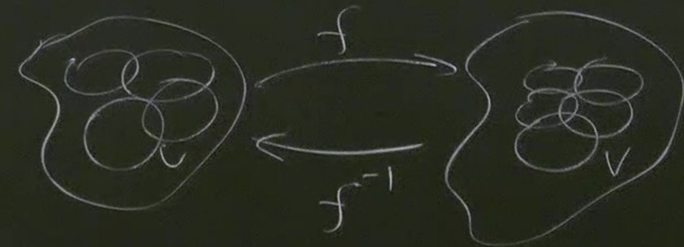
Recap

Topological spaces

open subset

- Topology on X $\quad \tau(X) \subset 2^X$
  - $\phi, X \in \tau(X)$
  - $\bigcap_{i, \text{fin}} U_i \in \tau(X)$
  - $\bigcup_i U_i \in \tau(X)$

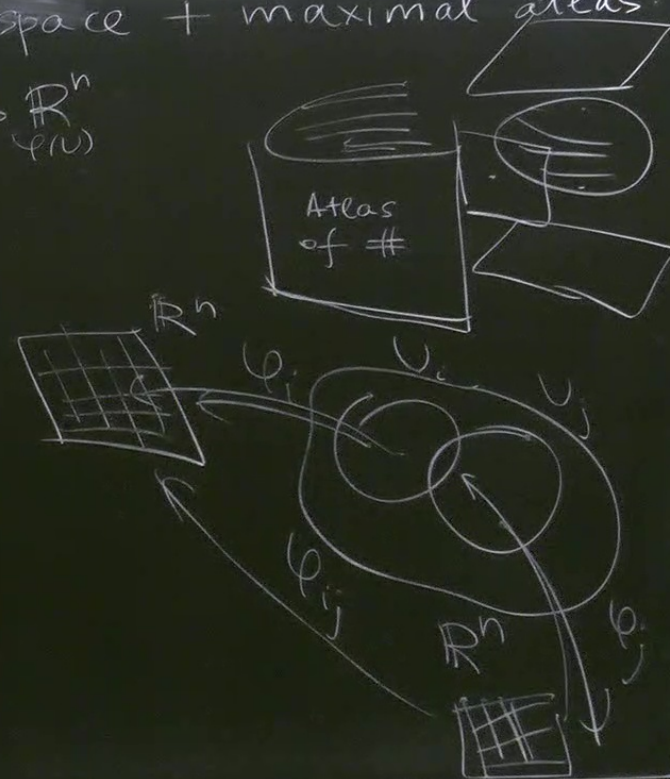


- Continuous maps: $f: X \to Y \quad \forall\, V \in \tau(Y): f^{-1}(V) \in \tau(X)$
- Homeomorphism: contin. f, bijective, f^{-1} also cont.

Differentiable manifolds $\rightsquigarrow$ topological space + maximal atlas

- Charts (U, φ), U - open subsets, $\varphi : U \to \underset{\varphi(U)}{\mathbb{R}^n}$
  - φ - injective, cont.
  - $\varphi(U)$ - open
- Atlas on M: $\mathcal{A}_M = \{(U_i, \varphi_i) \mid i \in I\}$ s.t.
  - $\bigcup_{i \in I} U_i = M$
  - $\varphi_i(U_{ij})$ is open in $\mathbb{R}^n$, $U_{ij} = U_i \cap U_j$
  - $\varphi_{ij} = \varphi_i \circ \varphi_j^{-1} \big|_{\varphi_j(U_{ij})}$ is C^∞ function.

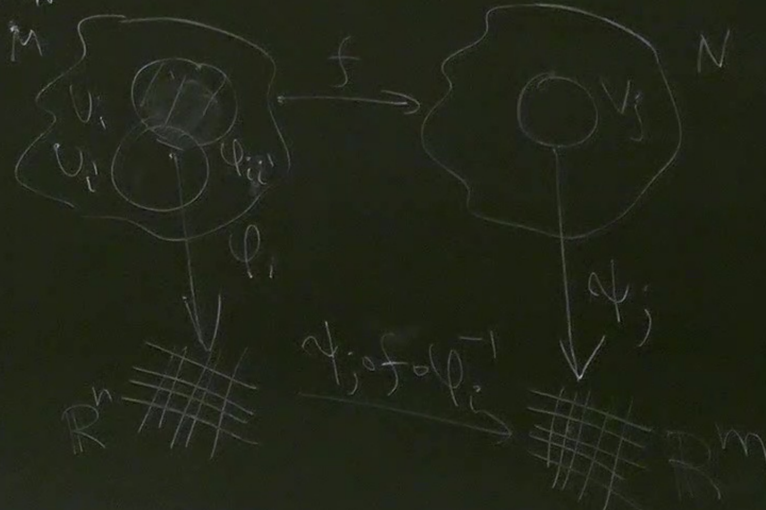
    transition map
- Maximal atlas - limiting atlas with all charts.



• Smooth maps: such continuous $f: M \to N$ that

$\forall \, (U_i, \varphi_i) \in \mathcal{A}_M$, $(V_j, \psi_j) \in \mathcal{A}_N$ s.t. $f(U_i) = V_i$: $\psi_j \circ f \circ \varphi_i^{-1} \in C^\infty(\varphi_i^{-1}(U))$

• It is enough to check smoothness only for one cover.



• There are also isomorphisms — diffeomorphisms.

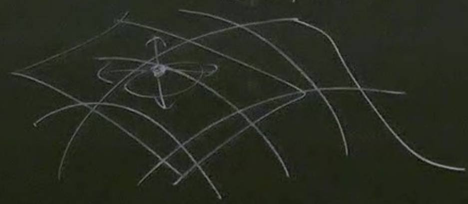
• f – is bijection

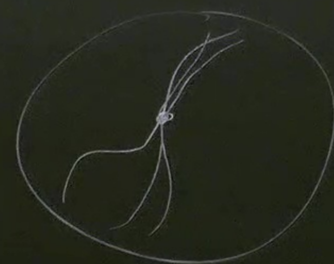
• f, f^{-1} are smooth.

$F(x, y, z) = 0$

$z = f(x, y)$

Tangent space





coordinate free

isms.

$\mathbb{R}^n$

• Look on how the functions transform infinitesimaly around points.

$$f(x_0 + \varepsilon v) = f(x_0) + \varepsilon v^i \frac{\partial f}{\partial x^i}(x_0) + O(\varepsilon^2)$$

← directional derivative

$\boxed{\text{Derivations at the point = tangent space.}}$

$$D_v : f \in C^\infty(\mathbb{R}^n, \mathbb{R}) \longmapsto v^i \frac{\partial f}{\partial x^i}(x_0) \in \mathbb{R}$$

↑ derivation.

• Look for all the curves which pass through x_0.
Identify similar ones.

• **Derivation** $X : C^\infty(M, \mathbb{R}) \longrightarrow \mathbb{R}$ at point x_0

satisfies: • $X(f \cdot g) = f(x_0)\, X(g) + X(f) \cdot g(x_0)$

• X is a linear operator on $C^\infty(M, \mathbb{R})$.

• **Claim** In case of $X = \mathbb{R}^n$ any derivation is a directional derivative.
(and vica versa).

1) $X(\alpha) = 0$, $\alpha \in \mathbb{R}$: $X(\alpha) = \alpha \cdot X(1)$, $\underbrace{X(1 \cdot 1)}_{= X(1)} = 1 \cdot X(1) + X(1) \cdot 1 = 2X(1)$

$\Rightarrow X(1) = 0$

$f \in C^\infty$

2) $f(x) = f(x_0) + (x-x_0)^i g_i(x)$, $g_i(x_0) = \frac{\partial f}{\partial x}(x_0)$, $g \in C^\infty$

$$f(x) = f(x_0) + f(x) - f(x_0) = f(x_0) + \int_0^1 \frac{d}{dt}\Big(f(x_0 + t(x-x_0))\Big)\Big|_{t=s} ds =$$

$$= f(x_0) + (x-x_0)^i \underbrace{\int_0^1 \frac{\partial f}{\partial x^i}(x_0 + s(x-x_0))\, ds}_{g_i} , \quad g_i(x) = \frac{\partial f}{\partial x^i}(x_0)$$

3) $$X(f(x)) = \underbrace{X(f(x_0))}_{0} + X((x-x_0)^i)\, g_i(x_0) + \underbrace{(x_0-x_0)^i X(g_i(x))}_{0} = X((x-x_0)^i) \frac{\partial f}{\partial x^i}(x_0)$$

$$\| $$

$$D_{X((x-x_0)^i), x_0}(f)$$

• Derivations at the point x_0 $(\mathbb{R}^n)$ $=$ directional derivatives at the point $=$ tangent vectors at the

$(\alpha X + \beta Y)(f) = \alpha X(f) + \beta X(f)$

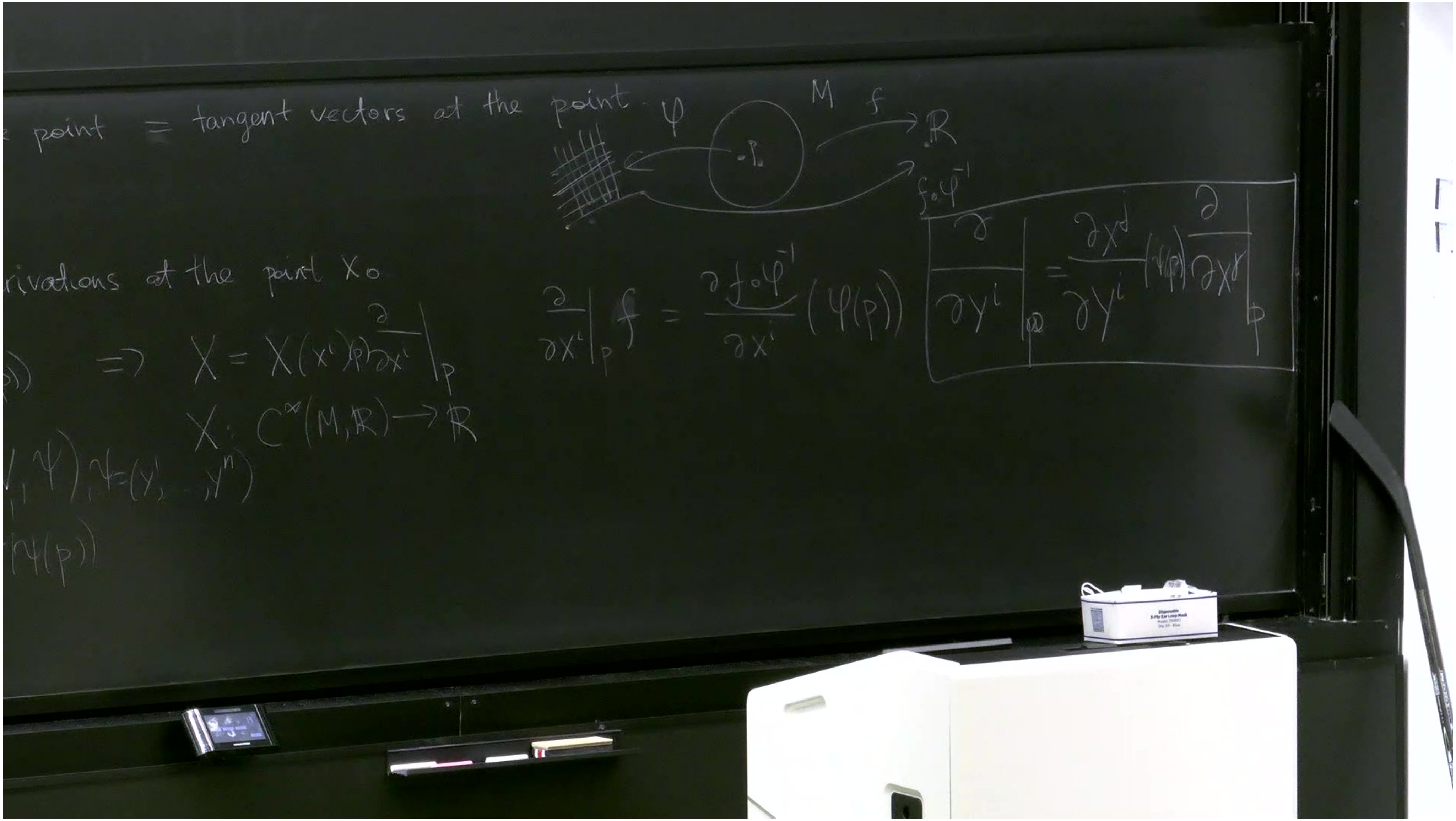
Def Tangent space at the point $p \in M$ is a space of derivations at the point x_0

Let's make it coordinate (U,φ) $\varphi: U \to \mathbb{R}^n$, $\varphi(p) = (x^1(p), \dots, x^n(p))$ $\Rightarrow$ $X = X(x^i)(p) \frac{\partial}{\partial x^i}\Big|_p$

$X: C^\infty(M,\mathbb{R}) \to \mathbb{R}$

Q How does it transform under the change of coordinate chart? (V,ψ), $\psi = (y^1, \dots, y^n)$

$$\frac{\partial}{\partial y^i}\Big|_p f = \frac{\partial f \circ \psi^{-1}}{\partial y^i}(\psi(p)) = \frac{\partial f \circ \varphi^{-1} \circ \varphi \circ \psi^{-1}}{\partial y^i}(\psi(p)) = \frac{\partial f \circ \varphi^{-1}}{\partial x^j}(\varphi(p)) \frac{\partial x^j \circ \psi^{-1}}{\partial y^i}(\psi(p))$$


point = tangent vectors at the point.
M
f
derivations at the point x₀
X = X(x^i)(p) ∂/∂x^i |_p
X : C^∞(M, R) → R
∂/∂x^i |_p f = ∂ f∘φ^{-1}/∂x^i (φ(p))
f∘φ^{-1}
∂/∂y^i |_p = ∂x^j/∂y^i (φ(p)) ∂/∂x^j |_p