**Title:** Lecture - QFT II, PHYS 603

**Speakers:** Francois David

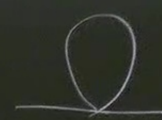
**Collection/Series:** Quantum Field Theory II (Core), PHYS 603, November 12 - December 11, 2024

**Subject:** Condensed Matter, Particle Physics, Quantum Fields and Strings

**Date:** December 10, 2024 - 9:00 AM

**URL:** https://pirsa.org/24120005

Gauge Theories SU(2)

$\phi^4$  $\to \frac{1}{2-d}$ SU(2)

- UV divergences at $d=4$
- Organize P.T  Gauge Part / Helicity structure

2pt  3pt,  4pt

$AA$  $AAA$  $AAAA$

$\bar{c}c$  $\bar{c}Ac$

$$\log \Lambda^2 \to \frac{1}{8\pi^2}\frac{1}{4-d}$$

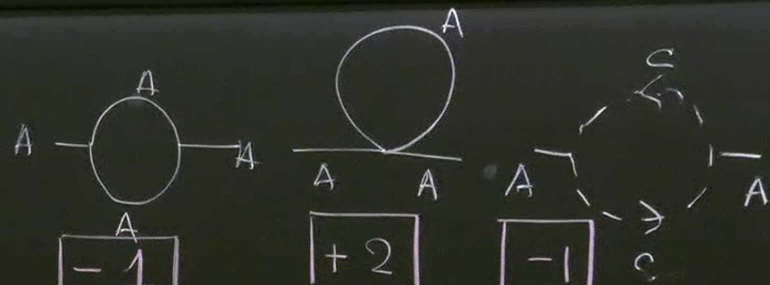
$$\Lambda^2 \to \frac{1}{2\pi}\frac{1}{2-d}$$

- dimensional regularization

Poincare, Gauge Sym., BRST, ~~Scale invariance~~

$\frac{1}{-d}$

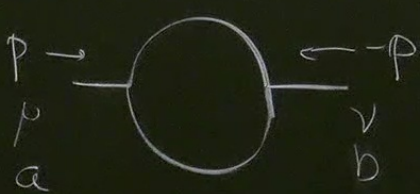
SU(2) $\Lambda^2$ potentially present




cancell ? if not

$\boxed{-1}$ $\boxed{+2}$ $\boxed{-1}$

mass generated for A. $m \sim g\,\Lambda \Rightarrow$ Causality + Unitarity




$$\boxed{\frac{1}{2\pi}\,\frac{1}{2-d}} \Leftarrow (-1)\,\delta^{\mu\nu}\,\delta^{ab}\int\frac{d^dk}{(2\pi)^d}\,\frac{1}{k^2}$$

$-1+2-1=0$ !

divergence $d \to 2$

$A^\mu A^\mu \frac{1}{2-d}$ not gauge invariant

Massive Vector boson $A_\mu$

$$(\partial_\mu A_\nu)^2 + b(\partial_\mu A_\mu)^2 + M^2 A_\mu A_\nu$$

3 $\perp$ modes mass $M$

1 // mode mass $M_\ell = \sqrt{1+b}\, M$

but $-$ sign

$(-+++)$ propagator

$$A_\mu K_{\mu\nu} A_\nu \qquad K_{\mu\nu}(p) = h_{\mu\nu}(p^2+M^2) + b\, p_\mu p_\nu$$

$$\sim\sim \quad (K^{-1})_{\mu\nu} = \frac{-1}{p^2+M^2}\begin{pmatrix} 0 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} + \frac{-1}{(1+b)p^2+M^2}\begin{pmatrix} -1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}$$

② Massive Vector boson $A_\mu$

$$(\partial_\mu A_\nu)^2 + b(\partial_\mu A_\mu)^2 + M^2 A_\nu A_\nu$$

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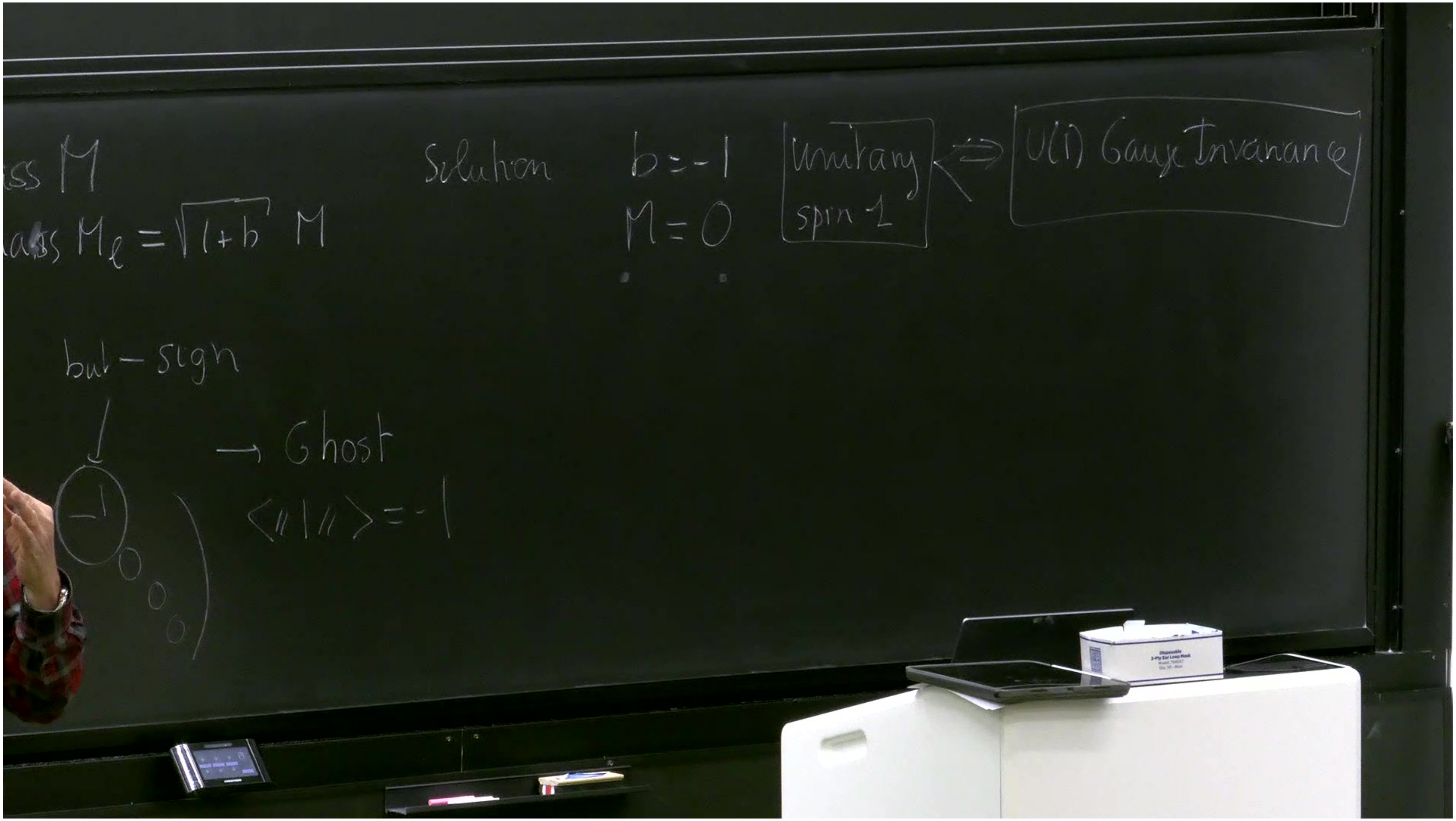
→ Gh

$\langle \parallel | \parallel$


ss M
ass M_ℓ = √(1+b) M
Solution
b = -1
M = 0
Unitary
spin-1
⇔
U(1) Gauge Invariance
but - sign
→ Ghost
⟨ | ⟩ = -1
-1

Poincare, Gauge Sym., BRST, ~~Scale invariance~~

(3) Another method to compute the $\beta$-function

1 loop Effective action

in "background Field" Gauge Fixing Method

$$A_\mu = A_\mu^{bg} + \tilde{A}_\mu$$

quantum — classical — quantum

enforce G.F on $\tilde{A}_\mu$: $D^{bg\,\mu} \tilde{A}_\mu = 0$

covariant derivative in the bg Field $A_\mu^{bg}$

$$\Gamma[A^{bg}] = S_{YM}[A^{bg}] + \frac{1}{2}\,\mathrm{tr}\left[\mathrm{Log}\left(S''[A^{bg}]\right)\right] - \mathrm{tr}\,\mathrm{Log}\left(-\partial^{\mu} D^{bg}_{\mu}\right)$$

quantum effective action

Hessian

gauge field

ghosts

ghost action Hessian

Diff operator acting on $\hat{A}$ but depending on $A^{bg}$

$\Downarrow$

Heat Kernel Expansion on short "time" for Diff Operators (Seeley-DeWitt)

important in Math $\longrightarrow$ Index Theorems . . . .

contains Log $\Lambda$ diverges or $\frac{1}{4-d}$ poles $\propto$ original action $S_{YM}[A^{bg}]$

$\Rightarrow$ renormalized action CT

quantum

covariant derivative
in the bg Field $A_\mu^{bg}$

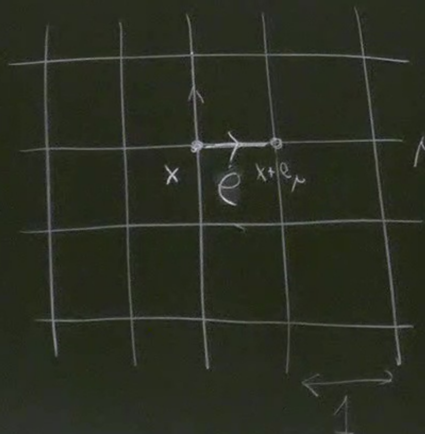
Heat Kernel
for Diff Operat...
important in ...

Wilson Loops & confinement, Lattice Gauge Theories

$\boxed{\text{K Wilson (74)}}$

F Wegner (72...

3d Ising ← dua...




lattice $\mathbb{Z}^d$ (square)

How to introduce gauge field

Matter Fields $\phi$ in fundamental representation

$$\phi = (\phi^i)_{i=1,2} \quad \text{for } SU(2) \qquad e_\mu = (0 \cdots \underset{\uparrow \mu}{1} \cdots 0)$$

$e_\mu$ elementary vector in direction $\mu$

$A_\mu dx^\mu \leftarrow$ connection

$\phi(x) \to \phi(y)$ covariant derivative

$$D_\mu = \partial_\mu - A_\mu$$

ordinary derivative

$$\phi(x+e_\mu) - \phi(x) = \ldots$$

quantum

covariant derivative
in the bg Field $A_\mu^{bg}$

Heat Kernel
for Diff Operat...
important in ...

Wilson Loops , Lattice Gauge Theories $\boxed{\text{K Wilson (74)}}$
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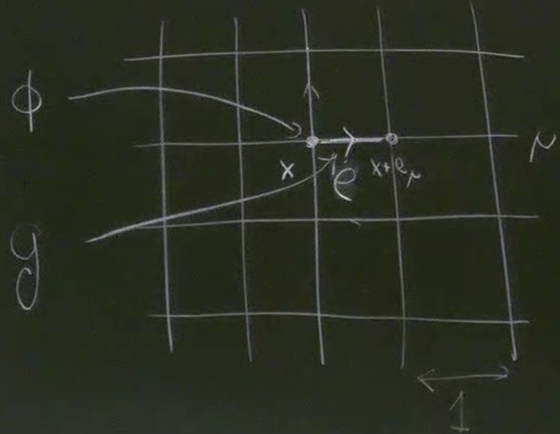

lattice $\mathbb{Z}^d$ (square)

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$A_\mu dx^\mu$ ← connection

$\phi(x) \to \phi(y)$ covariant derivative

$D_\mu = \partial_\mu - A_\mu$

ordinary derivative

$\phi(x+e_\mu) - \phi(x) = ...$

Heat Kernel Expansion in short "time" $\sim \frac{1}{4-d}$ poles $S_{YM}[A^{hg}]$

for Diff Operators (Seeley-DeWitt) $\Rightarrow$ renormalized action CT

important in Math $\longrightarrow$ Index Theorems . . . .

F Wegner (72)

3d. Ising $\underset{\text{dual}}{\longleftrightarrow}$ $\mathbb{Z}_2$ gauge theories

$\phi$ lives on sites

$\partial_\mu \phi$ lives on links

$\xrightarrow{e}$ oriented link (edge)

$\xleftarrow{}$ opposite oriented edge $e^*$

$$\boxed{\begin{array}{ll} e \Longleftrightarrow g(e) & g \in SU(2) \\ e^* & g(e^*) = g^{-1}(e) \end{array}} \quad G$$

connection

covariant derivative

$D_\mu = \partial_\mu - A_\mu$

$$D_\mu \phi(e_\mu) = \phi(x+e_\mu) - g(e_\mu)\phi(x)$$

ordinary derivative

$\phi(x+e_\mu) - \phi(x) = \partial_\mu \phi$

$x \xrightarrow{e} x+e_\mu$

$$g(e_\mu) = 1 + i\epsilon A_\mu e_\mu + \dots = e^{i\epsilon A_\mu dx^\mu)}$$

$\boxed{\text{small } \epsilon}$

ordinary gauge connection

What is a gauge transformation (local)

$$\phi(x) \longrightarrow h(x)\phi(x) \qquad h \in SU(2)$$

$$\boxed{\phi \to h \cdot \phi}$$

matter fields

$$\phi(x) \longrightarrow \left[1 + i\epsilon\,\alpha(x)\right]\phi(x)$$

what is it for $g(e)$

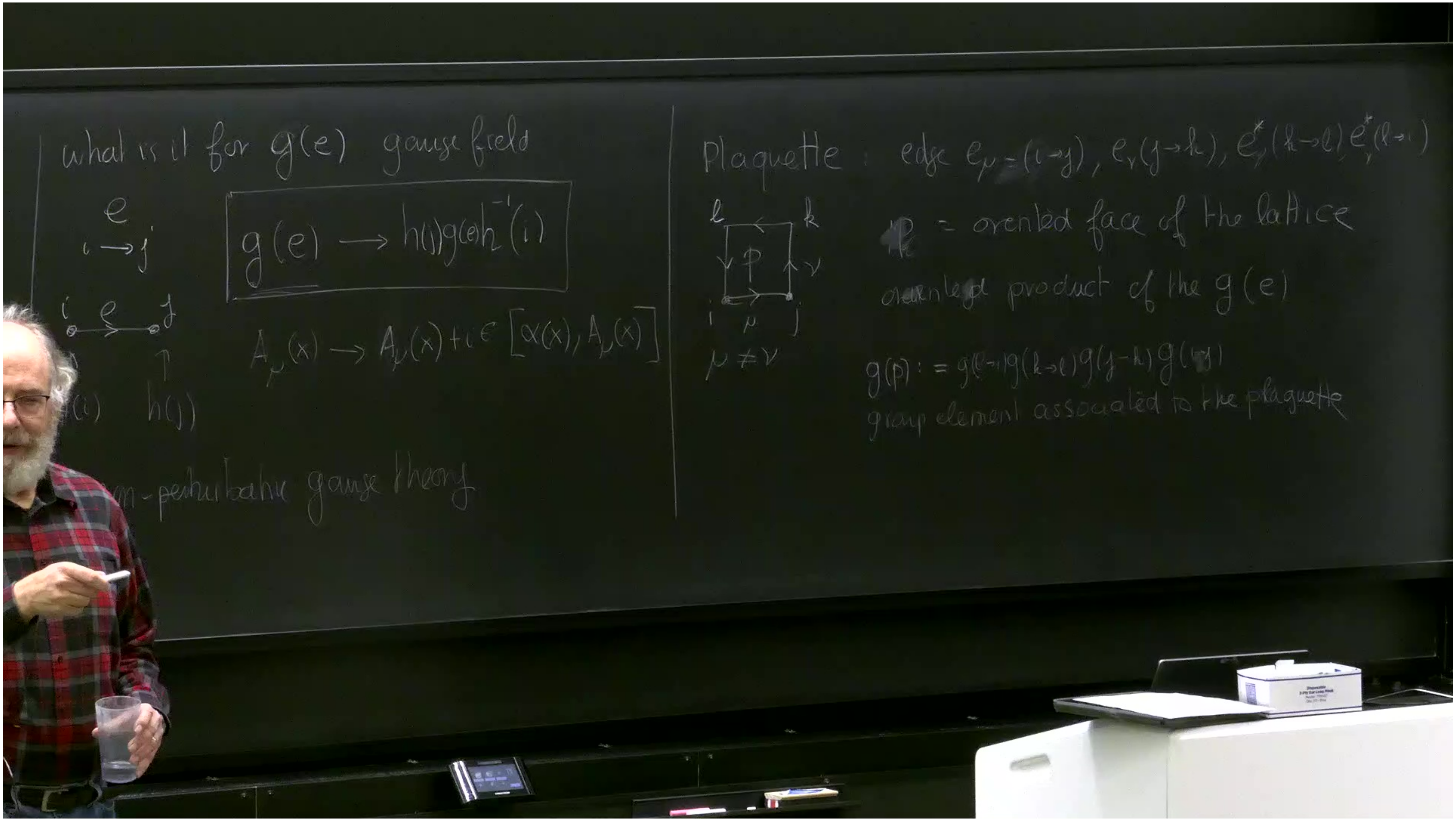
$e$

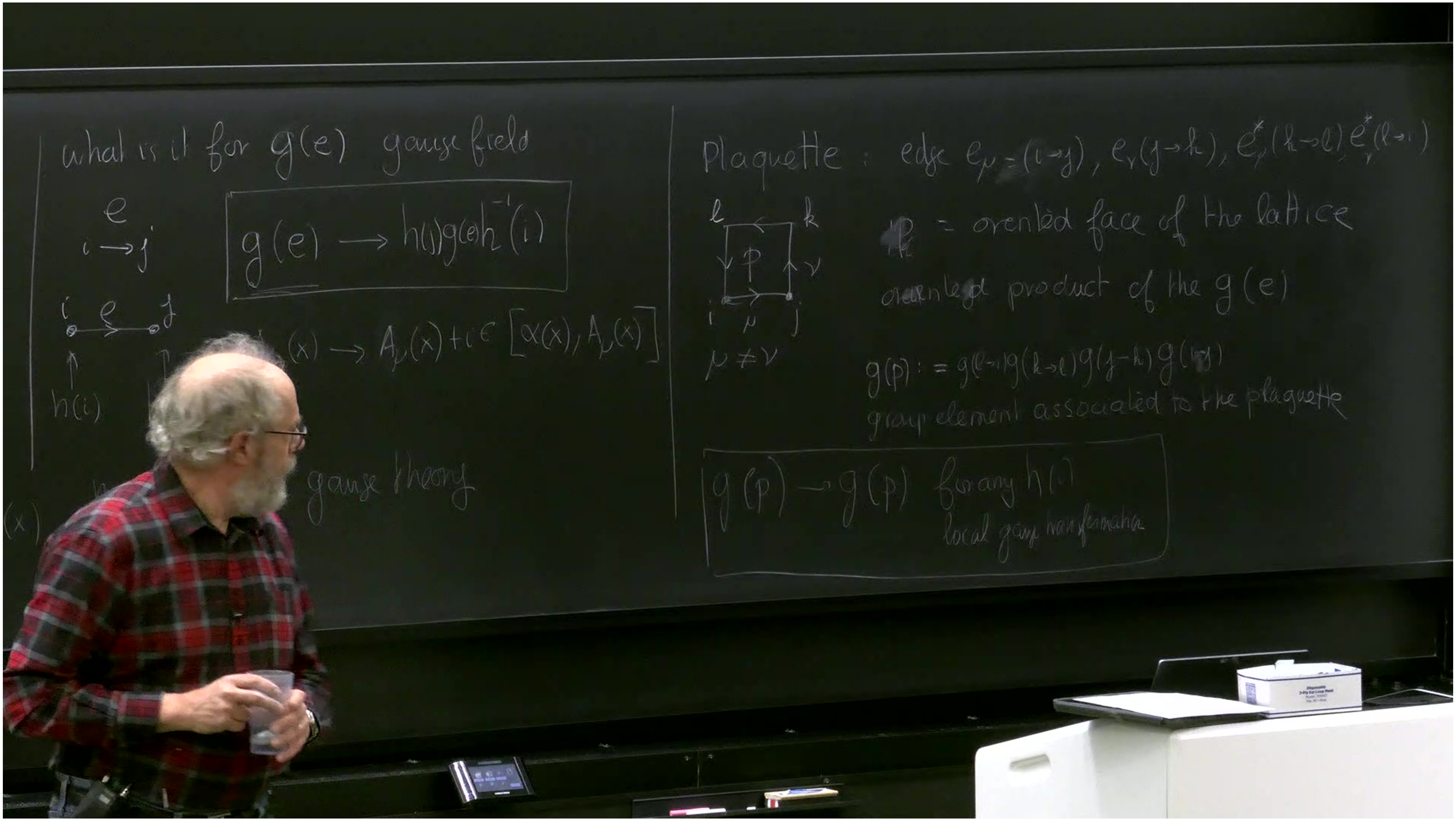
$i \longrightarrow j$

$$i \xrightarrow{\;e\;} j$$

$h(i) \qquad h(j)$

$$\boxed{g(e) \longrightarrow}$$


what is it for g(e) gauge field
g(e) → h(j)g(e)h^{-1}(i)
A_μ(x) → A_μ(x) + iε [α(x), A_μ(x)]
h(i) h(j)
non-perturbative gauge theory
Plaquette : edge e_μ = (i→j), e_ν(j→k), e_μ^*(k→l), e_ν^*(l→i)
p = oriented face of the lattice
oriented product of the g(e)
μ ≠ ν
g(p) := g(l→i)g(k→l)g(j→k)g(i→j)
group element associated to the plaquette



what is it for g(e) gauge field
g(e) → h(j)g(e)h⁻¹(i)
A_μ(x) → A_μ(x)+iε[α(x), A_μ(x)]
h(i)
gauge theory
Plaquette : edge e_μ=(i→j), e_ν(j→k), e*_μ(k→l), e*_ν(l→i)
p = oriented face of the lattice
oriented product of the g(e)
μ ≠ ν
g(p) := g(l→i)g(k→l)g(j→k)g(i→j)
group element associated to the plaquette
g(p) → g(p) for any h(i)
local group transformation

matter fields

$g(\square) \leftrightarrow$ curvature of the gauge connection

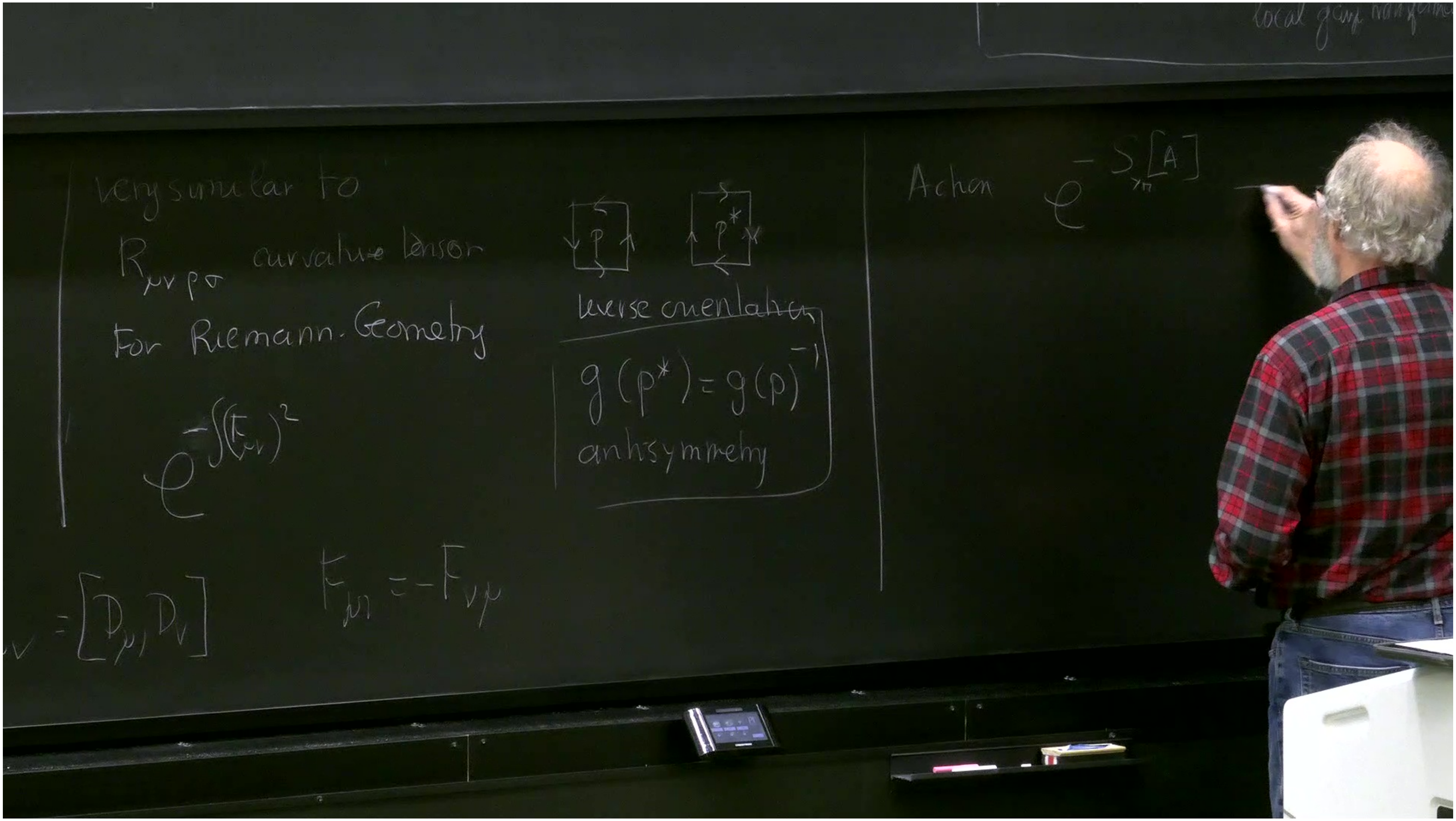
very similar to $R_{\mu\nu\rho\sigma}$ curvature tensor for Riemann. Geometry

$g(\rightarrow) = 1 + i\epsilon A(\rightarrow)$ $\quad A(\underset{\mu}{\rightarrow}) = A_\mu$

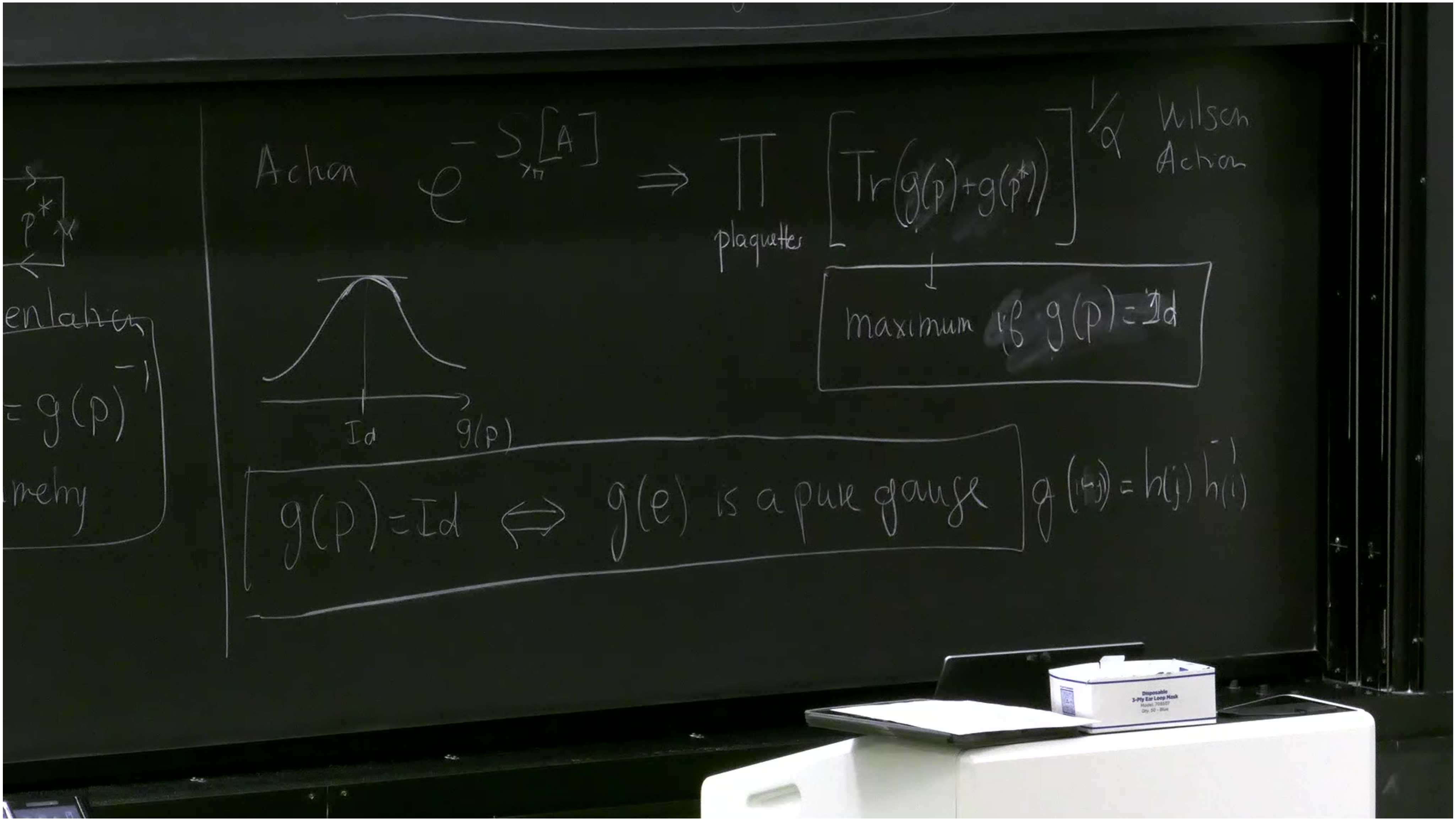
then

$$g(\square_{\mu\nu}) = 1 + i\epsilon F(\square_{\mu\nu})$$

$$F(\square_{\mu\nu}) = F_{\mu\nu} = [D_\mu, D_\nu]$$



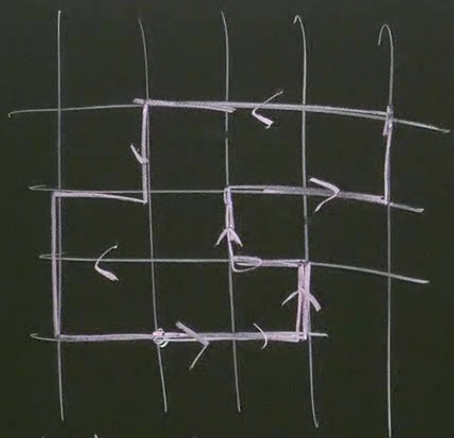
local gauge transformation
very similar to
$R_{\mu\nu\rho\sigma}$ curvature tensor
for Riemann. Geometry
$e^{-\int (F_{\mu\nu})^2}$
$p$
$p^*$
inverse orientation
$g(p^*) = g(p)^{-1}$
antisymmetry
Action $e^{-S_{YM}[A]}$
$F_{\mu\nu} = [D_\mu, D_\nu]$
$F_{\mu\nu} = -F_{\nu\mu}$


Action
e^{-S_{YM}[A]} ⇒ ∏_{plaquettes} [Tr(g(p)+g(p*))]^{1/α}
Wilson Action
maximum if g(p) = Id
Id
g(p)
g(p) = Id ⇔ g(e) is a pure gauge
g(i→j) = h(j) h(i)^{-1}
= g(p)^{-1}

$\phi = (\phi^i)_{i=1,2}$ for $SU(2)$ $\quad e_\mu = (0 \cdots \underset{\uparrow \mu}{1} \cdots 0)$ $\quad \phi(x+e_\mu) - \phi(x) = \partial_\mu \phi$

$e_\mu$ elementary vector on direction $\mu$

1



Wilson Loop

any oriented circuit $\mathscr{C}$ on the Lattice $\Rightarrow$ all gauge invariant observables of the theory

$$g(\mathscr{C}) = \prod_{e \in \mathscr{C}} g(e)$$

ordered product

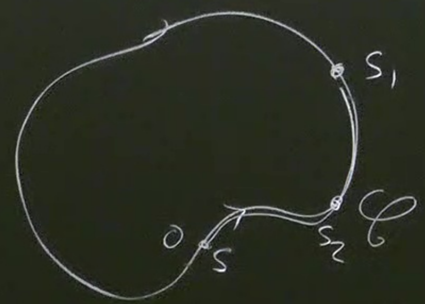
Non local

Gauge invariant quantity

Continuum

$\phi(x+e_\mu)-\phi(x)=\partial_\mu\phi$ $\quad x \xrightarrow{\mathcal{C}} x+e_\mu$

Continuum




oriented smooth loop

$$g(\mathcal{C}) \longrightarrow P\left[\exp\left(i\int_{\mathcal{C}} dx^\mu A_\mu(x)\right)\right]$$

Wilson loop operator

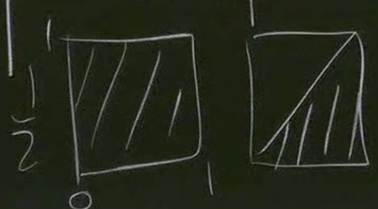
path ordered integral

magnetic flux

$$1 + i\int_0^1 ds\, \frac{dx^\mu}{ds} A_\mu(x) - \int_0^1 ds_1 \int_0^{s_1} ds_2\, \frac{dx^\mu}{ds_1}(s_1) A_\mu(x(s_1)) \frac{dx^\nu}{ds_2}(s_2) A_\nu(x(s_2))$$

$s \to X(s)$

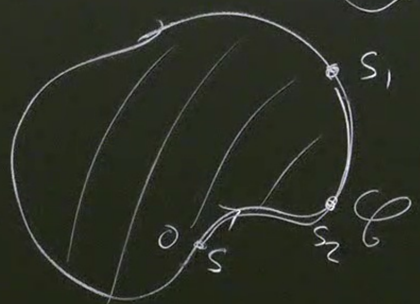
parametrization $S \in S^1$ $\quad s \in [0,1]$



$\phi(x+e_\mu) - \phi(x) = \partial_\mu \phi \qquad x \xrightarrow{e} x+e_\mu$

Continuum

$\longrightarrow \square = F_{\mu\nu}^2$




oriented smooth loop

$$g(\mathcal{C}) \longrightarrow \mathcal{P}\left[\exp\left(i\int_{\mathcal{C}} dx^\mu A_\mu(x)\right)\right]$$

Wilson loop operator

path ordered integral

magnetic flux

$$1 + i\int_0^1 ds \frac{dx^\mu}{ds} A_\mu(x) - \int_0^1 ds_1 \int_0^{s_1} ds_2 \frac{dx^\mu}{ds_1}(s_1) A_\mu(x(s_1)) \frac{dx^\nu}{ds_2}(s_2) A_\nu(x(s_2))$$

$s \to X(s)$

parametrization $s \in S^1$ $s \in [0,1]$