Title: Double groupoids and Generalized Kahler structures

Speakers: Marco Gualtieri

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Abstract:

The underlying holomorphic structure of a generalized Kahler manifold has been recently understood to be a square in the double category of holomorphic symplectic groupoids (or (1,1)-shifted symplectic stacks). I will explain what this means and how it allows us to describe the generalized Kahler metric in terms of a single real scalar function, resolving a conjecture made by physicists Gates, Hull, and Rocek in 1984. This is based on joint work with Yucong Jiang and Daniel Alvarez available at https://arxiv.org/abs/2407.00831.

Double groupoids and generalized Kähler structures

Math. Phys. seminar, PI.

Dec 12 2024

Marco Gualtieri
University of Toronto

TWISTED MULTIPLETS AND NEW SUPERSYMMETRIC NON-LINEAR σ-MODELS

S.J. GATES, Jr.* and C.M. HULL**

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139, USA

M. ROČEK***

Institute for Theoretical Physics, State University of New York, Stony Brook, NY 11794, USA

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A new $D = 2$ supersymmetric representation, the twisted chiral multiplet, is derived. Describing spins zero and one-half, the twisted multiplet is used to formulate supersymmetric nonlinear σ-models with $N = 2, 4$ extended supersymmetry. In general, the geometries of these new theories fall outside the classification given by Alvarez-Gaumé and Freedman. We give a complete description of the geometry of these new models; the scalar manifolds are *not Kähler* but are hermitian locally product spaces.

1. Introduction and Summary

The study of supersymmetric nonlinear sigma models has led to the discovery of a fascinating relation to complex manifold theory. (For a simple review of complex manifold theory, see [1].) In particular, for a certain class of theories, it has been

multiplet is independent of all θ's;
of $P_+\theta$ and $P_-\bar{\theta}$ ($P_\pm$ are proj
superfields are somewhat reminisce
We use this to construct new $N =$
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The Kähler geometry of the bos
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further SO(1, 1) action on tangent

The standard $N = 4$ models hav
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generate a smooth SU(2) action c
commuting quaternionic structures,
formed from products of any two o
an SL(4, R) algebra, the comple
SO(4) ≃ SU(2) × SU(2) subalgebra,
compact generators. Then the m
truncating to an $N = 2$ theory,
U(1) × U(1) × SO(1, 1), i.e., the su
subalgebra.

Remarkably, the metric of the
derivatives of a single real functi
potential of a Kähler manifold.

In section two, the results of A

representation. (The existence of this representation had been noted previously by W. Siegel.) We call this the "twisted chiral" multiplet. Whereas the usual chiral multiplet is independent of all $\bar{\theta}$'s; the $N=2$ twisted chiral multiplet is independent of $P_{+}\theta$ and $P_{-}\bar{\theta}$ ($P_{\pm}$ are projectors onto states of definite helicity). These superfields are somewhat reminiscent of the "Grassmann analytic" superfields of [7]. We use this to construct new $N=2,4$ supersymmetric nonlinear σ-models and the metrics on the associated manifolds. We also consider dimensional reduction of $D=4$ models to $D=3$ and 2. We study duality transformations and find terms analogous to the Wess-Zumino-Witten term [8]. (Such terms have also been found in connection with duality transformations in [9].) These theories generalize the results of [2, 3]. *In particular, they involve complex structures which are covariantly constant with respect to a connection with torsion.*

The Kähler geometry of the bosonic manifold of the usual $N=2$ models implies the existence of a complex structure $f_i{}^j$, squaring to *minus* the identity ($f_i{}^k f_k{}^j = -\delta_i{}^j$), which can be used to generate a smooth U(1) action on tangent vectors. The manifolds of our new theories, although not Kähler, possess *two commuting complex structures*, $f_{+i}{}^j$, $f_{-i}{}^j$, generating a $\mathrm{U}(1)\times\mathrm{U}(1)$ action. Their product, $\Pi_i{}^j \equiv f_{+i}{}^k f_{-k}{}^j$ squares to *plus* the identity ($\Pi_i{}^k \Pi_k{}^j = \delta_i{}^j$) and is known as an *almost product structure* [14]. The projectors $\frac{1}{2}(\mathbb{1} \pm \Pi)$ provide a natural decomposition of the tangent space and the manifold is a *locally product space* (defined in appendix C) [14], although in general irreducible. The almost product structure generates a further SO(1,1) action on tangent vectors.

The standard $N=4$ models have hyper-Kähler geometries that possess a quaternionic structure, i.e., three complex structures satisfying an SU(2) algebra. These generate a smooth SU(2) action on tangent vectors. Our $N=4$ models have *two*

$$S_{\pm ijk} = \tfrac{1}{4}\partial_{[i}h_{\pm jk]}. \tag{99}$$

Then (97) is equivalent to

$$\Pi_i{}^k h_{\pm kj} = -\Pi_j{}^k h_{\pm ki}. \tag{100}$$

Finally, a third way to impose this condition is to require that the submanifolds projected out by

$$\tfrac{1}{2}\left(\delta_i{}^j \pm \Pi_i{}^j\right) \tag{101}$$

are Kähler.

We emphasize that the discussion of locally product geometries is relevant only to the models formulated in sect. 5; if we do not assume that the complex structures f_+, f_- commute (76), we obtain a more general class of σ-models with action (58) and a hermitian scalar manifold that need not admit an almost product structure. We hope to analyze these models more fully in the future.

Additional supersymmetries require more complex structures $f_\pm{}^{(M)}{}_i{}^j$. These satisfy the conditions (68)–(70), while (67) and (71) are replaced by the more stringent conditions

$$f_\pm{}^{(M)}{}_i{}^j f_\pm^{(N)}{}_j{}^k + f_\pm^{(N)}{}_i{}^j f_\pm^{(M)}{}_j{}^k = -2\delta^{MN}\delta_i{}^k, \tag{102}$$

$$T_{ijk}\delta^{MN} = \tfrac{1}{2}T_{lm[i}f_\pm^{(M)}{}_j{}^l f_\pm^{(N)}{}_{k]}{}^m, \tag{103}$$

and thus for $N = 4$ supersymmetry the manifold has two quaternionic structures and therefore a dimensionality that is a multiple of four.

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Abstract

A description of the fundamental degrees of freedom underlying a generalized Kähler manifold, which separates its holomorphic moduli from the space of compatible metrics in a similar way to the Kähler case, has been sought since its discovery in 1984. In this paper, we describe a full solution to this problem for arbitrary generalized Kähler manifolds, which involves the new concept of a holomorphic symplectic Morita 2-equivalence between double symplectic groupoids, equipped with a Lagrangian bisection of its real symplectic core. Essentially, any generalized Kähler manifold has an associated holomorphic symplectic manifold of quadruple dimension and equipped with an anti-holomorphic involution; the metric is determined by a Lagrangian submanifold of its fixed point locus. This finally resolves affirmatively a long-standing conjecture by physicists concerning the existence of a generalized Kähler potential.

We demonstrate the theory by constructing explicitly the above Morita 2-equivalence and Lagrangian bisection for the well-known generalized Kähler structures on compact even-dimensional semisimple Lie groups, which have until now escaped such analysis. We construct the required holomorphic symplectic manifolds by expressing them as moduli spaces of flat connections on surfaces with decorated boundary, through a quasi-Hamiltonian reduction.

Def: (v.1) A GK structure is

$$(\underbrace{M, g}_{\text{Riemannian Mfld}}, \underbrace{I_+, I_-}_{\text{Pair of complex structures}}, H)$$

s.t. $d^c_+ \omega_+ + d^c_- \omega_- = 0$

$$d d^c_\pm \omega_\pm = 0$$

$$H := d^c_+ \omega_+$$

$$\left(d^c_\pm := [d, I^*_\pm] \qquad \omega_\pm := g I_\pm \right)$$

Examples

$M = G$ Even-dim real Lie group

g = bi-invt metric

$$I_\pm = \left(\begin{array}{ccc|ccc} & -i & & & +i & \\ \cdot & & & & & \cdot \\ & \cdot & & \cdot & & \end{array} \right) \text{Right/Left}$$

$$d^c_+ \omega_+ = - d^c_- \omega_- = H$$

Cartan 3-form

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Galois Symmetry

$$\begin{tikzcd}
(I_+, I_-) \arrow[r, dash] \arrow[d, dash] & (-I_+, I_-) \arrow[d, dash] \\
(I_+, -I_-) \arrow[r, dash] & (-I_+, -I_-)
\end{tikzcd}$$

$$\mathbb{Z}_2 \times \mathbb{Z}_2$$

Double complex conjugation

Def: (v.1) A GK structure is

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$$dd^c_\pm \omega_\pm = 0$$

$$H := d^c_+\omega_+$$

$$\left(d^c_\pm := [d, I^*_\pm] \qquad \omega_\pm := g I_\pm \right)$$

Def: (v.2) A GK structure is

$$(\underbrace{M, H}_{\text{Closed 3-form}}, \underbrace{\mathbb{J}_A, \mathbb{J}_B}_{\text{Generalized complex}})$$

s.t. $\quad \mathbb{J}_A \mathbb{J}_B = \mathbb{J}_B \mathbb{J}_A$

$$\mathbb{G} := \langle \mathbb{J}_A -, \mathbb{J}_B - \rangle > 0$$

$$\mathbb{J} : T_M \oplus T^*_M \circlearrowleft \qquad \mathbb{J}^2 = -1$$

$$\langle X+\xi, Y+\eta \rangle = \tfrac{1}{2}(\xi(Y) + \eta(X))$$

$$[X+\xi, Y+\eta] = [X,Y] + L_X\eta - i_Y d\xi + i_X i_Y H$$

Def: (v.1) A GK structure is

$$(M, g, I_+, I_-, H)$$

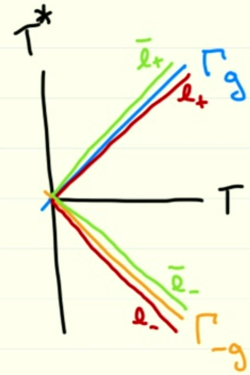
Thm (MG) Equivalence given by:

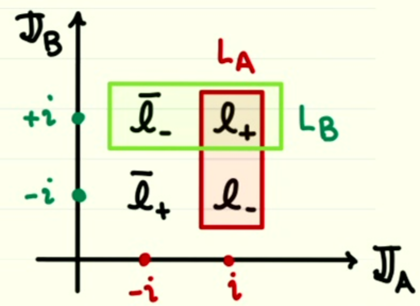
$$\mathbb{J}_{A/B} = \frac{1}{2}\begin{pmatrix} I_+ \mp I_+ & -(\omega_+^{-1} \pm \omega_-^{-1}) \\ \omega_+ \pm \omega_- & -(I_+^* \mp I_-^*) \end{pmatrix}$$

Def: (v.2) A GK structure is

$$(M, H, \mathbb{J}_A, \mathbb{J}_B)$$

$\mathbb{J}_A\mathbb{J}_B = \mathbb{J}_B\mathbb{J}_A$ $\quad \langle \mathbb{J}_A -, \mathbb{J}_B - \rangle > 0$





$$d^c_+\omega_+ + d^c_-\omega_- = 0 \quad \longleftrightarrow \quad \mathbb{J}_A, \mathbb{J}_B \text{ integrable}$$

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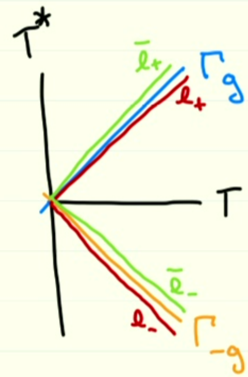

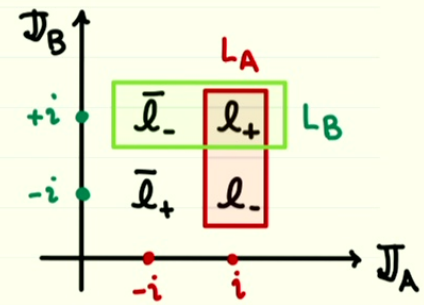

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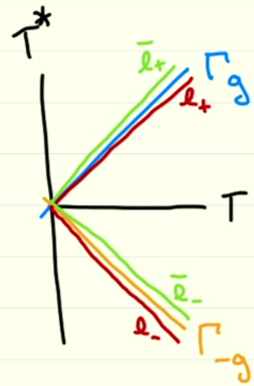

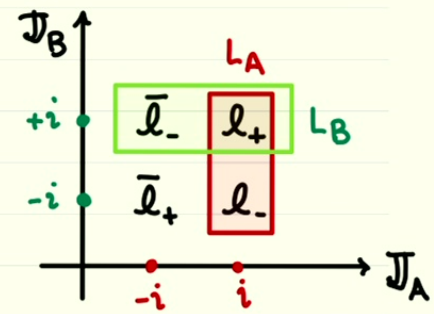

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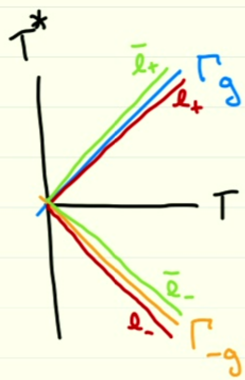
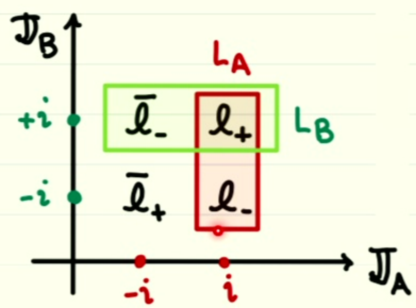




$\ell_+ \cong T_{1,0}(I_+) \qquad \ell_- \cong T_{1,0}(I_-)$

$T_{\mathbb{C}} \oplus T^*_{\mathbb{C}}$

all contain $\bar\ell_+ \cong T_{0,1}(I_+)$

$$\left.\begin{aligned} T_{\mathbb{C}} \oplus T^*_{\mathbb{C}} /\!/ \bar\ell_+ &= \mathcal{E}_+ \\ \bar L_A /\!/ \bar\ell_+ &= \mathcal{A}_+ \\ \bar L_B /\!/ \bar\ell_+ &= \mathcal{B}_+ \end{aligned}\right\}$$

Hol. Manin Triple over $X_+ = (M, I_+)$

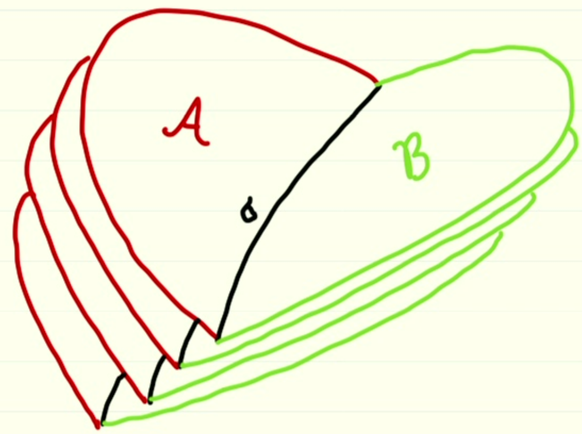
Thm On each cx mfld $X_\pm = (M, I_\pm)$, this defines a holomorphic Manin Triple

$$\mathcal{E}_\pm = \mathcal{A}_\pm \oplus \mathcal{B}_\pm$$

$$\mathcal{B}_\pm - \mathcal{A}_\pm = \Gamma_{\sigma_\pm}$$

$\sigma_\pm$ Holom. Poisson.

inducing on each $X_\pm$ a pair of transverse singular hol. foliations:



intersecting in the Hitchin poisson structure.

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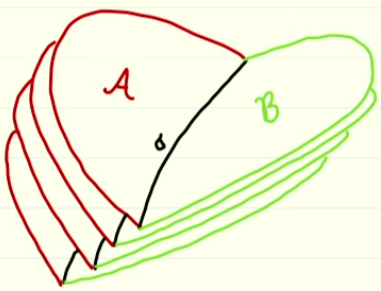
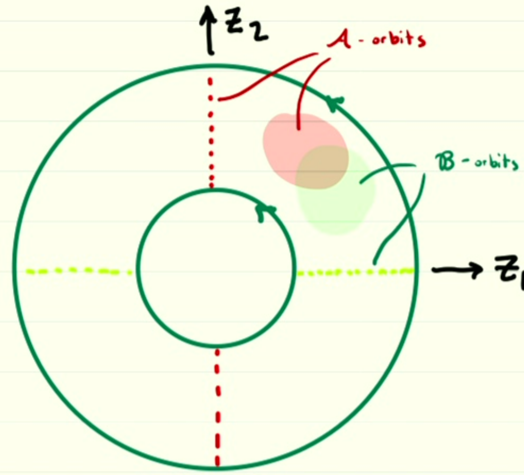


Example: $X = \mathbb{C}^2\backslash 0 / \vec{z} \sim 2\vec{z}$

$= SU(2) \times U(1)$



$$\sigma = z_1 z_2 \frac{\partial}{\partial z_1} \wedge \frac{\partial}{\partial z_2}$$

Main Questions:

① How are the holomorphic data $(X_+, \mathcal{E}_+, \mathcal{A}_+, \mathcal{B}_+)$

related to $(X_-, \mathcal{E}_-, \mathcal{A}_-, \mathcal{B}_-)$?

② How is the Gen. Kähler metric determined,

i.e. is there a potential function ?

1. Relation between Courant Algebroids

$\omega \in \Omega^{1,1} \qquad dd^c\omega = 0$

$H = d^c\omega = \mathrm{Re}\left(\mathcal{H} = -2i\,\partial\omega\right)$

$\bar{\mathcal{H}} \in \Omega^{1,2}$ $d\bar{\mathcal{H}} = 0$ $\bar{\mathcal{E}} \to \bar{X}$	$\mathcal{H} \in \Omega^{2,1}$ $d\mathcal{H} = 0$ $\mathcal{E} \to X$

$$\bar{\mathcal{H}} - \mathcal{H} = d\,(2i\omega)$$

gauge equivalence of Matched Pairs of $\mathcal{E}, \bar{\mathcal{E}}$

In GK case

$$d^c_+\omega_+ = -d^c_-\omega_- = H$$

$$\begin{tikzcd}
\bar{\mathcal{H}}_- & & \mathcal{H}_+ \\
& H \arrow[ul, "-i\omega_-"'] \arrow[ur, "-i\omega_+"] \arrow[dl, "i\omega_+"] \arrow[dr, "i\omega_-"'] & \\
\bar{\mathcal{H}}_+ & & \mathcal{H}_-
\end{tikzcd}$$

$$\begin{tikzcd}
\bar{\mathcal{E}}_- & \mathcal{E}_+ \arrow[l, "i(\omega_+-\omega_-)"', "\cong"] \\
\bar{\mathcal{E}}_+ \arrow[u, "-i(\omega_++\omega_-)", "\cong"'] & \mathcal{E}_- \arrow[l, "\cong"', "i(\omega_+-\omega_-)"] \arrow[u, "\cong", "-i(\omega_++\omega_-)"']
\end{tikzcd}$$

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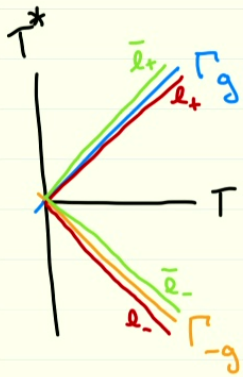

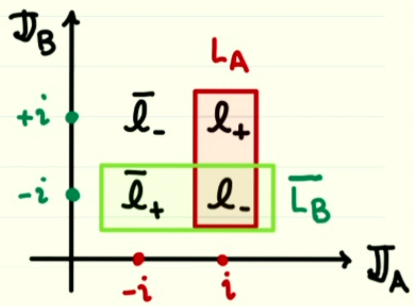

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$\bar{\mathcal{E}} \to \bar{X}$	$\mathcal{E} \to X$

$$\bar{\mathcal{H}} - \mathcal{H} = d\,(2i\omega)$$

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\end{tikzcd}$$

$$\begin{tikzcd}
\bar{\mathcal{E}}_- & & \mathcal{E}_+ \arrow[ll, "i(\omega_+-\omega_-)"', "\cong"] \\
\bar{\mathcal{E}}_+ \arrow[u, "-i(\omega_++\omega_-)", "\cong"'] & & \mathcal{E}_- \arrow[ll, "\cong"', "i(\omega_+-\omega_-)"] \arrow[u, "\cong", "-i(\omega_++\omega_-)"']
\end{tikzcd}$$

2. Relation between Manin triples

$$
\begin{tikzcd}
& \bar{\mathcal{B}}_- & \mathcal{B}_+ \arrow[l, "i(\omega_+ - \omega_-)"', "\cong"] & \\
\bar{\mathcal{A}}_- & \bar{\mathcal{E}}_- & \mathcal{E}_+ \arrow[l] & \mathcal{A}_+ \\
\bar{\mathcal{A}}_+ \arrow[u, "-i(\omega_+ + \omega_-)", "\cong"'] & \bar{\mathcal{E}}_+ \arrow[u] & \mathcal{E}_- \arrow[l] \arrow[u] & \mathcal{A}_- \arrow[u, "\cong", "-i(\omega_+ + \omega_-)"'] \\
& \bar{\mathcal{B}}_+ & \mathcal{B}_- \arrow[l, "\cong"', "i(\omega_+ - \omega_-)"] &
\end{tikzcd}
$$

Gauge equivalence of Matched pairs

Thm (D. Álvarez, M.G., Y. Jiang)

Let $(\mathcal{E}_\pm, A_\pm, B_\pm)$ hol. Manin Triples and $F_1, F_2 \in \Omega^2(M,\mathbb{R})$ s.t.

$$\begin{array}{ccccc} & \bar{B}_- & \xleftarrow[\cong]{} & B_+ & \\ \bar{A}_- & \bar{\mathcal{E}}_- & \xleftarrow[iF_2]{} & \mathcal{E}_+ & A_+ \\ \uparrow\cong & \uparrow iF_1 & & iF_1 \uparrow & \uparrow\cong \\ \bar{A}_+ & \bar{\mathcal{E}}_+ & \xleftarrow{iF_2} & \mathcal{E}_- & A_- \\ & \bar{B}_+ & \xleftarrow{\cong} & B_- & \end{array}$$

gauge equivalences of Matched Pairs.

Then:

$$\tfrac{1}{2}(F_1 \mp F_2) I_\pm = g_\pm + b_\pm$$

is s.t. $g_+ = g_- = g$,

$b_+ = -b_- = b$,

and $Re(\mathcal{H}_+) - db = Re\,\mathcal{H}_- - db =: H$

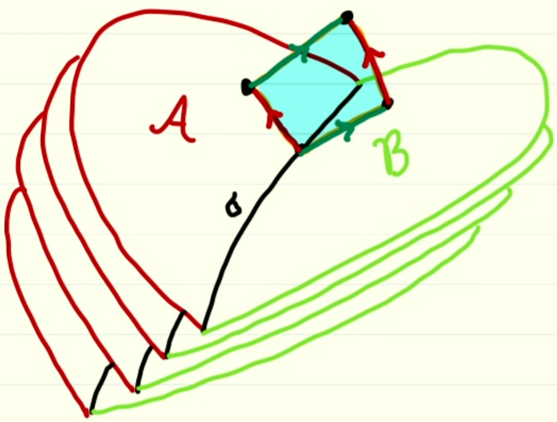
is s.t. $\pm d^c_\pm \omega_\pm = H$.

$\Rightarrow$ Gen. Kähler if g pos. def.

Key Idea (Lu - Weinstein, Mackenzie - Xu)

Manin triples are the infinitesimal objects of

Symplectic Double Lie Groupoids $\subset$ (1,1) - shifted symplectic stacks



$$\begin{tikzcd} D \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & B \arrow[d, shift left] \arrow[d, shift right] \\ A \arrow[r, shift left] \arrow[r, shift right] & X \end{tikzcd}$$

horizontal & vertical composition.

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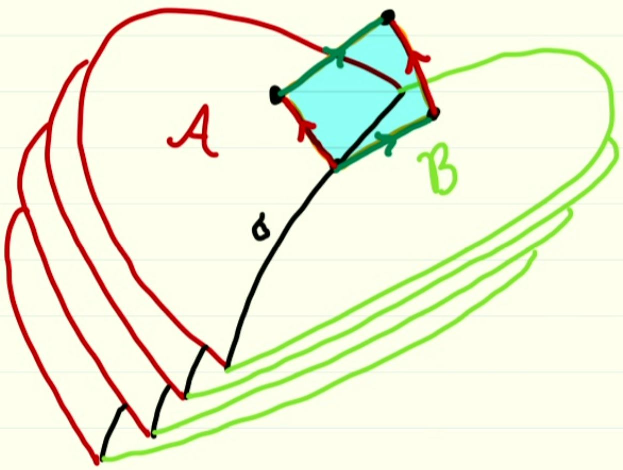
Symplectic Double Lie Groupoids $\subset$ (1,1) - shifted symplectic stacks



$$\begin{array}{ccc} D & \rightrightarrows & B \\ \downdownarrows & & \downdownarrows \\ A & \rightrightarrows & X \end{array}$$

horizontal & vertical composition.

Generalized morphisms

For groupoids:

$$
\begin{tikzcd}
G_X \arrow[d, shift left] \arrow[d, shift right] & & Z \arrow[dll] \arrow[drr] & & G_Y \arrow[d, shift left] \arrow[d, shift right] \\
X & & & & Y
\end{tikzcd}
$$

Morita equivalence

(bi-principal bibundle)

For double groupoids:

$$
\begin{tikzcd}
D' \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & A' \arrow[d, shift left] \arrow[d, shift right] & W_1 \arrow[l] \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & A \arrow[d, shift left] \arrow[d, shift right] & D \arrow[l, shift left] \arrow[l, shift right] \arrow[d, shift left] \arrow[d, shift right] \\
B' \arrow[r, shift left] \arrow[r, shift right] & X' & W_0 \arrow[l] \arrow[r] & X & B \arrow[l, shift left] \arrow[l, shift right] \\
 & & & Z_0 \arrow[u] \arrow[d] & Z_1 \arrow[l, shift left] \arrow[l, shift right] \arrow[u] \arrow[d] \\
 & & & X'' & B'' \arrow[l, shift left] \arrow[l, shift right] \\
 & & & A'' \arrow[u, shift left] \arrow[u, shift right] & D'' \arrow[l, shift left] \arrow[l, shift right] \arrow[u, shift left] \arrow[u, shift right]
\end{tikzcd}
$$

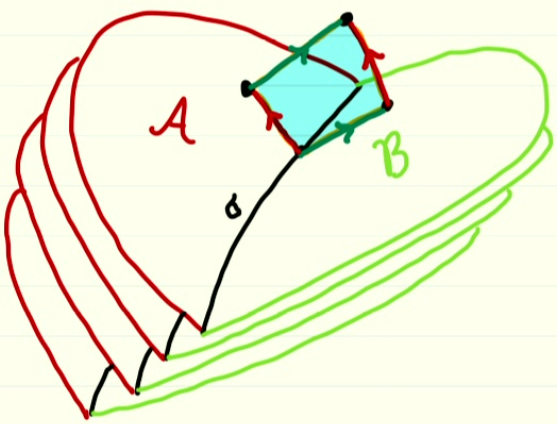
(1,0) and (0,1)

gen. morphisms,

Key Idea (Lu - Weinstein, Mackenzie - Xu)

Manin triples are the infinitesimal objects of

Symplectic Double Lie Groupoids $\subset$ $(1,1)$-shifted symplectic stacks



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\end{tikzcd}$$

(1,0) and (0,1) gen. morphisms,

Def: self-adjoint $(1,1)$-morphism:

$$\begin{tikzcd}
\overline{D}_- & & D_+ \arrow[ll, "W"'] \\
& \arrow[ur, Rightarrow, "C"'] & \\
\overline{D}_+ \arrow[uu, "\overline{Z}^T"] & & D_- \arrow[ll, "\overline{W}^T"] \arrow[uu, "Z"']
\end{tikzcd}$$

$$\begin{array}{ccccccccc}
\overline{D}_- & \rightrightarrows & \overline{A}_- & \leftarrow & W_1 & \rightarrow & A_+ & \leftleftarrows & D_+ \\
\downdownarrows & & \downdownarrows & & \downdownarrows & & \downdownarrows & & \downdownarrows \\
\overline{B}_- & \rightrightarrows & \overline{X}_- & \leftarrow & W_0 & \rightarrow & X_+ & \leftarrow & B_+ \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
\overline{Z}_1 & \rightrightarrows & \overline{Z}_0 & \leftarrow & C & \rightarrow & Z_0 & \leftleftarrows & Z_1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\overline{B}_+ & \rightrightarrows & \overline{X}_+ & \leftarrow & \overline{W}_0 & \rightarrow & X_- & \leftleftarrows & B_- \\
\upuparrows & & \upuparrows & & \upuparrows & & \upuparrows & & \upuparrows \\
\overline{D}_+ & \rightrightarrows & \overline{A}_+ & \leftarrow & \overline{W}_1 & \rightarrow & A_- & \leftleftarrows & D_-
\end{array}$$

A square of $(1,0)$, $(0,1)$-morphisms

filled with a Symplectic double Morita bimodule

and equipped with real structure $\tau : \square \to \overline{\square}^T$, $\quad \tau \overline{\tau}^T = \mathrm{Id}.$

$$\tau^* \overline{\Omega} = \Omega$$

Real Structure inherited by C:

$$\sigma : (C, \Omega_C) \to (\bar{C}, \bar{\Omega}_C)$$

Fixed Point set $(S, \omega_S) \subset C$

Symplectic Core

$$\begin{tikzcd} & S \arrow[dl] \arrow[dr] & \\ \mathrm{Re}\, X_+ & & \mathrm{Re}\, X_- \end{tikzcd}$$

Lagrangian bisections

= GK metrics

Potentials = Generating functions

Thm (D.Á., M.G., Y.J)

The global structure governing generalized Kähler geometry is

- self-adjoint (1,1) morphism of hol. symp. Double groupoids

  (G. Kähler class)

- real Lagrangian bisection of real symplectic core.

  (G. Kähler metric)

Application: Quantization

Gen. Kähler case (symplectic type)

$$
\begin{array}{ccccc}
(G, \Omega_G) & & (Z, \Omega_Z) & & (G, \Omega_G) \\
\downdownarrows & \swarrow & \uparrow & \searrow & \downdownarrows \\
(X, \sigma) & & \mathcal{L} & & (X, \sigma)
\end{array}
$$

(Z, Ω_Z) = Hol. symplectic Morita equivalence

$\mathcal{L} \subset Z$ — $\mathrm{Im}\,\Omega_Z$-Lagrangian, C^∞ bisection.

Real symplectic mfld: $(Z, \mathrm{Im}\,\Omega)$

Branes:

$$B_0 = (Z, \mathrm{Re}\,\Omega = \mathrm{curv}\nabla/2\pi i)$$

$$B_1 = \mathcal{L}$$

$$\mathrm{Hom}_A(B_0, B_1) = H^0(X, \underset{\substack{\uparrow \\ \text{prequantum bundle}}}{L_\omega})$$

$Z \underset{X}{\times} \cdots \underset{X}{\times} Z$ — k-twisted cotangent

$$\mathrm{Hom}(B_0^{*k}, B_1^{*k}) = H^0(X, L_\omega^k)$$

$\Rightarrow$ graded noncommutative algebra

Application: Quantization

Gen. Kähler case (symplectic type)

$$
\begin{tikzcd}
(G,\Omega_G) \arrow[d, shift left] \arrow[d, shift right] & (Z,\Omega_Z) \arrow[dl] \arrow[dr] & (G,\Omega_G) \arrow[d, shift left] \arrow[d, shift right] \\
(X,\sigma) & \mathcal{L} \arrow[u, hook] & (X,\sigma)
\end{tikzcd}
$$

(Z,Ω_Z) = Hol. symplectic Morita equivalence

$\mathcal{L} \subset Z$ $\quad$ $\operatorname{Im}\Omega_Z$ - Lagrangian

C^∞ bisection.

Real symplectic mfld: $(Z, \operatorname{Im}\Omega)$

Branes:

$$B_0 = (Z, \operatorname{Re}\Omega = \operatorname{curv}\nabla/2\pi i)$$

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prequantum bundle

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$Z \times_X \cdots \times_X Z$ $\quad$ k-twisted cotangent

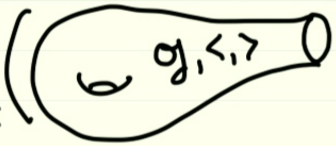
$$\operatorname{Hom}(B_0^{*k}, B_1^{*k}) = H^0(X, L_\omega^k)$$

$\Rightarrow$ graded noncommutative algebra

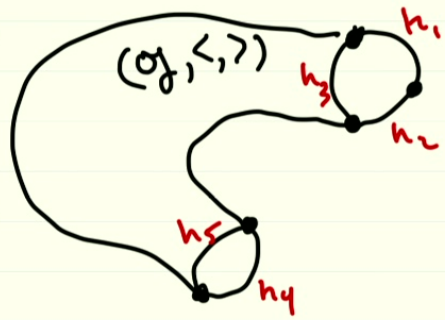
Constructing Double Morita equivalences: Moduli of flat connections

$\mathcal{M}_{flat}\Bigg($  $\Bigg)$

symplectic

$\mathcal{M}_{flat}\Bigg($  $\Bigg)$

Poisson

$\mathcal{M}_{flat}\Bigg($  $\Bigg)$

Symplectic

$h_i \subset \mathfrak{g}$ Lagrangian

$h_+ \cap h_- = 0$

GK str on Lie groups

K compact even-dim^{l} group

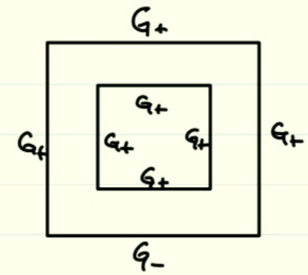
$G = K_{\mathbb{C}}$

I complex str on $\mathfrak{k}$ with

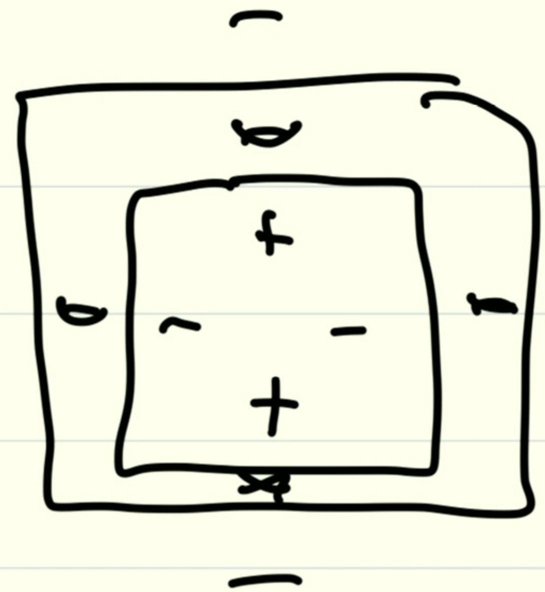
$$\mathfrak{g}_{10} = \mathfrak{g}_+ = \mathfrak{n}_+ \oplus \mathfrak{t}_{1,0}$$

$$\mathfrak{g}_{01} = \mathfrak{g}_- = \mathfrak{n}_- \oplus \mathfrak{t}_{0,1}$$

$(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ Manin triple

$\mathcal{M}^{\text{flat}}_{G}\Big($  $\Big)$

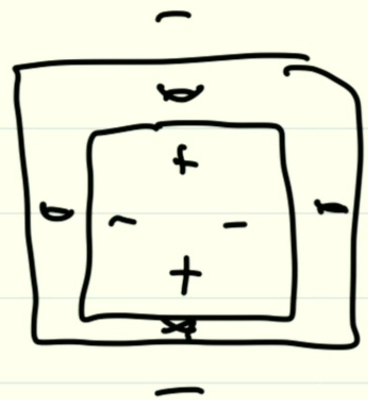
$$\mathcal{M}^{\text{flat}}_{G}\Big(+ $$



$$ + \Big)$$

ps

lim^{d} group

$$\mathcal{M}^{flat}_{G}\Big(+ $$  $$ + \Big)$$

with

$M_{+} \oplus t_{1,0}$