Title: The Gaudin model in the Deligne category Rep GL_t

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Abstract:

Deligne's category D_t is a formal way to define the category of finite-dimensional representations of the group GL_n with $n=t$ being a formal parameter (which can be specialized to any complex number). I will show how to interpolate the construction of the higher Hamiltonians of the Gaudin quantum spin chain associated with the Lie algebra $\mathfrak{gl}_n$ to any complex n, using D_t. Next, according to Feigin and Frenkel, Bethe ansatz equations in the Gaudin model are equivalent to no-monodromy conditions on a certain space of differential operators of order n on the projective line. We also obtain interpolations of these no-monodromy conditions to any complex n and prove that they generate the relations in the algebra of higher Gaudin Hamiltonians for generic complex n. I will also explain how it is related to the Bethe ansatz for the Gaudin model associated with the Lie superalgebra $\mathfrak{gl}_{m|n}$.

This is joint work with Boris Feigin and Filipp Uvarov,
https://arxiv.org/abs/2304.04501.

Gaudin model in Deligne's category Rep GL_t.

1 $\mathfrak{g}$ - s/s Lie algebra / $\mathbb{C}$

$L(\lambda)$ - f.d. irr. rep of $\mathfrak{g}$, w h.w. λ.

$\{x_a\}$, $\{x^a\}$ - dual bases of $\mathfrak{g}$

Hamiltonians: $H_i(\underline{z})$ $L(\lambda_1) \otimes \ldots \otimes L(\lambda_n) \circlearrowleft$

$\underline{z} = (z_1, z_2, \ldots, z_n)$ $H_i(\underline{z}) := \sum_{j \neq i} \frac{\Omega_{ij}}{z_i - z_j}$ $\Omega_{ij} = \sum x_a^{(i)} x^{a(j)}$

$\underline{z} = (z_1, z_2, \ldots, z_n)$ $\quad H_i(\underline{z}) := \sum_{j\neq i} \frac{\Omega_{ij}}{z_i - z_j}$ $\quad \Omega_{ij} = \sum \ldots x_a \ldots x$

Prop: $[H_i(\underline{z}), H_j(\underline{z})] = 0$

Pf: $[\Omega_{ij}, \Omega_{kl}] = 0$ if $\#\{i,j,k,l\} = 4$

$[\Omega_{ij}, \Omega_{ik} + \Omega_{kj}] = 0$

Problem: simultaneous diagonalization of $H_i(\underline{z})$. $\quad L(\lambda_1) \otimes \ldots \otimes L(\lambda_n)$

highest

$\mathrm{Hom}_{\mathfrak{g}}(L(\mu), L(\lambda_1) \otimes \ldots \otimes L(\lambda_n))$

Bethe ansatz method:

Ex: $\mathfrak{g} = \mathfrak{sl}_2$.

$$\Omega_{ij} = e^{(i)} f^{(j)} + f^{(i)} e^{(j)} + \frac{1}{2} h^{(i)} h^{(j)}.$$

$L(\lambda_1) \otimes \ldots \otimes L(\lambda_n)$

$\lambda_i \in \mathbb{Z}_{\geq 0}$.

Ψ

$v_{\lambda_1} \otimes \ldots \otimes v_{\lambda_n} = v_\Lambda$ - highest

$$f(w) := \sum_i \frac{f^{(i)}}{w - z_i}$$

$$v_\Lambda(w_1, \ldots, w_k) = f(w_1) f(w_2) \ldots f(w_k) v_\Lambda$$

Prop: $v_\Lambda(w_1, \ldots, w_k)$ is an eigenvecto

Ex: $\mathfrak{g} = \mathfrak{sl}_2$.

$$\Omega_{ij} = e^{(i)} f^{(j)} + f^{(i)} e^{(j)} + \frac{1}{2} h^{(i)} h^{(j)}.$$

$$L(\lambda_1) \otimes \ldots \otimes L(\lambda_n)$$

$$\cup$$

$$v_{\lambda_1} \otimes \ldots \otimes v_{\lambda_n} = v_\Lambda \quad \text{- highest.}$$

$\lambda_i \in \mathbb{Z}_{\geq 0}$.

$$f(w) := \sum_i \frac{f^{(i)}}{w - z_i}$$

$$v_\Lambda(w_1 \ldots w_k) = \underline{f(w_1) f(w_2) \ldots f(w_k)\, v_\Lambda}$$

Prop: $v_\Lambda(w_1 \ldots w_k)$ is an eigenvector $\Longleftrightarrow$

$$\Longleftrightarrow \left\{ \sum_{i=1}^{n} \frac{\lambda_i}{w_j - z_i} - \sum_{s \neq j} \frac{2}{w_j - w_s} = 0. \right.$$

$$v_\Lambda(w_1, \dots, w_k) = f(w_1) f(w_2) \dots f(w_k) v_\Lambda$$

$$\iff \left\{ \sum_{i=1}^{n} \frac{\lambda_i}{w_j - z_i} - \sum_{s \neq j} \frac{2}{w_j - w_s} = 0 \quad \forall j \right.$$

Bethe ansatz eqs.

Thm: such eigenvectors form a basis in $L(\lambda_1) \otimes \dots \otimes L(\lambda_n)$

$v_\Lambda(w_1,\dots,w_k) = f(w_1)\, f(w_2) \dots f(w_k)\, v_\Lambda$

$$\Leftrightarrow \left\{ \sum_{i=1}^{n} \frac{\lambda_i}{w_j - z_i} - \sum_{s \neq j} \frac{2}{w_j - w_s} = 0 \quad \forall j \right.$$

Bethe ansatz eq.

Thm: such eigenvectors form a basis in $L(\lambda_1)\otimes \dots \otimes L(\lambda_n)$

Generalization: $w_1, \dots, w_k$; $\alpha_1, \dots, \alpha_k$ $\rightsquigarrow$ $v_\Lambda(w_1,\dots,w_k) \in L(\lambda_1)\otimes \dots \otimes L(\lambda_n)_{\sum \lambda_i - \sum_{j=1}^{k} \alpha_j}$

$$\forall j \quad \sum \frac{(\lambda_i, \alpha_j)}{w_j - z_i} - \sum_{s \neq j} \frac{(\alpha_j, \alpha_s)}{w_j - w_s} = 0.$$

Complete for generic $\underline{z}$.

Stabilization phenomenon: $\mathfrak{g} = \mathfrak{gl}_N$ $\quad N \to \infty$

λ_i are Young diagrams. $L(\lambda) = S^\lambda(\mathbb{C}^N)$ - Schur functor.

Irreducible components in $S^\lambda(\mathbb{C}^N) \otimes S^\mu(\mathbb{C}^{N*})$.

$\underline{z} = (z_1, z_2, \dots, z_n)$ $\quad H_i(\underline{z}) := \sum_{j \neq i} \frac{\Omega_{ij}}{z_i - z_j}$ $\quad \Omega_{ij} = \sum x_a \dots x \dots$

Stabilization: $\dim \mathrm{Hom}_{\mathfrak{gl}_N}\left(L(\lambda,\mu),\ L(\lambda_1,\mu_1)\otimes \dots \otimes L(\lambda_n,\mu_n)\right)$ stabilizes

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| $

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad V^{\lambda,\mu}_{\underline{\lambda},\underline{\mu}}$

as $N \to \infty$.

Q: What happens to Bethe ansatz eq ...

Make $V^{\lambda,\mu}_{\underline{\lambda},\underline{\mu}}$ depending on $N \in \mathbb{C}$.

Rep GL_t $\quad t \in \mathbb{C}$??

tensor category (Deligne).

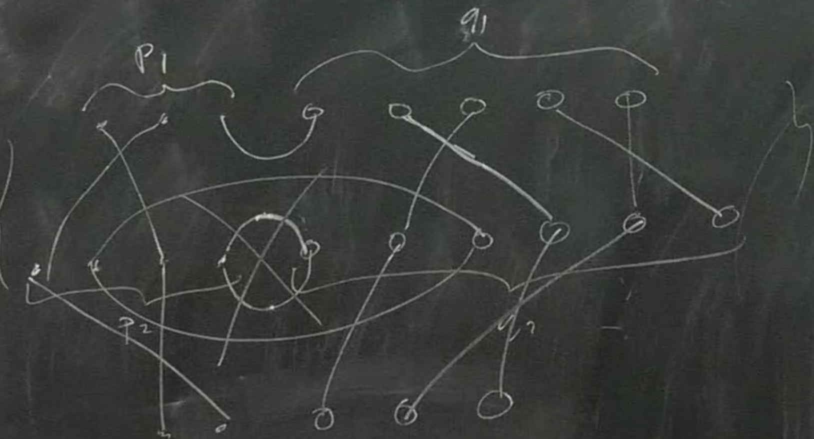
Rep GL_N is generated as a $\otimes$ category by $V = \mathbb{C}^N$ and V^*.

(can find any irred GL_N-rep in $V^{\otimes p} \otimes V^{*\otimes q} = \boxed{T^{p,q}}$)

λ_i are Young diagrams. $L(\lambda) = S^\lambda(\mathbb{C}^N)$ = Schur functor.

Highest irreducible component $\underbrace{S^\lambda(\mathbb{C}^N)\otimes S^\mu(\mathbb{C}^{N^*})}\supset L(\lambda,\mu)$.

Schur-Weyl duality $\Longrightarrow$ $\mathrm{Hom}_{GL_N}(T^{p_1 q_1}, T^{p_2 q_2}) = \left(V^{\otimes(q_1+p_2)}\otimes V^{*\otimes(p_1+q_2)}\right)^{GL_N}$

$=$ Span $\Bigg\{$ 

$N\cdot$

$\mathbb{C} \to V\otimes V^* \xrightarrow{tr} \mathbb{C}$

$1 \longrightarrow Id \longrightarrow N\cdot Id$.

Can make $N = t \in \mathbb{C}$.

Can make $N = t \in \mathbb{C}$.

Karoubian envelope: add object $\mathrm{Im}\, P$ $\quad \forall \in P \in \mathrm{End}(T^{p,q})$, $P^2 = P$

Thm: $t \notin \mathbb{Z}$ get a semisimple category, w simple obj.

$L(\lambda, \mu)$ $\quad \lambda, \mu$ any Young diagrams.

$t = N \in \mathbb{Z}_{>0}$ $\quad \mathrm{Rep}\, GL_t \longrightarrow \mathrm{Rep}\, GL_{N_0|N_1}$ $\quad N_0 - N_1 = N$

$V \rightsquigarrow \mathbb{C}^{N_0|N_1}$

$$H_i(\underline{z}) : \mathrm{Hom}_{GL_t}\left(L(\lambda,\mu), L(\lambda_1,\mu_1)\otimes \dots \otimes L(\lambda_n,\mu_n)\right)$$

t

Prop: can express Ω_{ij} in terms of the diagrams.

$$\mathrm{Hom}(\mathbb{C}, V\otimes V^*)$$

$\underline{z} = (z_1, \dots, z_n)$, $\lambda_1, \dots, \lambda_n$ $\quad w_1, \dots, w_k$.

$$a(z) = \sum \frac{\lambda_i/2}{z - z_i} - \sum \frac{1}{z - w_j}$$

rational fn w 2nd poles at z_i.
1st order pole at w_j.

$$\big(\partial_z - a(z)\big)\big(\partial_z + a(z)\big) = \partial_z^2 - q(z),$$

BA eqs $\Longleftrightarrow$ no pole at w_j?

$$q(z) = a^2(z) + a'(z).$$

$$q(z) = \sum \frac{\frac{\lambda_i(\lambda_i+2)}{4}}{(z - z_i)^2} + \frac{q_i}{z - z_i},$$

"no monodromy" condition.

$$q(z) = a^2(z) + a'(z).$$

$$q(z) = \sum \frac{\frac{\lambda_i(\lambda_i+2)}{4}}{(z-z_i)^2} + \frac{q_i}{z-z_i}, \quad \text{"no monodromy condition"}$$

$$\partial_z^N + \sum_{m=1}^{N} b_m(z)\,\partial_z^{N-m}$$

$b_m(z)$ has deg m pole at z_i.

+ no-monodromy condition at z_i.

Thm: For GL_t

$$\partial_z^t + \sum_{m=1}^{\infty} b_m(z)\,\partial_z^{t-m}$$

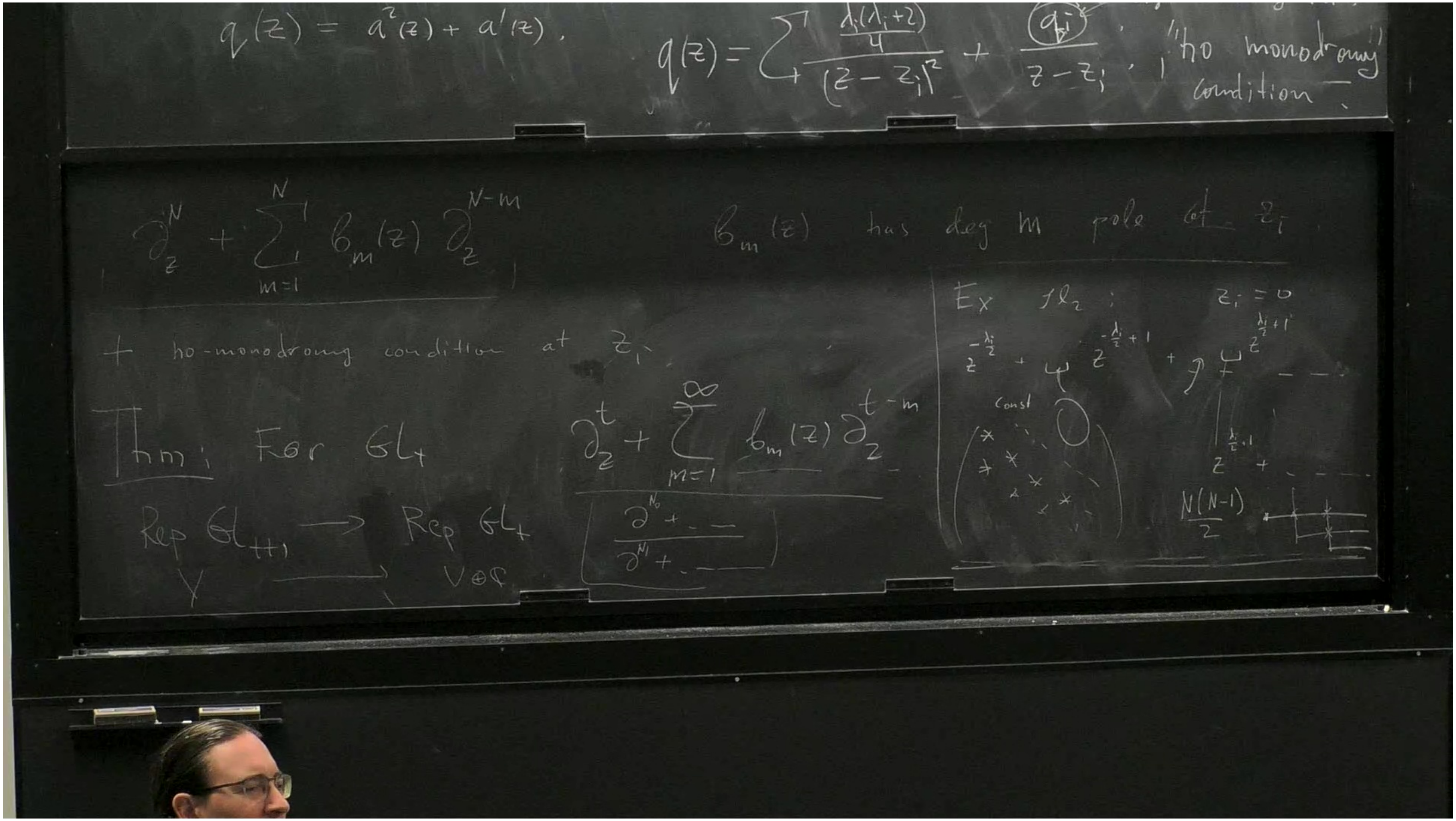
Ex sl_2: $z_i = 0$

$$z^{-\frac{\lambda_i}{2}} + \varphi\, z^{-\frac{\lambda_i}{2}+1} + \ldots\, z^{\frac{\lambda_i}{2}+1} - \ldots$$

const

$$z^{\frac{\lambda_i}{2}+1} + \ldots$$

$$\frac{N(N-1)}{2}$$



$q(z) = a^2(z) + a'(z)$.
$q(z) = \sum \frac{\frac{\lambda_i(\lambda_i+2)}{4}}{(z-z_i)^2} + \frac{q_i}{z-z_i}$, "no monodromy" condition
$\partial_z^N + \sum_{m=1}^{N} b_m(z)\, \partial_z^{N-m}$
$b_m(z)$ has deg m pole at z_i
+ no-monodromy condition at z_i
Ex sl_2 : $z_i = 0$
$z^{-\frac{\lambda_i}{2}} + \cdots z^{-\frac{\lambda_i}{2}+1} + \cdots z^{\frac{\lambda_i}{2}+1}$
const
$z^{\frac{\lambda_i}{2}+1} + \cdots$
$\frac{N(N-1)}{2}$
Thm: For GL_t
$\partial_z^t + \sum_{m=1}^{\infty} b_m(z)\, \partial_z^{t-m}$
$\frac{\partial^{N_0} + \cdots}{\partial^{N_1} + \cdots}$
Rep $GL_{t+1} \longrightarrow$ Rep GL_t
$V \longrightarrow V \oplus \mathbb{C}$

pole at w_j?

eigenvalues of H_i.

"no monodromy condition".

Can make $N = t \in \mathbb{C}$.

$N \cdot \mathrm{Id}$.

Deligne's GL_t is Langlands dual to

Etingof: Feigin's $\mathfrak{gl}_t$ $\cong$ + Khesin-Malikov.

$$\mathfrak{gl}_N = \mathrm{Mat}_N \Longleftarrow \left(U(\mathfrak{sl}_2) \big/ C = \tfrac{(N-1)(N+1)}{4} \right)$$

$Z(U(\mathfrak{gl}_t))$ Lie algebra in $\mathrm{Rep}\, GL_t$.