Title: Lecture - Mathematical Physics, PHYS 777-

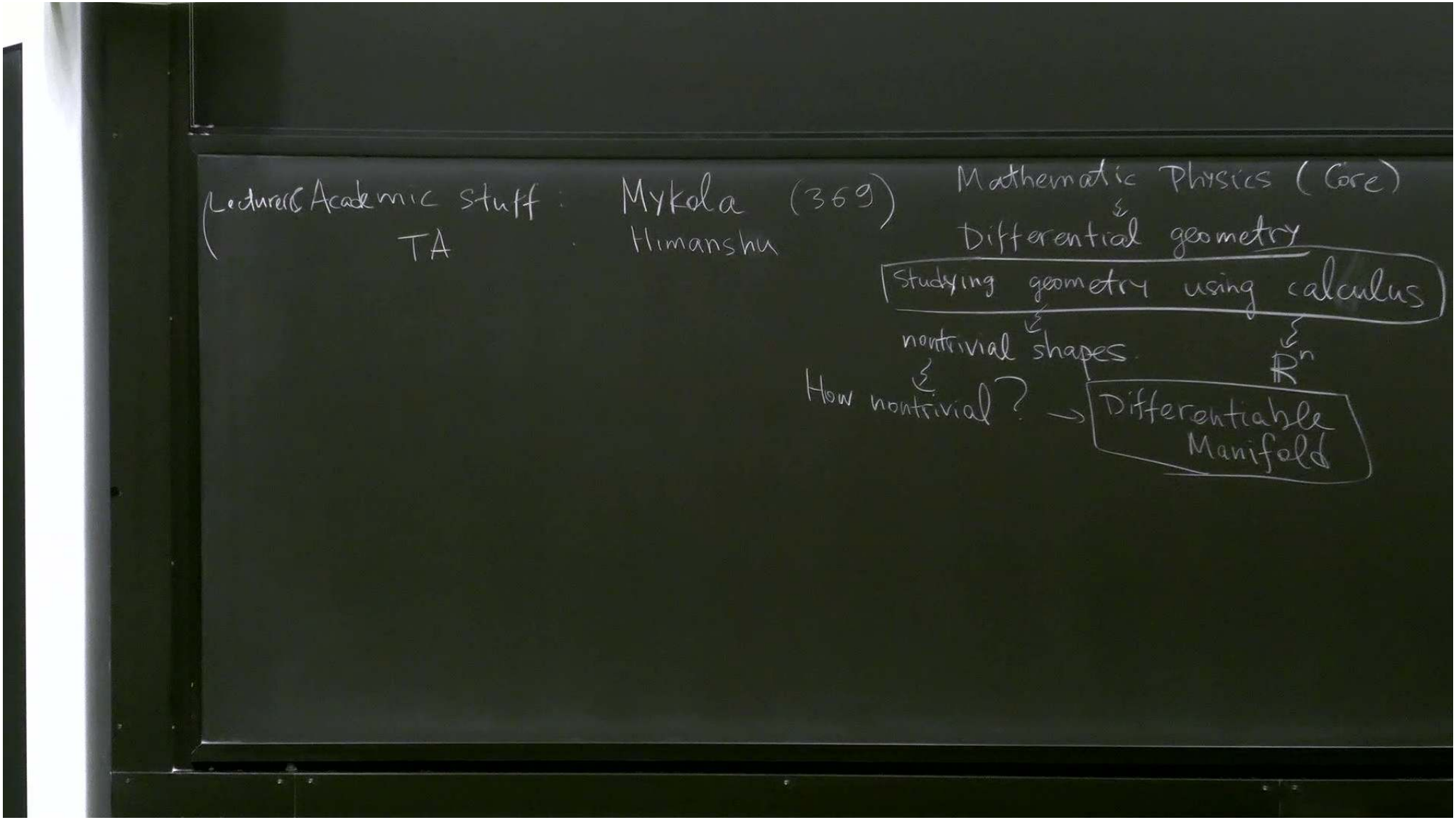
Speakers: Mykola Semenyakin

Collection/Series: Mathematical Physics (Core), PHYS 777-, January 6 - February 5, 2025

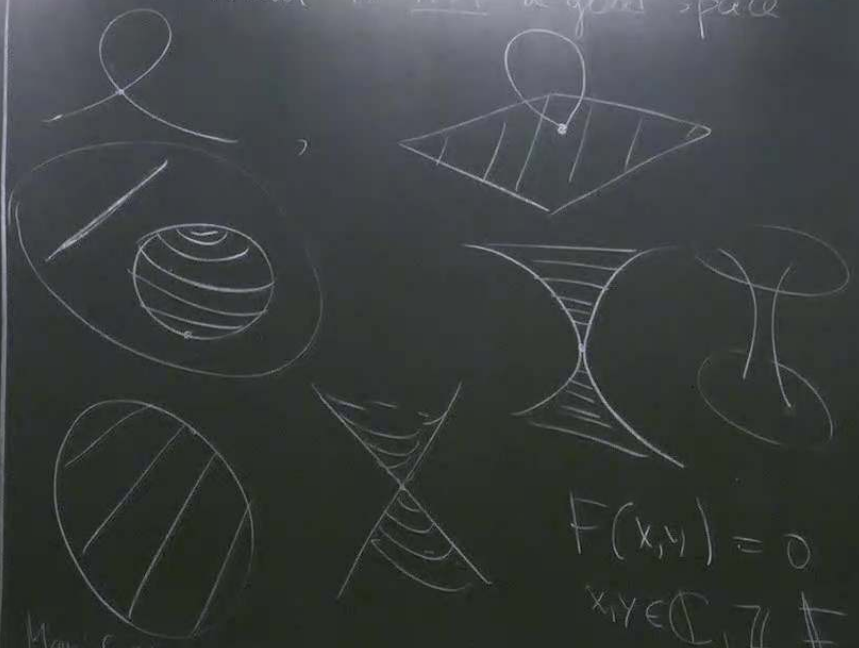
Subject: Mathematical physics

Date: January 07, 2025 - 9:00 AM

URL: https://pirsa.org/25010001


(Lecturer & Academic stuff : Mykola (369)
TA : Himanshu
Mathematic Physics (Core)
&
Differential geometry
Studying geometry using calculus
&
nontrivial shapes
&
How nontrivial? → Differentiable Manifold
$\mathbb{R}^n$

What is not a good space



Manifold with boundary

$F(x,y) = 0$

$x, y \in \mathbb{C}, \mathbb{Z}, \mathbb{F}_q$

algebraic geometry

What is a good space

Spaces which locally look like $\mathbb{R}^n$, n the same everywhere.

$\mathbb{R}^n$, system of m equation $F_i(x_1, \dots, x_n) = 0$, $i = 1, \dots, m$

if system is non-degenerate $\Rightarrow$ this defines n-m-dim manifold

Structures on spaces

$x^2 + y^2 + z^2 = 1 \subset \mathbb{R}^3$

→ Metric $g = g_{ij} dx^i \otimes dx^j$

→ Differential forms.

→ something that can be integr. $dx^1 \wedge dx^2 \wedge dx^3$

→ Tensors.

→ Symplectic form. - important in mechanics.

- Tangent bundle
- Fibre bundles
- Yang-Mills theory

☆ Topological space "something continuous"

X - set, topology on X: $\tau \subset 2^X$

e.g. $X = \{1,2,3,4\}$

$2^X = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{1,3\}, \ldots, \{1,2,3\}, \ldots, \{1,2,3,4\}\}$

$\mathbb{R}$

$[0,1] \setminus \{0,1\}$

open subsets, topology

- $\emptyset, X \in \tau$
- any (finite or infinite) union of $U_i \in \tau$ is in τ.
- any finite intersection of $U_i \in \tau$ is in τ.

$\{0\} = \bigcap_{n \geq 1} \left(-\frac{1}{n}, \frac{1}{n}\right)$

0 1

0.01

Useful example

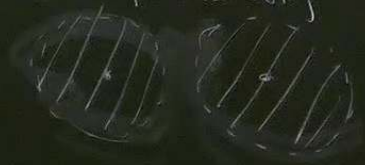
Any subset of $\mathbb{R}^n$ with induced topology

Useless example

$X = \{1,2,3,4\}$ $\tau = \{\emptyset, \{1,2,3,4\}, \{2\}, \{1,2\}, \{2,3\}, \{1,2,3\}\}$

- Continuous mapping $f: X \to Y$ $\quad f^{-1}(U)$ is open in X for $U \in \tau(Y)$, $U \subset Y$

- We are going to deal with top spaces (and the mappings of them) ☆☆ which are Hausdorff and second-countable.



"topology is finite" in some way

with induced topology. $\tau = \{\emptyset, \{1,2,3,4\}, \{1,2\}, \{1,2\}, \{2,3\}, \{1,2,3\}\}$

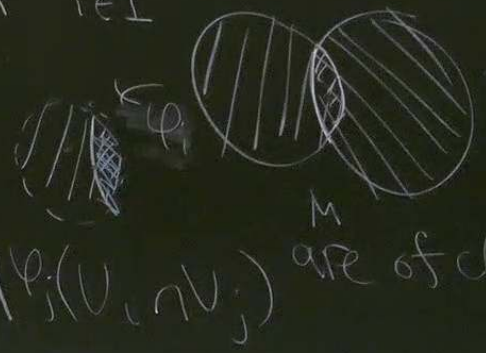
Def Chart (U, φ) $\varphi: U \longrightarrow \mathbb{R}^n$ (continuous)

M - topological space. $U \in \tau(M)$

- φ - injective
- its image is open

Def Atlas is a collection of charts on M (U_i, φ_i), $i \in I$:

- Charts cover all of M: $\bigcup_{i \in I} U_i = M$
- $\varphi_i(U_i \cap U_j)$ is open in $\mathbb{R}^n$
- the transition maps. $\varphi_{ij} = \varphi_i \circ \varphi_j^{-1}$ are of class C^∞



$\varphi_j(U_i \cap U_j)$

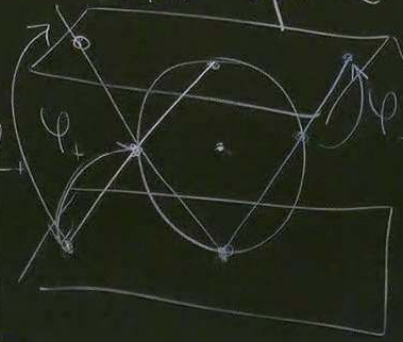
Definition A smooth n-manifold is a set M with n-dimensional smooth atlas (An). We say that n is dimension of M, $\dim M = n$.

differentiable structure

Remark One can replace C^∞ condition with C^0 condition, differentiable manifold → topological manifolds

Examples · $U \subset \mathbb{R}^n$

· Unit sphere: $S^2 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 = 1\} = U_+ \cup U_-$



$U_+ = S^2 \setminus \{(0,0,+1)\}$ $\varphi_+ =$

$U_- = S^2 \setminus \{(0,0,-1)\}$ $\varphi_- =$

$U_+ \cap U_- = S^2 \setminus \{(0,0,\pm 1)\}$

{tutorial 1, stereographic projection}