Title: Phase Spaces and Operator Algebras for Subregions in Gauge Theory and Quantum Gravity

Speakers: Marc Klinger

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Abstract:

What does it mean to specify a subregion in a diffeomorphism invariant fashion? This subtle question lies at the heart of many deep problems in quantum gravity. In this talk, we will explore a program of research aimed at answering this question. The two principal characters of the presentation are the extended phase space and the crossed product algebra. The former furnishes a symplectic structure which properly accounts for all of the degrees of freedom necessary to invariantly specify a subregion in gauge theory and gravity, while the latter serves as a quantization of this space into an operator algebra which formalizes the observables of the associated quantum theory. The extended phase space and the crossed product were originally motivated by the problems of the non-invariance/non-integrability of symmetry actions in naive subregion phase spaces, and the non-factorizability/divergence of entanglement entropy in naive subregion operator algebras. The introduction of these structures resolves these issues, while the correspondence between them unifies these resolutions. To illustrate the power of our framework, we demonstrate how the modular crossed product of semiclassical quantum gravity can be reproduced via this approach. We then provide some remarks on how this construction may be augmented in the non-perturbative regime, leading to the notion of a `fuzzy subregion'. We conclude with remarks on currently ongoing and future work, which includes applications to asymptotic and corner symmetries, quantum reference frames, generalized entropy, and the definition of quantum diamonds.

Phase Spaces and Operator Algebras for Subregions in Gauge Theory and Quantum Gravity

with applications to corner symmetry and generalized entropy

Marc Klinger[1]

[1]University of Illinois at Urbana-Champaign

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Introduction

- What does it mean to define a subregion in quantum gravity?
- We can't just draw a boundary and say that our theory lives inside; general diffeomorphisms will move and distort the boundary.
- This seems to lead us to the idea of a 'fuzzy subregion'. In QG we shouldn't think of a subregion as being fixed, but rather as occupying a 'cloud'.
- In this talk, I will provide some insight into how one might quantify this 'cloud' formulation of local physics in gauge theory and quantum gravity.

Outline of Talk

1. Physical Motivation
2. The Extended Phase Space
3. The Crossed Product
4. Looking Forward

The Non-Invariance of the Symplectic Form

- Classically, we study subregion physics by specifying a (possibly bounded) co-dimension one submanifold of spacetime $R \subset M$, and constructing a phase space X_R comprised of the field degrees of freedom therein.
- The Poisson algebra of functions on X_R is governed by the symplectic form Ω_R, which is typically an integral over a density $\omega_R \in \Omega^2(X_R, \Omega^{d-1}(R))$.
- In gravity, we expect that diffeomorphisms should act upon the phase space as a symmetry. In symplectic geometry, this means that we expect diffs should preserve the symplectic form.
- However, because the action of general diffeomorphisms will distort the subregion, the symplectic form is not left invariant under diffeomorphisms.

The Non-Factorizability of the Hilbert Space

- Quantumly, we aim to study subregion physics by constructing an algebra of operators affiliated with the subregion A_R.
- We may also be interested in understanding the Hilbert space representations of this algebra, which furnish a set of quantum states.
- Suppose that $\Sigma \subset M$ is a Cauchy surface which 'factorizes' as $\Sigma = R \cup_S R^c$.
- Here, R^c is the causal complement of R and S is a shared boundary which we will refer to as the entangling surface.
- Naively, we might hope that $H_\Sigma = H_R \otimes H_{R^c}$. But, in general, Hilbert spaces do not factorize across subregions.

Towards a Resolution

- The two problems we have addressed are closely related to each other.
- In the classical phase space analysis, the symplectic form is not invariant under general diffeomorphisms due to flux that leaks out through the entangling surface.
- The non-factorizability of the Hilbert space disallows the definition of reduced density operators for the subregion, which prevents the computation of entanglement observables.
- A proposal ([DF16]): We should promote degrees of freedom living on the boundary of a subregion to be dynamical.
- $$X_R \mapsto X_R \times_S G_S, \qquad H_\Sigma \simeq H_R \otimes_S H_{R^c}. \tag{1}$$

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Sketch of the Extended Phase Space

- Consider a Lagrangian theory with fields Φ and momenta Π defined on a bounded subregion R admitting an action by a group G.
- To complete the phase space, we introduce $\varphi_{(1)}$ which encodes an embedding of R along with an action of the group G, and restricts to a map $\varphi_{(2)} \equiv \varphi_{(1)}|_{\partial R}$.
- The extended symplectic potential is of the form

$$\Theta_R^{ext} = \int_R \varphi_{(1)}^* \left(\tilde{\Pi} \wedge \delta\tilde{\Phi} + M_A \wedge \delta\varphi_{(1)}^A \right) + \int_{\partial R} \varphi_{(2)}^* Q_A \wedge \delta\varphi_{(2)}^A. \quad (2)$$

- Here, $J = M + dQ$ is the Noether current derived from the invariance of the action under transformations by the group G, $(\tilde{\Phi}, \tilde{\Pi})$ are 'dressed fields', and $\delta\varphi_{(i)}^A$ are infinitesimal generators of group transformations.

Gauge Symmetries and Global Symmetries

- The dynamical fields of the extended phase space are $(\tilde{\Phi}, \tilde{\Pi}, \varphi^A_{(1)}, M_A, \varphi^A_{(2)}, Q_A)$.
- In a typical gauge theory, the part of the Noether current supported at co-dimension one furnishes a set of gauge constraints. We expect that $M_A \hat{=} 0$, e.g. these constraints vanish on shell.*
- By contrast, the corner supported charges Q_A needn't vanish on shell and therefore do not coincide with constraints. Rather, these charges encode a 'global' symmetry which acts upon the phase space even on shell.
- As a technical aside, $\delta\varphi^A_{(1)}$ and $\delta\varphi^A_{(2)}$ are flat connections for the groups $G_{(1)}$ and $G_{(2)}$ which encode the restriction of the symmetry group G to its action at co-dimension one and two.

Benefits of Extension

- As a result of extension, the action of G on the phase space is guaranteed to preserve the symplectic form (e.g. it preserves Poisson brackets) and to be integrable.
- Recall, a symmetry action is integrable if, for each infinitesimal generator $\underline{\mu} \in \mathfrak{g}$, the vector field that generates the action of $\underline{\mu}$ on phase space, $\underline{V}_{\underline{\mu}} \in TX_R^{ext}$, is Hamiltonian:

$$\exists H_{\underline{\mu}} \in C^\infty(X_R^{ext}), \text{ s.t. } \delta H_{\underline{\mu}} + I_{\underline{V}_{\underline{\mu}}} \Omega_R^{ext} = 0. \tag{3}$$

- Moreover, the action of G is equivariant in the sense that the Poisson algebra of the Hamiltonian functions is homomorphic to that of $\mathfrak{g}$:

$$\{H_{\underline{\mu}}, H_{\underline{\nu}}\} = H_{[\underline{\mu},\underline{\nu}]}. \tag{4}$$

A 'Fuzzy' Subregion

- Although we have labeled the extended phase space by the subregion R, the physics it describes is not restricted therein.
- Constructing the extended phase space can be regarded as ensuing in a sequence of two stages:
  1. First, we extend the subregion $R \mapsto R \cup N(\partial R)$, where $N(\partial R)$ is the tubular neighborhood of the boundary.
  2. Second, we incorporate the embedding fields $\varphi_{(1)}$ and $\varphi_{(2)}$ as dynamical modes.
- One can think of $N(\partial R)$ as the geometric structure which is left invariant under the corner supported part of the symmetry group. By construction, the phase space defined on $X_{R \cup N(\partial R)}$ is invariant under G.
- On the other hand, by promoting the embedding degrees of freedom to dynamical variables we ensure that the symmetry is implemented equivariantly on the phase space.

An Aside on Asymptotic Symmetries

- In the course of this discussion we have made no assumptions about the nature of R, it could be unbounded, have a finite distance boundary, or an asymptotic boundary.
- If R possesses an asymptotic boundary, the phase space we have constructed can be brought into correspondence with the symplectic approach to soft symmetries and memories in gauge theory and gravity presented in, e.g., [Str17].
- In that context, soft charges are symplectically paired with flat connections which generate so-called large gauge transformations.
- Constructing an appropriate Hilbert spaces for scattering in gauge theory and gravity with memory effects is an open problem [PSW22].

Outline of Talk

Background

- What is the quantum version of the story we have told?
- In [KL24], we argued that the extended phase space quantizes to an algebra called a crossed product.
- The crossed product algebra has occupied a very central role in recent operator algebraic approaches to gravity and gauge theory.
- This enterprise was largely inspired by [Wit22], who argued that quantum effects at finite N predicate an enlargement of subregion algebras in AdS/CFT.
- In subsequent work, it has been shown that this enlarging of the algebra has the effect of curing divergences that typically occur in the computation of entropy.
- More to the point, the entropy of states in Witten's crossed product is actually the generalized gravitational entropy.

Anatomy of the crossed product

- One can think of the crossed product as an algebraic generalization of a group extension.
- Let M be a von Neumann algebra represented on a Hilbert space $\pi : M \to B(H)$, and suppose that a group G acts upon M as automorphisms $\alpha : G \to \mathrm{Aut}(M)$.
- To form the crossed product, we first enlarge the Hilbert space $H \mapsto H_{ext} \equiv L^2(G; H)$, and introduce a pair of representations $\pi_\alpha : M \to B(H_{ext})$ and $\lambda : G \to U(H_{ext})$ such that

$$\pi_\alpha \circ \alpha_g(x) = \lambda(g)\pi_\alpha(x)\lambda(g)^{-1}. \tag{5}$$

- Then, we can define the crossed product to be the von Neumann algebra which is generated by products of the form $\pi_\alpha(x)\lambda(g)$.

From Extended Phase Space to Crossed Product

- Recall that our extended phase space was coordinatized by $(\tilde{\Phi}, \tilde{\Pi}, \varphi_{(1)}, M, \varphi_{(2)}, Q)$.
- We can imagine performing a geometric quantization on this phase space by choosing the position polarization $q \equiv (\tilde{\Phi}, \varphi_{(1)}, \varphi_{(2)})$.
- We can interpret the resulting Hilbert space as consisting of square integrable functions from G into what would have been the Hilbert space of the non-extended theory. This coincides with what we called H_{ext} in the previous slide.
- $$f(\tilde{\Phi}, \tilde{\Pi}) \mapsto \pi_\alpha(x), \qquad f(\varphi_{(1)}, \varphi_{(2)}) \mapsto \lambda(g) \tag{6}$$

The Crossed Product in Semiclassical QG

- To see the correspondence between the extended phase space and the crossed product in action, let us consider the following simple example.
- Let R be a wedge region in a flat spacetime, and denote by M_R the operator algebra associated to this subregion.
- In seminal work, Araki demonstrated that M_R is a type III von Neumann algebra, meaning it does not admit density operators or entropies for states.
- From a physical point of view, the type III nature of the wedge algebra recognizes the appearance of UV divergences when one attempts to 'partial trace' the degrees of freedom in the complementary region to the wedge.
- This is because the wedge and its complement are highly entangled across their shared boundary.

The Crossed Product in Semiclassical QG

- A theorem by Bisognano and Wichmann establishes that the modular automorphism of the wedge algebra coincides with action on quantum fields of boosts along the entangling surface.
- A von Neumann algebra is type III if and only if its modular automorphism is outer, meaning that although the boost acts upon the subregion algebra as a symmetry (e.g. it preserves the subregion) it is not generated by any elements inside the algebra itself.
- From the point of view of the classical phase space associated with the wedge region, X_R, the boost diffeomorphism preserves the symplectic form but is not associated with any Hamiltonian function e.g. it is not integrable.
- We may therefore wonder what would happen if we formulate and quantize an extended phase space in which the boost generator is directly incorporated.

The Crossed Product in Semiclassical QG

- In fact, the resulting von Neumann algebra is precisely the celebrated modular crossed product.
- Given any von Neumann algebra, its crossed product with the modular automorphism is automatically semifinite.
- This means that the modular crossed product admits a semifinite trace, density operators, and allows for the computation of finite entanglement and entropy.
- The entropy of so-called classical-quantum states* on the extended wedge algebra reproduces the generalized entropy:

$$S(\tilde{\varphi}) = S(\varphi) + \frac{A}{4G_N}. \tag{7}$$

- From our point of view, we regard $A/4G_N$ as the Noether charge associated with the boost diffeomorphism.

The Crossed Product in Non-Perturbative QG

- It is important to note, however, that the conclusions of the previous slides rested largely upon the starting point of being able to uniquely specify a subregion algebra.
- Indeed, this result is only valid in semiclassical quantum gravity.
- On a related note, when we extended the phase space and the operator algebra, we only did so by including the generator of boosts of the entangling surface.
- This is because the starting point of the crossed product is an algebra preserving action, and only the boost preserves the fixed subregion algebra.
- Naturally, we are left with the question: How do we incorporate the rest of the diffeomorphisms? Surely, these are important in non-perturbative QG.

‘Fuzzy’ Subregion Algebras

- The answer is that we need to revise our starting point.
- In QG it is not reasonable to begin with a fixed subregion algebra and then lift that algebra to a crossed product, since the very act of fixing the subregion already eliminates many important features.
- Instead, we should take as our starting point the full extended phase space we discussed previously and quantize this space wholesale.
- Recalling the two step procedure for specifying the extended phase space, we can imagine first defining an algebra for the fuzzy subregion, $M_{R \cup N(\partial R)}$, in which the full score of corner supported diffeomorphisms act as symmetries.
- Then, we can enlarge this algebra by incorporating the generators of the corner symmetries, resulting in a crossed product, $M_{R \cup N(\partial R)} \rtimes_\alpha G_{N(\partial R)}$.

Outline of Talk

Overview

- In this talk we have demonstrated the utility of the extended phase space in properly accounting for all of the dynamical degrees of freedom in subregion field theories.
- We have argued for a correspondence between the extended phase space and the crossed product, which is an algebraic construction for extending operator algebras by the generators of symmetries acting therein.
- Semiclassically, the crossed product extends a wedge region by the generators of boosts of its entangling surface. These boosts generate the modular automorphism of the algebra, and their inclusion cures diverges that otherwise appear in the computation of entropic measures like entanglement.
- Non-perturbatively, we must begin with our fuzzy subregion at which point we may incorporate the full score of corner symmetries.

The Area Operator and Emergent Geometry

- Present analyses of the modular crossed product in semiclassical quantum gravity restrict attention to classical-quantum states, which assume that field theory degrees of freedom can be completely disentangled from the regulated area operator.
- In this way, the area operator is made to be a central element in the algebra, at least at leading order in G_N.
- In recent work [AAK24], we demonstrated how, for more general states, and even more exotic algebras, one can recover a fully quantum notion of the area operator. This is closely related to the notion of approximate quantum error correction.
- Beyond the semiclassical limit, it is no longer trivial to identify the area operator with a geometric notion. Instead, it becomes more natural to interpret the geometric area as emerging from entropy factorization for states in the gravitational algebra.

The Semifiniteness of the Fuzzy Subregion Algebra

- An important feature of the crossed product which is appropriate for semiclassical gravity is that it is automatically semifinite as a von Neumann algebra.
- Determining the type of the algebra which is appropriate to describe a 'subregion' in non-perturbative quantum gravity is an important open question.
- In [AAKL24], we proved a theorem which significantly generalizes upon Takesaki's seminal result on the semifiniteness of the modular crossed product.
- In particular, we showed that the crossed product of any von Neumann algebra by a group G is semifinite, if there exists an embedding $\gamma : \mathbb{R} \hookrightarrow G$ such that $\alpha \circ \gamma : \mathbb{R} \to \text{Aut}(M)$ is KMS with respect to a state which is also (quasi) invariant under the action of the group.

Memory Effects and G-Framed Algebras

- More broadly, it is important to ask whether a single crossed product algebra will be sufficient to cover the full set of observables in a general gauge theory.
- In [AACKL24a], we showed that the crossed product can be interpreted as an algebraic formulation of a quantum reference frame.
- However, we also demonstrated that a single crossed product cannot encode multiple non-isomorphic reference frames.
- In [AACKL24b] we formalized this observation by demonstrating that crossed product algebras are equivalent to the notion of a trivial quantum principal bundle.
- To rectify this we introduced the G-Framed algebra which is a collection of crossed product charts sewn together along their algebraic overlaps. In other words, a non-trivial quantum principal bundle (or quantum orbifold).

Memory Effects and G-Framed Algebras

- A nice example which illustrates the need for more sophisticated algebras is the memory effect in gauge theory and quantum gravity.
- In general it is challenging to construct a single separable Hilbert space which includes states with all possible memories.
- One proposal, [PSW22], for circumventing this issue is to work instead with an abstract algebra whose predual furnishes a complete set of states with all possible memories.
- A Hilbert space of states with a fixed subset of memories is a representation of the abstract algebra with respect to a state with support over such memories. The von Neumann algebra obtained by completing the algebra in the weak operator topology of this Hilbert space can be interpreted as the quantum reference frame of a single observer.

Quantum Diamonds

- [CFL24] argued that in the quantization of gravity on null hypersurfaces representations of null translations are centrally extended, implying that the quantum operators which implement these transformations do not act trivially on physical states.
- In currently ongoing work with Rob Leigh, we are investigating the implications of this central extension in the definition of a fully quantum notion of subregion which we call the quantum diamond.
- Starting with the algebra of the tubular neighborhood of a corner, the centrally extended representations of null translations furnish a pair of vacuum modules which resemble the structure of a $2d$-CFT.
- We argue that this can be regarded as a generalized notion of holography for arbitrary null hypersurfaces.