Title: Lecture - Relativity, PHYS 604

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Collection/Series: Relativity (Core), PHYS 604, November 12 - December 11, 2024

Subject: Cosmology, Strong Gravity

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Abstract:

Cosmology

homogenous & isotropic

§ 1 Killing Vectors

$$x' \to x \quad : \quad g = \frac{\partial x}{\partial x'} \frac{\partial x}{\partial x'} \, g$$

$$x' = x - \varepsilon \xi$$

$$\boxed{\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0}$$

Rmk.

Rmk.

$$\nabla_\sigma \nabla_\rho \xi_\mu = - R^\alpha{}_{\sigma\rho\mu} \xi_\alpha \longleftarrow \nabla^n \xi \text{ are linear in } \xi, \nabla\xi$$

$$\Rightarrow \frac{d(d+1)}{2} \quad d: \xi^\mu(X)$$

$$\frac{d(d-1)}{2}: \nabla^\mu \xi^\nu(X)$$

Hom: $\xi^{(\alpha)}_\mu = \delta^\alpha_\mu, \quad \nabla\xi = 0$

Isotropy: $\xi^{(\alpha\beta)}_\mu(X) = 0$

$$\nabla_\mu \xi^{(\alpha\beta)}_\nu(X) = \delta^\alpha_\mu \delta^\beta_\nu - \delta^\beta_\mu \delta^\alpha_\nu$$

Theorem I. Uniqueness

Let $(\mathcal{M}, g)$, $(\mathcal{M}', g')$ be two spaces (maximally symmetric),

s.t. (i) g & g' have the same signature and

(ii) they have the same Ricci Scalar curvature scalar

then $\varphi: \mathcal{M} \to \varphi(\mathcal{M})$ isometry. Moreover.

$$R_{\mu\nu} = -(d-1) K g_{\mu\nu}$$

$$R_{\lambda\rho\sigma\nu} = K(g_{\sigma\rho} g_{\lambda\nu} - g_{\nu\rho} g_{\lambda\sigma})$$

$$\nabla_\mu \xi_\nu^{(\alpha\beta)}(X) = \delta^\alpha_\mu \delta^\beta_\nu - \delta^\beta_\mu \delta^\alpha_\nu$$

Theorem II Maximally Symm Tens

We call a tensor A symm under ξ:

$$\mathcal{L}_\xi A = 0.$$

If the space is maximally symm, this gives

(i) All scalars are constant

(ii) All vectors vanish

(iii) All rank tensors are proportional to the metric

Theorem III : MSS

$$\mathcal{M} = \bigcup_{\{v^a\}} \Sigma(\{v^a\}) \longleftarrow \text{maximally symmetric coordinates } \{u^\mu\}$$

$$ds^2 = g_{ab}(v^a)\, dv^a dv^b + f(v)\, \gamma_{\mu\nu}(u)\, du^\mu du^\nu \quad \Rightarrow$$

$\uparrow$ maximally symm metric

(iii) All rank tensors are proportional to the metric

$$\Rightarrow \quad ds^2 = -dt^2 + a(t)^2 \gamma_{ij}(\vec{x})\, dx^i dx^j$$

maximally symmetric metric on euclidean 3-space

$$d\ell^2 = d\vec{x}^2 + k\,\frac{(\vec{x}\cdot d\vec{x})^2}{1-k\vec{x}^2}, \qquad k = \begin{cases} +1 & \text{pos. curv} \\ 0 & \text{no curv} \\ -1 & \text{neg. curv} \end{cases}$$

$$\boxed{ds^2 = -dt^2 + a(t)^2\left(dx^2 + k\frac{(x\cdot dx)^2}{1-kx^2}\right)}$$

FRW metric

$$G_{00} = 3(\dot{a}^2 + k)\,a^2$$

$$R = 6(k + a\ddot{a} + \dot{a}^2)/a^2$$

$$T^0{}_0 = -\rho(t)$$

$$T^0{}_i = 0$$

$$T^i{}_j = p(t)\,\delta^i{}_j$$

$$T^\mu{}_\nu = (\rho + p)\,u^\mu u_\nu + p\,\delta^\mu{}_\nu$$

$$u^\mu = \begin{pmatrix} 1 \\ \vec{0} \end{pmatrix}$$

$\rho(t)$

0

$p(t)\,\delta^i_j$

$u^\mu u_\nu + p\,\delta^\mu{}_\nu$

$\begin{pmatrix}1\\ \vdots \\ 0\end{pmatrix}$

$$0 = T\,dS = d(\rho a^3) + p\,da^3$$

$$= a^3\left(d\rho + 3(p+\rho)\,da/a\right)$$

$$\Rightarrow \boxed{\frac{d\rho}{dt} + 3H(\rho+p) = 0} \longleftrightarrow \nabla_\mu T^{\mu\nu} = 0$$

Hubble

$$\boxed{H = \frac{1}{a}\frac{da}{dt}}$$

$$dE = T\,dS - p\,dV$$

tiny box, but in co-space no!

confinement phenomenon

only gauge singlets $\Leftrightarrow$ asymptotic states

$$\boxed{ds^2 = -dt^2 + a(t)^2\left(dx^2 + k\frac{(x\cdot dx)^2}{1-kx^2}\right)}$$

FRW metric

$$G_{00} = 3(\dot a^2 + k)\,a^2$$

$$R = 6(k + a\ddot a + \dot a^2)/a^2$$

$$\boxed{H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}} \quad (*1)$$

$$T^0{}_0 = -\rho(t$$

$$T^0{}_i = 0$$

$$T^i{}_j = p(t)$$

$$T^\mu{}_\nu = (\rho+p)u^\mu u_\nu +$$

$$u^\mu = \begin{pmatrix}1\\ \vec 0\end{pmatrix}$$

results

SUSY $\mathcal{N}=2$

but in a very tiny

$$p = w\,\rho$$

$$\begin{cases} \text{relativistic}: & w = 1/3 \\ \text{cdm}: & w = 0 \\ \text{de}: & w = -1 \end{cases} \underset{(k=0)}{\Longrightarrow}$$

$w \neq -1$ $\qquad$ $w = -1$

$$\rho = \frac{\rho_0}{a^{3(1+w)}} \quad , \quad \rho = \rho_0$$

$$a(t) = a_0 \left(\frac{t}{t_0}\right)^{2(1+w)/3} \quad , \quad a(t) = a_0 e^{H_0 t}$$

(de Sitter)

$$\rho_k \equiv \frac{3k}{8\pi G} \frac{1}{a^2}$$

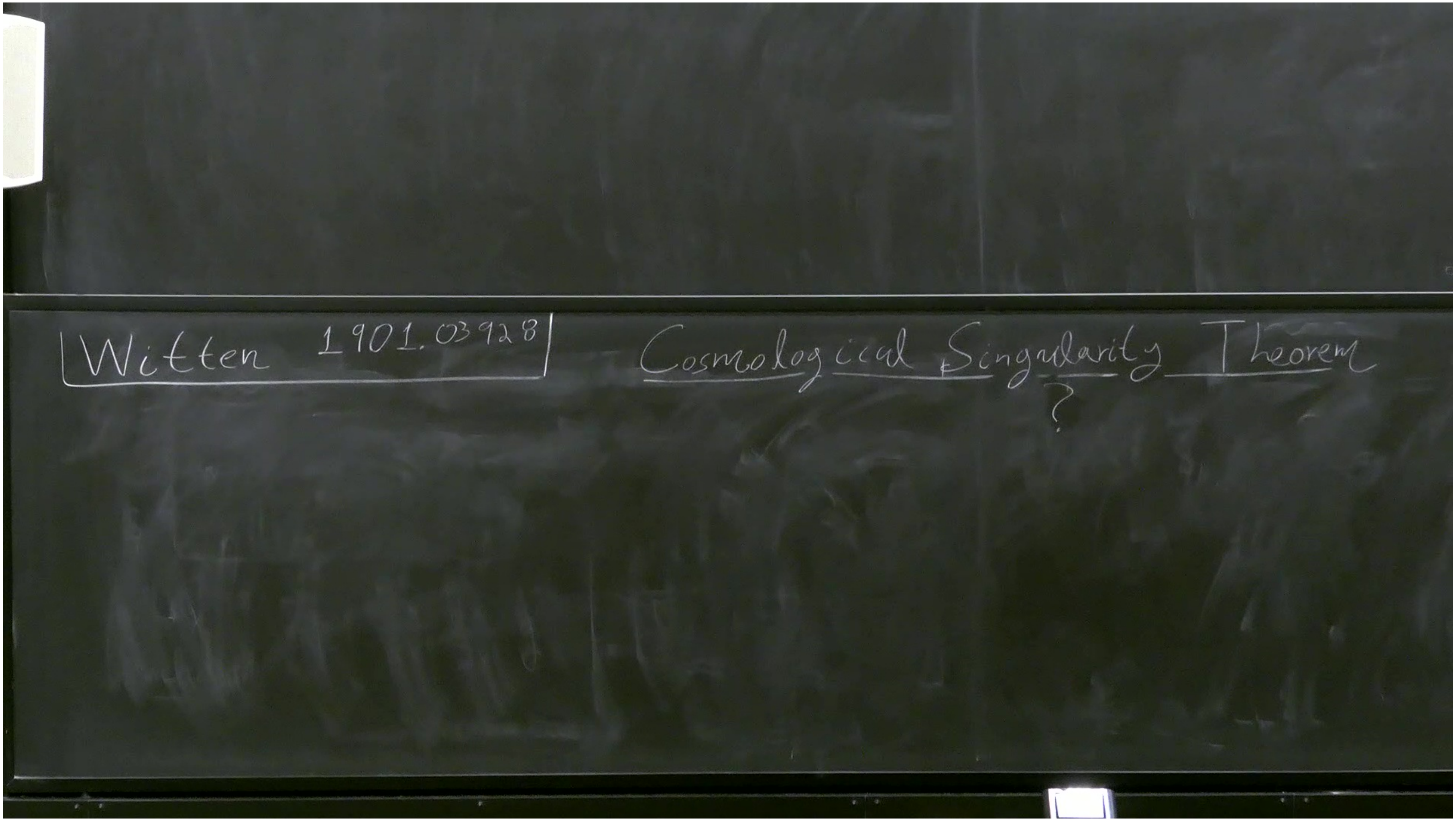
$a_0 e^{H_0 t}$

itter)

ΛCDM

10^{-4} 30% $|\cdot| \leq 10^{-3}$ 70%

$$\rho(a) = \frac{\rho_{r,0}}{a^4} + \frac{\rho_{cdm,0}}{a^3} + \frac{\rho_{k,0}}{a^2} + \rho_{de,0} \rightarrow t_{univ} \simeq 13.7\ \mathrm{Gyr}$$


Witten 1901.03928
Cosmological Singularity Theorem
?

$$\lim_{t \to 0} R = \infty$$

$$\lim_{t \to 0} R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} = \infty$$

$$a(t) \sim t^{\lambda}$$

Defs

1) Spacetime = a smooth Lorentzian manifold

2) Causal Path: $x^\mu(s)$

$$-g_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} \geq 0$$

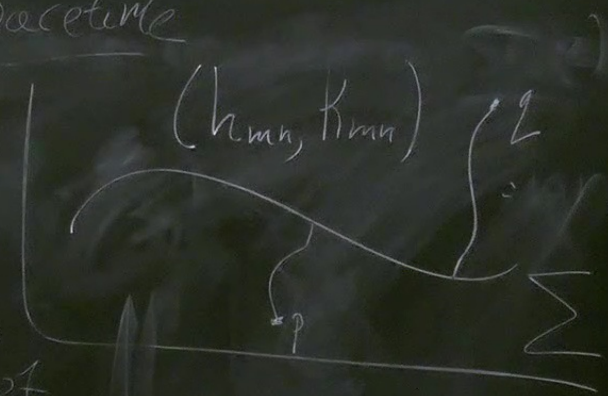
...en manifold

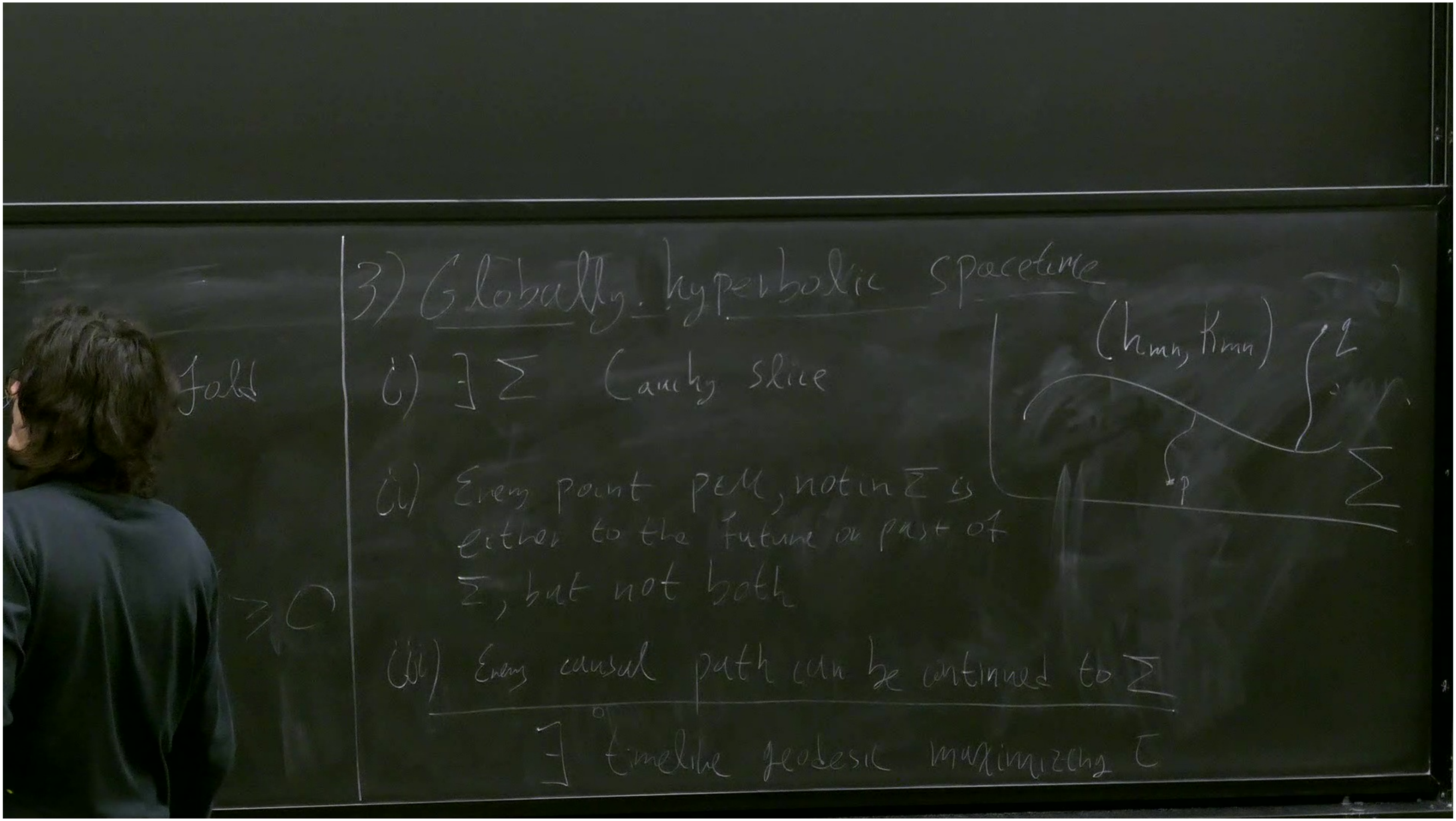
$$\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} \geq 0$$

3) Globally hyperbolic spacetime

i) $\exists\, \Sigma$ Cauchy slice

ii) Every point $p \in M$, not in Σ is either to the future or past of Σ, but not both





Fold
3) Globally hyperbolic spacetime
i) ∃ Σ Cauchy slice
ii) Every point p∈M, not in Σ is either to the future or past of Σ, but not both
iii) Every causal path can be continued to Σ
∃ timelike geodesic maximizing τ
(h_mn, K_mn)
Σ
p

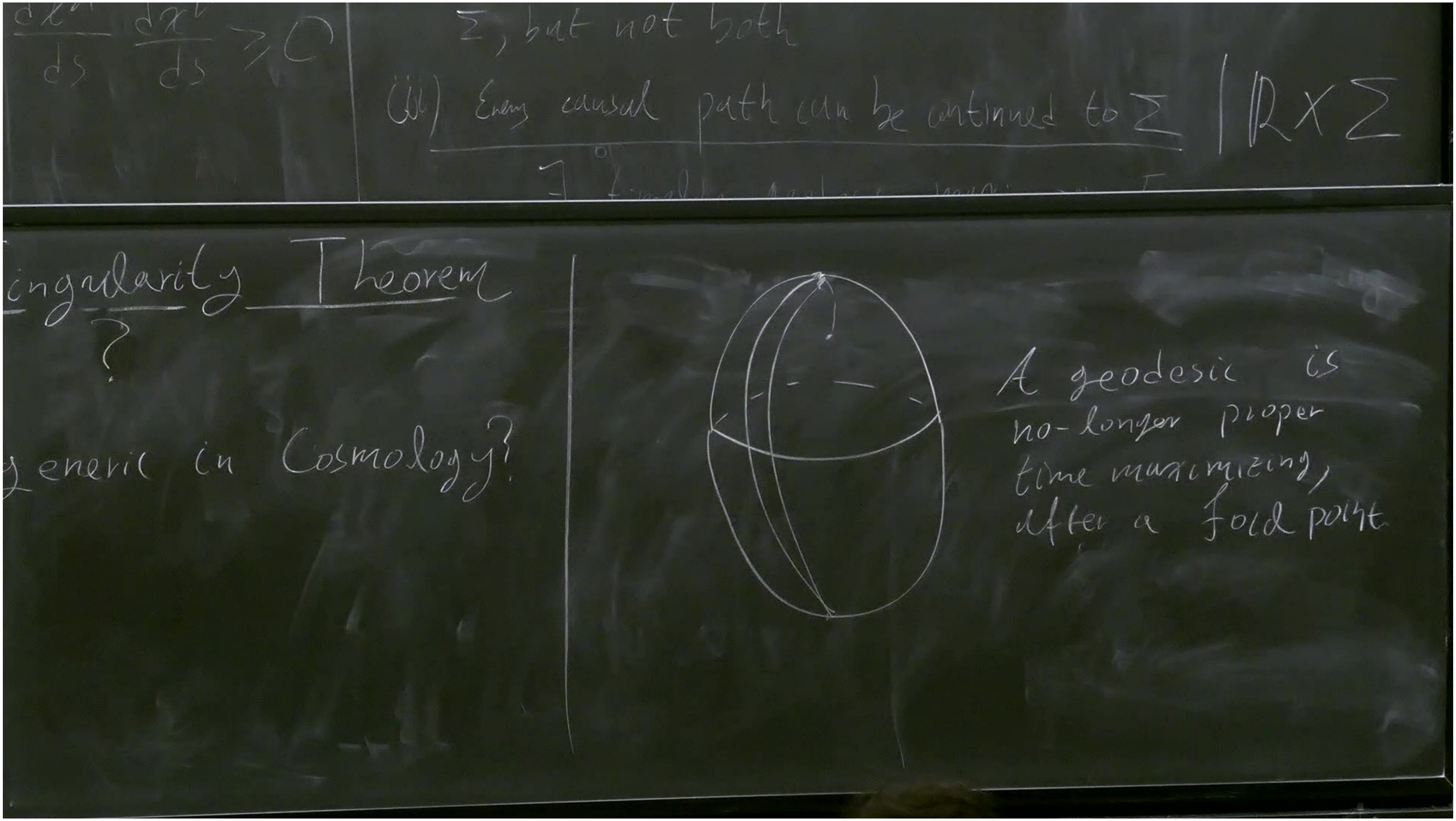
Witten 1901.03928

4) Complete manifold: every geodesic can be extended infinitely in both directions

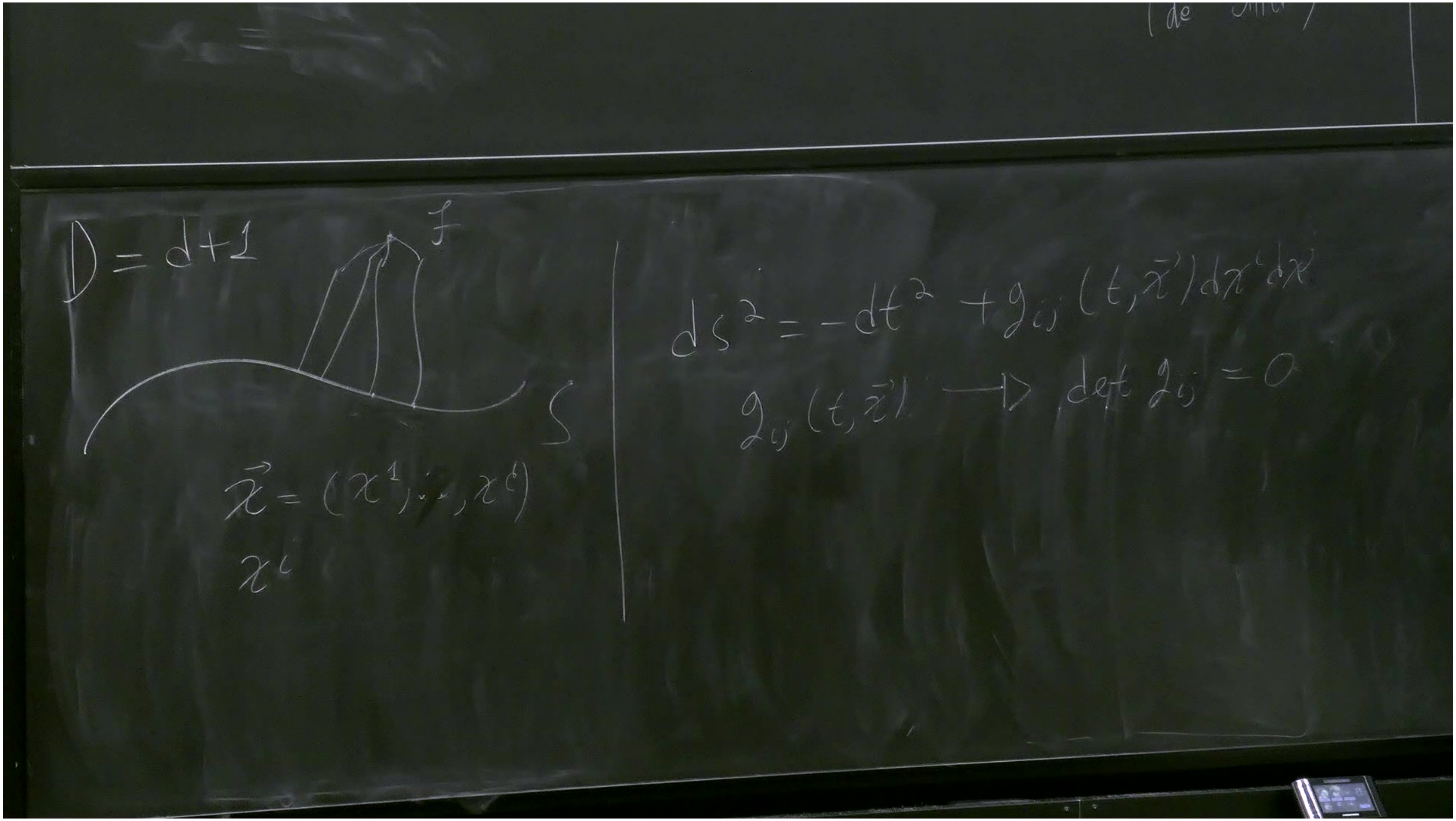
5) Focal Point

Cosmological Singularity ?

Are singularities generic in



$\frac{dx^\mu}{ds} \frac{dx^\nu}{ds} \geq 0$
Σ, but not both
(iii) Every causal path can be continued to Σ
$\mathbb{R} \times \Sigma$
Singularity Theorem
?
generic in Cosmology?
A geodesic is no-longer proper time maximizing, after a focal point


$D = d+1$
f
Σ
$\vec{x} = (x^1, \dots, x^d)$
x^i
$ds^2 = -dt^2 + g_{ij}(t,\vec{x})dx^i dx^j$
$g_{ij}(t,\vec{x}) \longrightarrow \det g_{ij} = 0$

$$R_{\mu\nu} = 8\pi G \underbrace{\left(T_{\mu\nu} - \frac{1}{D-2} g_{\mu\nu} T\right)}_{\hat{T}_{\mu\nu}} \equiv 8\pi G \hat{T}_{\mu\nu}$$

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

$$R_{tt} = -\frac{1}{2} \partial_t \mathrm{Tr}\, g^{-1}\dot{g} - \frac{1}{4} \mathrm{Tr}(g^{-1}\dot{g})^2$$

$$\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} \geq 0$$

Σ, but not both

(iii) Every causal path can be continued to Σ | $\mathbb{R} \times \Sigma$

$\exists$ timelike geodesic maximizing τ

$$V = \sqrt{\det(g)} \quad \text{Volume}$$

$$\theta = \frac{\dot{V}}{V} = \frac{1}{2} \operatorname{Tr} g^{-1} \dot{g} \quad \text{expansion}$$

$$\sigma^i{}_j = \frac{1}{2}\left(g^{ik} \dot{g}_{kj} - \frac{1}{d} \delta^i{}_j \operatorname{Tr} g^{-1} \dot{g} \right)$$

Strong energy $\hat{T}_{tt} \geq 0$

Raychaudhuri

$$R_{tt} = 8\pi G \hat{T}_{tt}$$

$$R_{tt} = -\dot{\theta} - \frac{\theta^2}{d} - \operatorname{Tr} \sigma^2$$

$$\boxed{\dot{\theta} + \frac{\theta^2}{d} = -\operatorname{Tr} \sigma^2 - 8\pi G \hat{T}_{tt}}$$

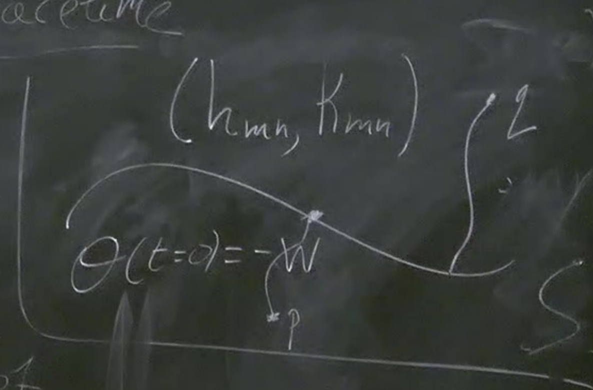
$$\boxed{\dot{\theta} + \frac{\theta^2}{d} \leq 0}$$

en manifold

$$\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} \geq 0$$

3) Globally hyperbolic spacetime

i) $\exists\, \Sigma$ Cauchy slice



ii) Every point $p \in M$, not in Σ is either to the future or past of Σ, but not both

iii) Every causal path can be continued to Σ | $\mathbb{R} \times \Sigma$

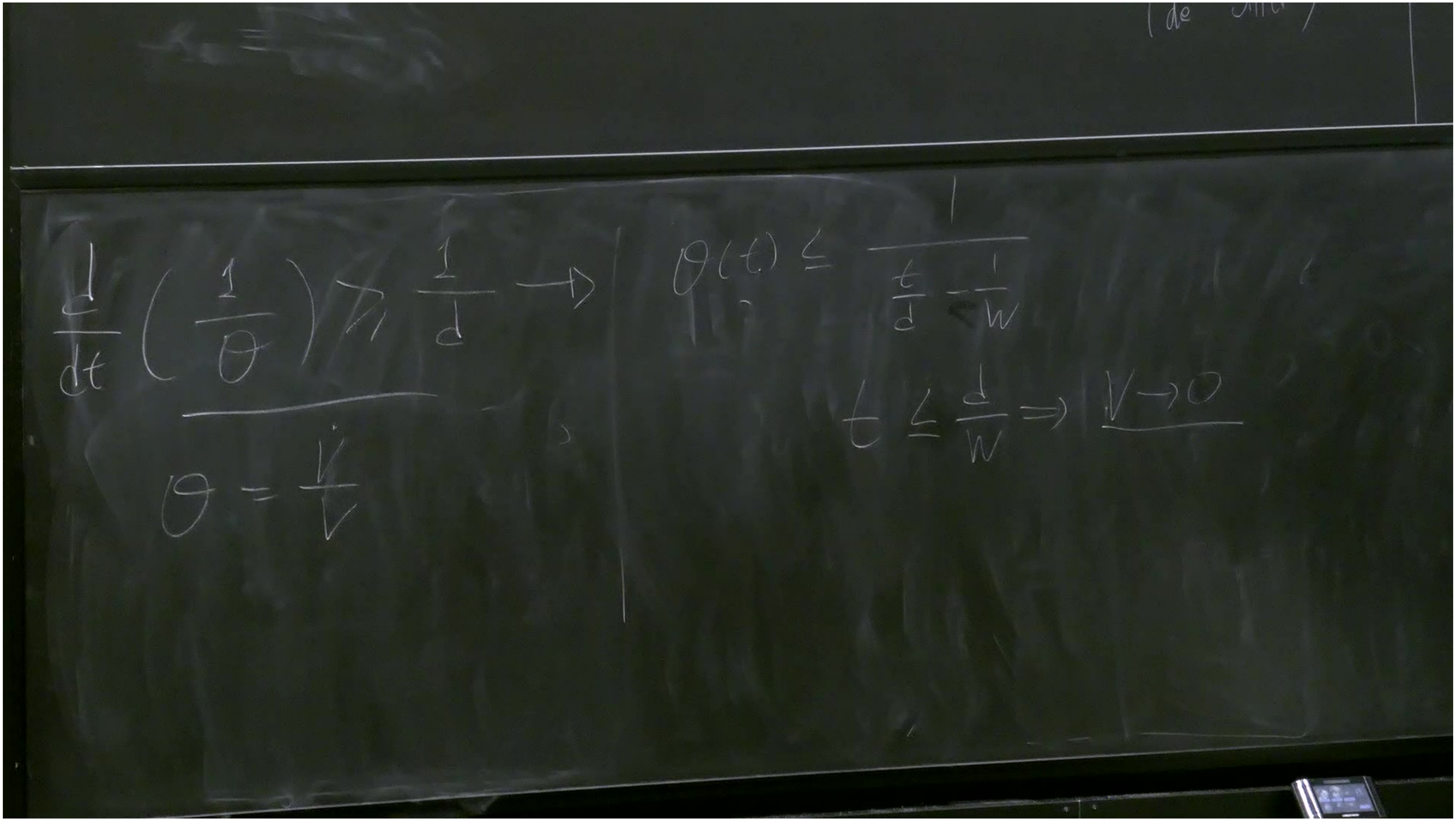
$\exists$ timelike geodesic maximizing τ

$$\sigma^i{}_j = \frac{1}{2}\left(g^{ik} g_{kj} - \frac{1}{d}\delta^i{}_j \operatorname{Tr} g^{-1} g\right)$$

Strong energy $\hat{T}_{tt} \geq 0$

$$\boxed{\dot{\theta} + \frac{\theta^2}{d} = -\operatorname{Tr}\sigma^2 - 8\pi G \hat{T}_{tt}}$$

$$\boxed{\dot{\theta} + \frac{\theta^2}{d} \leq 0}$$


$\frac{d}{dt}\left(\frac{1}{\theta}\right) \geq \frac{1}{d} \longrightarrow$
$\theta(t) \leq \frac{1}{\frac{t}{d} - \frac{1}{W}}$
$\theta = \frac{\dot{V}}{V}$
$t \leq \frac{d}{W} \Rightarrow V \to 0$

Hawking's Thm (1965)

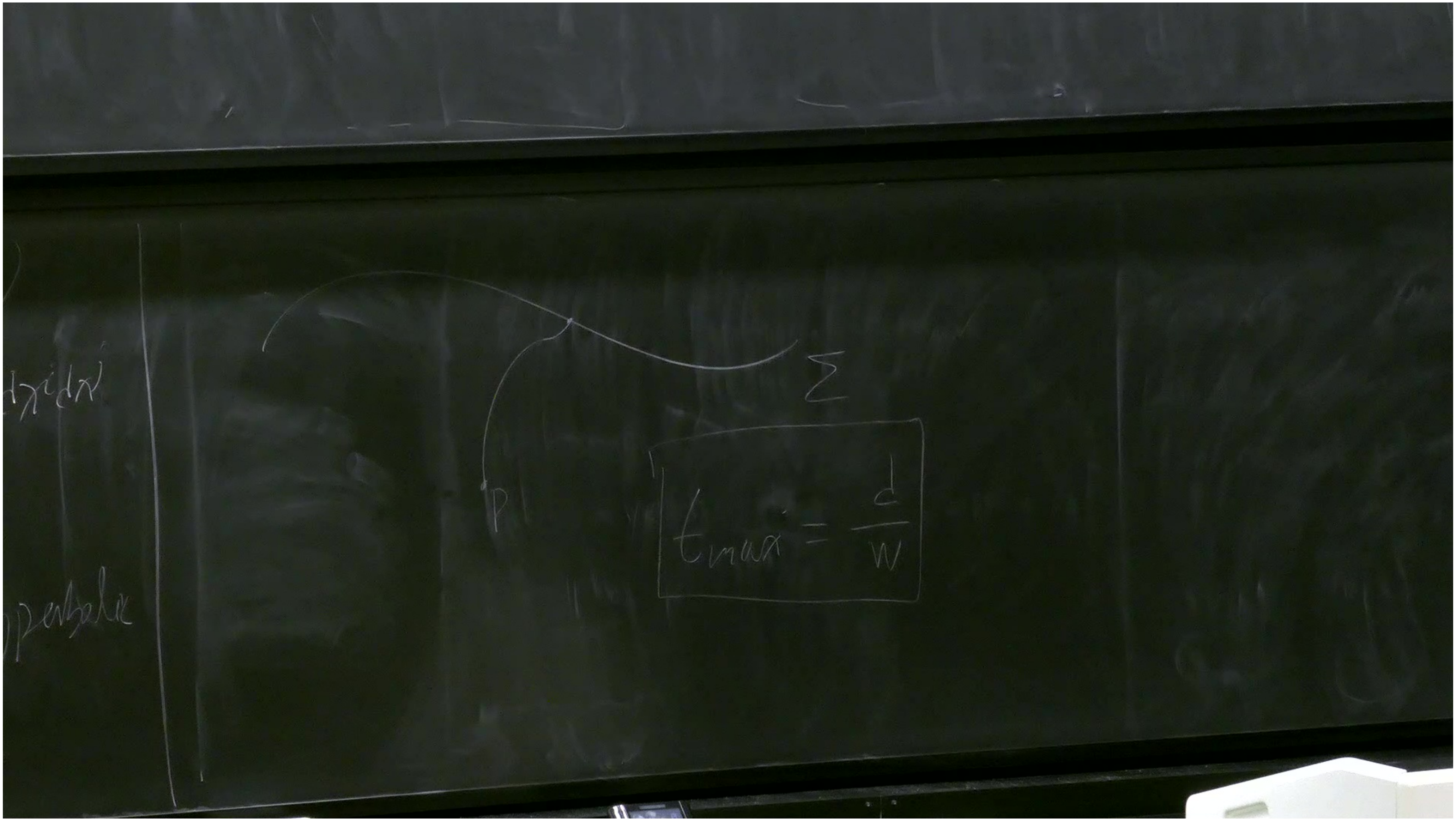
$$ds^2 = -dt^2 + g_{ij}(t, \vec{x}) dx^i dx^j$$

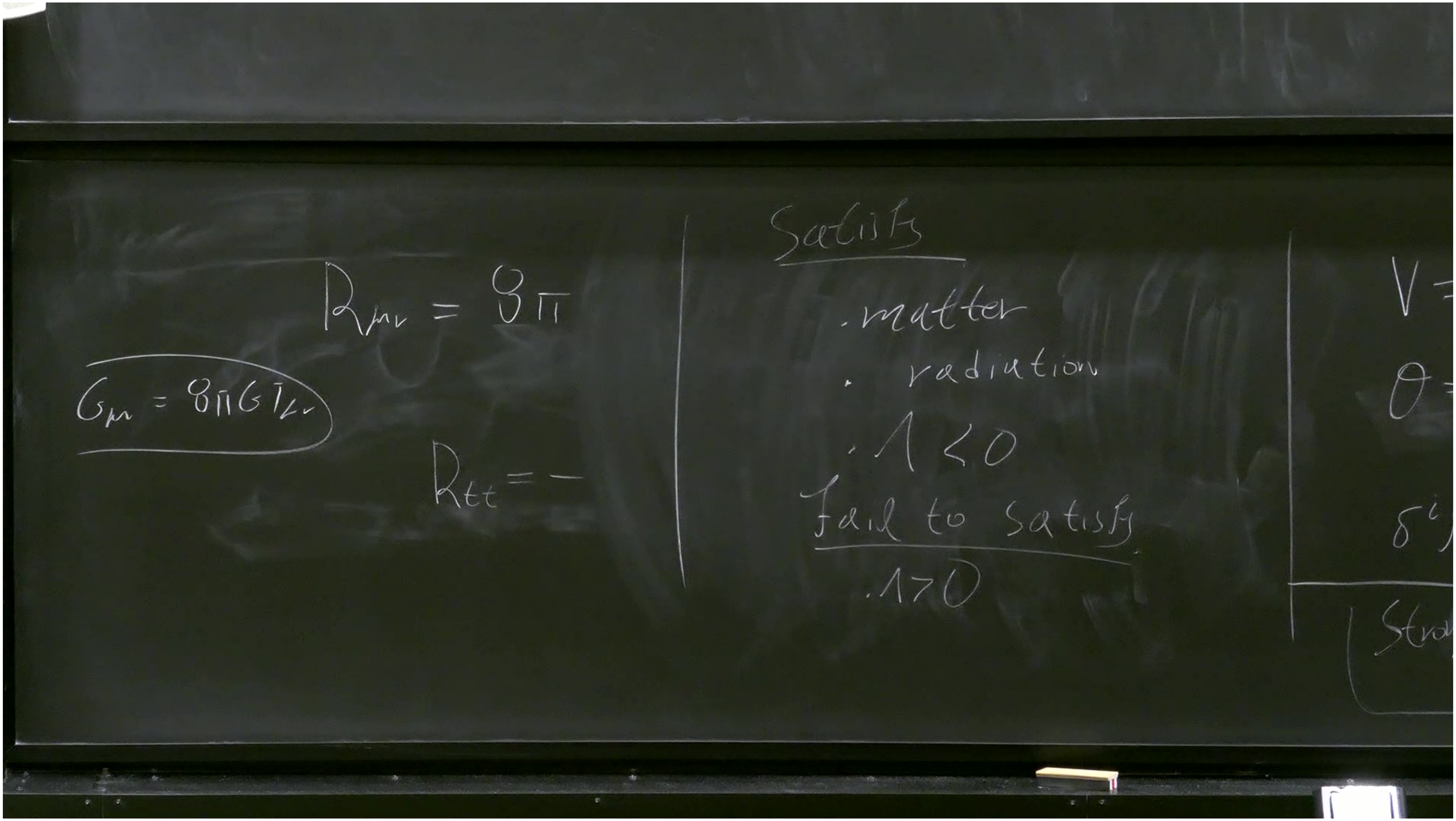
Assumptions:

(i) Universe is globally hyperbolic

(ii) Strong energy condition

(iii) GR


Σ
p
t_max = d / w


$R_{\mu\nu} = 8\pi$
$G_{\mu\nu} = 8\pi G T_{\mu\nu}$
$R_{tt} = -$
Satisfy
. matter
. radiation
. $\Lambda < 0$
fail to satisfy
. $\Lambda > 0$
$V =$
$\theta =$