Title: An SPT-LSM theorem for weak SPTs with non-invertible symmetry

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Abstract:

Like ordinary symmetries, non-invertible symmetries can characterize Symmetry-Protected Topological (SPT) phases. In this talk, we will discuss weak SPTs protected by projective non-invertible symmetries. Projective symmetries are ubiquitous in quantum spin models and can be leveraged to constrain their phase diagram and entanglement structure, e.g., Lieb-Schultz-Mattis (LSM) theorems. We will show how, surprisingly, projective non-invertible symmetries do not always imply LSM theorems. We will first discuss a simple, exactly solvable 1+1D quantum spin model in an SPT phase protected by both translation and non-invertible symmetries forming a non-trivial projective algebra. We will then generalize this example to a class of projective non-invertible Rep(G) x G x translation symmetries. For some finite groups G, this projectivity implies an LSM theorem. When it does not, we prove it still provides a constraint through an SPT-LSM theorem: any unique and gapped ground state is necessarily a non-invertible weak SPT state with non-trivial entanglement. [This talk is based on arXiv:2409.18113]

An SPT-LSM theorem for weak SPTs with non-invertible symmetry

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arXiv:2409.18113

Quantum phases and symmetry

A fundamental problem in quantum condensed matter physics is to understand quantum phases

1. How do we diagnose different quantum phases?
2. What are the allowed possible quantum phases?

Sometimes, phases are characterized by a symmetry

- ➤ Superfluids by U(1) boson number conservation
- ➤ Topological insulators by $U(1)_f$ and $\mathbb{Z}_2^T$ symmetries

For such phases, symmetries provide answers to questions (1) and (2).

Generalized symmetries

There has been a recent flurry of interest in generalizing the notion of symmetries

- Ordinary symmetries transform local operators in an invertible manner (e.g., $c_r^\dagger \to e^{i\theta} c_r^\dagger$)
- So-called generalized symmetries modify this definition

Non-invertible symmetries have non-invertible transformations

[Bhardwaj, Tachikawa 2017; Chang, Lin, Shao, Wang, Yin 2018; ⋯]

- Can arise at critical points from Kramers-Wannier dualities
  [⋯ ; Choi, Córdova, Hsin, Lam, Shao 2021; ⋯ ; Seiberg, Shao 2023; **SP**, Delfino, Lam, Aksoy 2024]
- Can emerge in ordered phases (are symmetries of nonlinear sigma models) [**SP** 2023; **SP**, Zhu, Beaudry, X-G Wen 2023]

Generalized symmetries

Q: Why should we consider these as symmetries?

A: They pass the duck test!

If it looks like a duck, swims like a duck, and quacks like a duck, then it probably is a duck.

- Have conservation laws
- Can constrain phase diagrams (be anomalous)
- Can characterize SSB and SPT phases

Quantum phases + generalized symmetry

Which quantum phases are characterized by generalized symmetries?

Build-a-phase recipe

(1) Choose your generalized symmetries adjectives

$$\mathrm{a}_1\text{–}\mathrm{a}_2\text{–}\mathrm{a}_3\text{–}\cdots \text{ Symmetry}$$

(2) Specify SSB and SPT pattern

Ordered phases *Topological insulators*

Topological order *Maxwell phases* *Higgs phases* *Fracton phases*

Phases we have yet to name!

TL;DR for this talk

In this talk, we explore Symmetry Protected Topological (SPT) phases characterized by non-invertible symmetries

- Find a new class of entangled weak SPTs characterized by projective non-invertible symmetries

Outline:

1. Review SPTs (from a symmetry defect perspective)
2. Simple example of an entangled weak SPT characterized by a projective non-invertible symmetry
3. General discussion on (SPT-)LSM theorems from projective non-invertible symmetries

What are SPTs

An SPT phase is a gapped quantum phase described by a symmetry with a unique ground state on all closed spatial manifolds [Chen, Gu, Liu, Wen 2011; Wang, Senthil 2013; Else, Nayak 2014; ⋯]

- Interesting physics can arise on boundaries and interfaces between SPTs (e.g., topological order, gapless excitations)

SPTs are characterized in the bulk by their response to static probes

- Background gauge fields and symmetry defects

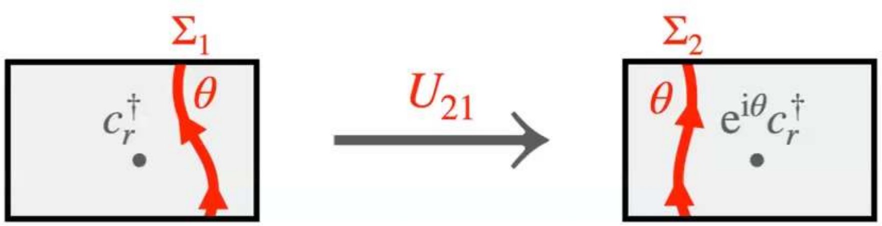
Symmetry defects

Symmetry defects (domain walls) are localized modifications to the Hamiltonian $H^{(\Sigma)}_{\text{defect}} = H + \delta H(\Sigma)$ and other operators

➤ Moved using unitary operators (are topological defects)

$$H^{(\Sigma_2)}_{\text{defect}} = U_{21} H^{(\Sigma_1)}_{\text{defect}} U^{\dagger}_{21}$$

➤ Implement the symmetry transformation across space



➤ Twisted boundary conditions $(T_{\perp})^L$ = Symmetry operator

Example: $\mathbb{Z}_2 \times \mathbb{Z}_2$ SPTs

1d periodic lattice with two qubits on each site $j \sim j + L$ acted on by Pauli operators X_j, Z_j and $\tilde{X}_j$, $\tilde{Z}_j$.

$$H_p = -\sum_{j=1}^{L} (X_j + \tilde{X}_j) \qquad H_c = -\sum_{j=1}^{L} \left(\tilde{Z}_{j-1} X_j \tilde{Z}_j + Z_j \tilde{X}_j Z_{j+1}\right)$$

$$|\mathrm{GS}_p\rangle = |+ + \cdots +\rangle \qquad |\mathrm{GS}_c\rangle = \tilde{Z}_{j-1} X_j \tilde{Z}_j |\mathrm{GS}_c\rangle = Z_j \tilde{X}_j Z_{j+1} |\mathrm{GS}_c\rangle$$

- Both models have a unique symmetric gapped ground state
- There is a $\mathbb{Z}_2 \times \tilde{\mathbb{Z}}_2$ symmetry $U = \prod_j X_j$ and $\tilde{U} = \prod_j \tilde{X}_j$ with $U|\mathrm{GS}_\bullet\rangle = \tilde{U}|\mathrm{GS}_\bullet\rangle = |\mathrm{GS}_\bullet\rangle$

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H_p and H_c are both in a $\mathbb{Z}_2 \times \tilde{\mathbb{Z}}_2$ SPT phase

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Distinguishing $\mathbb{Z}_2 \times \mathbb{Z}_2$ SPTs

Are H_p and H_c in different $\mathbb{Z}_2 \times \tilde{\mathbb{Z}}_2$ SPT phases?

We can check by inserting a U symmetry defect at $\langle L,1 \rangle$

➤ Gives rise to U-twisted boundary conditions: $Z_{j+L} = -Z_j$

1. H_p is unaffected, so its ground state still satisfies

$$U\,|\mathrm{GS}_{p;U}\rangle = +\,|\mathrm{GS}_{p;U}\rangle \qquad \tilde{U}\,|\mathrm{GS}_{p;U}\rangle = +\,|\mathrm{GS}_{p;U}\rangle$$

2. H_c becomes $H_c + 2\,Z_L\,\tilde{X}_L\,Z_1$, and its ground state satisfies

$$U\,|\mathrm{GS}_{c;U}\rangle = +\,|\mathrm{GS}_{c;U}\rangle \qquad \tilde{U}\,|\mathrm{GS}_{c;U}\rangle = -\,|\mathrm{GS}_{c;U}\rangle$$

Distinguishing $\mathbb{Z}_2 \times \mathbb{Z}_2$ SPTs

Are H_p and H_c in different $\mathbb{Z}_2 \times \tilde{\mathbb{Z}}_2$ SPT phases?

We c

➤ G

> Different domain wall decorations imply that H_p and H_c are in different $\mathbb{Z}_2 \times \tilde{\mathbb{Z}}_2$ SPT phases
>
> [Chen, Lu, Vishwanath 2013; Gaiotto, Johnson-Freyd 2017; Wang, Ning, Cheng 2021]

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$$U\,|\mathrm{GS}_{c;U}\rangle = +\,|\mathrm{GS}_{c;U}\rangle \qquad \tilde{U}\,|\mathrm{GS}_{c;U}\rangle = -\,|\mathrm{GS}_{c;U}\rangle$$

Example: $\mathbb{Z}_2$ weak SPTs

1d periodic lattice with a qubit on each site $j \sim j + L$

$$H_+ = -\sum_j X_j \quad \text{vs.} \quad H_- = +\sum_j X_j$$

- Both have a unique gapped ground state $|\mathrm{GS}_\pm\rangle = \otimes_j |\pm\rangle$
- Symmetries: $\mathbb{Z}_2 \times \mathbb{Z}_L$ with $U = \prod_j X_j$ and $T: j \to j+1$

H_+ and H_- are both in $\mathbb{Z}_2 \times \mathbb{Z}_L$ SPT phases

SPTs characterized by translations are called weak SPTs

> H_+ and H_- are both in $\mathbb{Z}_2$ weak SPT phases

Example: $\mathbb{Z}_2$ weak SPTs

Are H_+ and H_- in different $\mathbb{Z}_2$ weak SPT phases?

Let's insert a $U = \prod_j X_j$ symmetry defect at $\langle L,1 \rangle$

- Neither H_+ or H_- are modified by $Z_{j+L} = -Z_j$
- Translation operator becomes $T = X_1 T_{\text{defect-free}}$ $(T^L = U)$

	Even L	Even L, $\mathbb{Z}_2$ symmetry defect
$U\vert\text{GS}_\pm\rangle =$	$+\vert\text{GS}_\pm\rangle$	$+\vert\text{GS}_\pm\rangle$
$T\vert\text{GS}_\pm\rangle =$	$+\vert\text{GS}_\pm\rangle$	$\pm\vert\text{GS}_\pm\rangle$

Different $\mathbb{Z}_2$ weak SPTs

Example: $\mathbb{Z}_2$ weak SPTs

Are H_+ and H_- in different $\mathbb{Z}_2$ weak SPT phases?

Translation defect carries $\mathbb{Z}_2$ symmetry charge in $|\mathrm{GS}_-\rangle$

- Inserting a translation defect is done by

$$T^L = 1 \to T^L = T \implies L \to L-1$$

- Translation operator becomes $T = X_1 T_{\text{defect-free}}$ $(T^L = U)$

	Even L	Even L, $\mathbb{Z}_2$ symmetry defect	Odd L
$U\lvert\mathrm{GS}_\pm\rangle =$	$+\lvert\mathrm{GS}_\pm\rangle$	$+\lvert\mathrm{GS}_\pm\rangle$	$\pm\lvert\mathrm{GS}_\pm\rangle$
$T\lvert\mathrm{GS}_\pm\rangle =$	$+\lvert\mathrm{GS}_\pm\rangle$	$\pm\lvert\mathrm{GS}_\pm\rangle$	$+\lvert\mathrm{GS}_\pm\rangle$

A curious Hamiltonian

1d periodic lattice with a single qubit and $\mathbb{Z}_4$ qudit on each site $j \sim j + L$ [**SP**, Lam, Aksoy arXiv:2409.18113]

- σ^x, σ^z act on qubits: $(\sigma^x)^2 = (\sigma^z)^2 = 1$ and $\sigma^z\sigma^x = -\sigma^x\sigma^z$
- X, Z act on $\mathbb{Z}_4$ qudits: $X^4 = Z^4 = 1$ and $ZX = \mathrm{i}\, XZ$

$$H = -\sum_j \sigma_j^x\, C_{j+1}\, \sigma_{j+1}^x + \frac{1}{4}\sum_j (Z_j - Z_j^\dagger)\, \sigma_j^z\, (Z_{j+1} - Z_{j+1}^\dagger)$$

- C acts as $X \to X^\dagger$ and $Z \to Z^\dagger$
- Is a sum of commuting terms and has a unique gapped ground state

Some curious symmetries

$$H = -\sum_j \sigma_j^x C_{j+1} \sigma_{j+1}^x + \frac{1}{4}\sum_j (Z_j - Z_j^\dagger)\, \sigma_j^z \,(Z_{j+1} - Z_{j+1}^\dagger)$$

What are the symmetries of H?

- $\mathbb{Z}_L$ lattice translations $T: j \to j+1$
- Three $\mathbb{Z}_2$ symmetry operators

$$U = \prod_j X_j^2, \qquad R_1 = \prod_j \sigma_j^z, \qquad R_2 = \prod_j Z_j^2$$

- 🤷‍♂️ symmetry operator

$$R_{\mathsf{E}} = \frac{1}{2}\left(1 + R_1\right)\left(1 + R_2\right)\prod_j Z_j^{\prod_{k=1}^{j-1}\sigma_k^z}$$

Some curious symmetries

$$H = -\sum_j \sigma_j^x C_{j+1} \sigma_{j+1}^x + \frac{1}{4} \sum_j (Z_j - Z_j^\dagger)\, \sigma_j^z \,(Z_{j+1} - Z_{j+1}^\dagger)$$

What are the symmetries of H?

R_{E} is a non-invertible symmetry operator

- $R_{\mathsf{E}} \,|\psi\rangle = 0$ if $R_1 \,|\psi\rangle = -\,|\psi\rangle$ or $R_2 \,|\psi\rangle = -\,|\psi\rangle$
- R_{E} have zero-eigenvalues $\implies R_{\mathsf{E}}$ is non-invertible

- 🤷‍♂️ symmetry operator

$$R_{\mathsf{E}} = \frac{1}{2}\left(1 + R_1\right)\left(1 + R_2\right) \prod_j Z_j^{\prod_{k=1}^{j-1} \sigma_k^z}$$

A curious SPT

These symmetry operators obey

$$U^2 = 1, \quad R_i^2 = 1, \quad R_{\mathsf{E}}^2 = 1 + R_1 + R_2 + R_1 R_2, \quad R_{\mathsf{E}} R_i = R_i R_{\mathsf{E}} = R_{\mathsf{E}}$$

$$U\, R_{\mathsf{E}} = (-1)^L R_{\mathsf{E}}\, U$$

➤ Form a (projective) $\mathbb{Z}_2 \times \mathsf{Rep}(D_8)$ symmetry

Dihedral group of order 8 $D_8 \simeq \langle r, s \mid r^2 = s^4 = 1, rsr = s^3 \rangle$

➤ Four 1d reps $\mathbf{1}$, $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3 = \mathbf{P}_1 \otimes \mathbf{P}_2$ and one 2d irrep E

$$\mathbf{P}_i \otimes \mathbf{P}_i = \mathbf{1} \qquad \mathsf{E} \otimes \mathsf{E} = \mathbf{1} \oplus \mathbf{P}_1 \oplus \mathbf{P}_2 \oplus \mathbf{P}_3 \qquad \mathsf{E} \otimes \mathbf{P}_i = \mathbf{P}_i \otimes \mathsf{E} = \mathsf{E}$$

Some curious symmetries

$$H = -\sum_j \sigma_j^x \, C_{j+1} \, \sigma_{j+1}^x + \frac{1}{4} \sum_j (Z_j - Z_j^\dagger) \, \sigma_j^z \, (Z_{j+1} - Z_{j+1}^\dagger)$$

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- Form a (projective) $\mathbb{Z}_2 \times \mathsf{Rep}(D_8)$ symmetry

Ground state satisfies

$$T\,|\mathrm{GS}\rangle = +\,|\mathrm{GS}\rangle \qquad U\,|\mathrm{GS}\rangle = +\,|\mathrm{GS}\rangle \qquad R_1\,|\mathrm{GS}\rangle = +\,|\mathrm{GS}\rangle$$

$$R_2\,|\mathrm{GS}\rangle = \begin{cases} +\,|\mathrm{GS}\rangle, & L \text{ even} \\ -\,|\mathrm{GS}\rangle, & L \text{ odd} \end{cases} \qquad R_{\mathsf{E}}\,|\mathrm{GS}\rangle = \begin{cases} +2\,|\mathrm{GS}\rangle, & L \text{ even} \\ 0, & L \text{ odd} \end{cases}$$

A curious SPT

These sym

$U^2 = 1,$ $R_i R_{\mathsf{E}} = R_{\mathsf{E}}$

H is in a $\mathbb{Z}_2 \times \mathsf{Rep}(D_8)$ weak SPT phase

➤ Translation defects carry $\mathsf{Rep}(D_8)$ symmetry charge in $|\mathrm{GS}\rangle$

➤ Form a

Ground state satisfies

$$T|\mathrm{GS}\rangle = +|\mathrm{GS}\rangle \qquad U|\mathrm{GS}\rangle = +|\mathrm{GS}\rangle \qquad R_1|\mathrm{GS}\rangle = +|\mathrm{GS}\rangle$$

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The surprising lack of an 't Hooft anomaly

Inserting U or R_E symmetry defects leads to the projective algebras

U *symmetry defect*	R_E *symmetry defect*
$R_E T = -T R_E$	$TU = -UT$

For invertible symmetries, such projective algebras imply an 't Hooft anomaly (e.g., the type III anomaly $(-1)^{\int_{M_3} a \cup b \cup c}$)

[Matsui 2008; Yao, Oshikawa 2020; Seifnashri 2023; Kapustin, Sopenko 2024]

➤ This is not true for non-invertible symmetries!

A curious projective algebra

This SPT is characterized by a projective symmetry:

$$U R_{\mathsf{E}} = -R_{\mathsf{E}} U \qquad \text{(odd } L\text{)}$$

Projective unitary symmetries $U_1 U_2 = e^{i\theta} U_2 U_1$ forbid SPTs

- Assume non-degenerate symmetric ground state:

  1. $\langle \psi | U_1 U_2 | \psi \rangle = \langle \psi | \psi \rangle = 1$
  2. $\langle \psi | U_1 U_2 | \psi \rangle = e^{i\theta} \langle \psi | U_2 U_1 | \psi \rangle = e^{i\theta}$

  } *Contradicts assumption*

Projective non-invertible symmetries are compatible with SPTs

- Loophole: symmetry operator has zero-eigenvalues
- $U R_{\mathsf{E}} = (-1)^L R_{\mathsf{E}} U$ enforces $R_{\mathsf{E}} | \mathrm{GS}_{\mathrm{SPT}} \rangle = 0$ when L is odd

The surprising lack of an 't Hooft anomaly

Inserting U or R_{E} symmetry defects leads to the projective algebras

U *symmetry defect*	R_{E} *symmetry defect*
$R_{\mathsf{E}}\, T = -\, T R_{\mathsf{E}}$	$T U = -\, U T$

Fails because of $R_{\mathsf{E}} = 0$ loophole

Fails because the degeneracy is encoded in the defect's quantum dimension

Projective $Z(G) \times \mathsf{Rep}(G)$ symmetry

The projective $\mathbb{Z}_2 \times \mathsf{Rep}(D_8)$ symmetry is a special case of a more general projective $Z(G) \times \mathsf{Rep}(G)$ symmetry

- $Z(G)$ is the center of a finite group G
- $\mathsf{Rep}(G)$ is the fusion category of representations of G

$Z(G)$ symmetry operator U_z, with $z \in Z(G)$, satisfies

$$U_{z_1} U_{z_2} = U_{z_1 z_2}$$

$\mathsf{Rep}(G)$ symmetry operator R_Γ, with Γ an irrep of G, satisfies

$$R_{\Gamma_a} \times R_{\Gamma_b} = R_{\Gamma_a \otimes \Gamma_b} = R_{\oplus_c N_{ab}^c \Gamma_c} = \sum_c N_{ab}^c R_{\Gamma_c}$$

- Non-invertible symmetry when G is non-Abelian

Projective $Z(G) \times \mathsf{Rep}(G)$ symmetry

The projectivity arises through the relation

$$R_\Gamma U_z = (e^{i\phi_\Gamma(z)})^L \, U_z R_\Gamma \quad \text{with } e^{i\phi_\Gamma(z)} = \mathrm{Tr}[\Gamma(z)] \, / \, d_\Gamma$$

e.g., $e^{i\phi_\Gamma(z)}$ when $G = \mathbb{Z}_2$ $(Z(\mathbb{Z}_2) = \mathbb{Z}_2)$

z \ Γ	**1**	sign
$+1$	$+1$	$+1$
-1	$+1$	-1

Projective $Z(G) \times \mathsf{Rep}(G)$ symmetry

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e.g., $\mathrm{e}^{\mathrm{i}\phi_\Gamma(z)}$ when $G = D_8$ $(Z(D_8) = \mathbb{Z}_2)$

$z \backslash \Gamma$	$\mathbf{1}$	$\mathbf{1}_1$	$\mathbf{1}_2$	$\mathbf{1}_3$	E
$+1$	$+1$	$+1$	$+1$	$+1$	$+1$
-1	$+1$	$+1$	$+1$	$+1$	-1

Explicit expressions of U_z and R_Γ for the Hilbert space $\bigotimes_j \mathbb{C}^{|G|}$

$$U_z = \sum_{\{g_j\}} |zg_1, \cdots, zg_L\rangle\langle g_1, \cdots, g_L| \qquad R_\Gamma = \sum_{\{g_j\}} \mathrm{Tr}[\Gamma(g_1 \cdots g_L)]\,|g_1, \cdots, g_L\rangle\langle g_1, \cdots, g_L|$$

Projective $Z(G) \times \mathsf{Rep}(G)$ symmetry

The projectivity arises through the relation

$$R_\Gamma U_z = (\mathrm{e}^{\mathrm{i}\phi_\Gamma(z)})^L \, U_z R_\Gamma \quad \text{with } \mathrm{e}^{\mathrm{i}\phi_\Gamma(z)} = \mathrm{Tr}[\Gamma(z)] \,/\, d_\Gamma$$

e.g., $\mathrm{e}^{\mathrm{i}\phi}$

Projective algebras also arise from inserting symmetry defects [SP, Lam, Aksoy arXiv:2409.18113]

$z \in Z(G)$ *defect*	$\Gamma \in \mathsf{Rep}(G)$ *defect*
$R_\Gamma T^{(z)}_{\mathrm{tw}} = \mathrm{e}^{\mathrm{i}\phi_\Gamma(z)} T^{(z)}_{\mathrm{tw}} R_\Gamma$	$T^{(\Gamma)}_{\mathrm{tw}} U_z = \mathrm{e}^{\mathrm{i}\phi_\Gamma(z)} U_z T^{(\Gamma)}_{\mathrm{tw}}$

Explic $\bigotimes_j \mathbb{C}^{|G|}$

$$U_z = \sum_{\{g_j\}} |z g_1, \cdots, z g_L\rangle\langle g_1, \cdots, g_L| \qquad R_\Gamma = \sum_{\{g_j\}} \mathrm{Tr}[\Gamma(g_1 \cdots g_L)] \, |g_1, \cdots, g_L\rangle\langle g_1, \cdots, g_L|$$

(SPT)-LSM theorems

$$R_\Gamma U_z = (\mathrm{e}^{\mathrm{i}\phi_\Gamma(z)})^L U_z R_\Gamma$$

There is an Lieb-Schultz-Mattis (LSM) theorem when $\mathrm{e}^{\mathrm{i}\phi_\Gamma(z)}$ is non-trivial for a unitary R_Γ

[⋯; Matsui 2008; Chen, Gu, Wen 2010; Yao, Oshikawa 2020; Ogata, Tasaki 2021; Seifnashri 2023; Kapustin, Sopenko 2024]

- The LSM theorem forbids SPT phases
- The ground state always has long-range entanglement

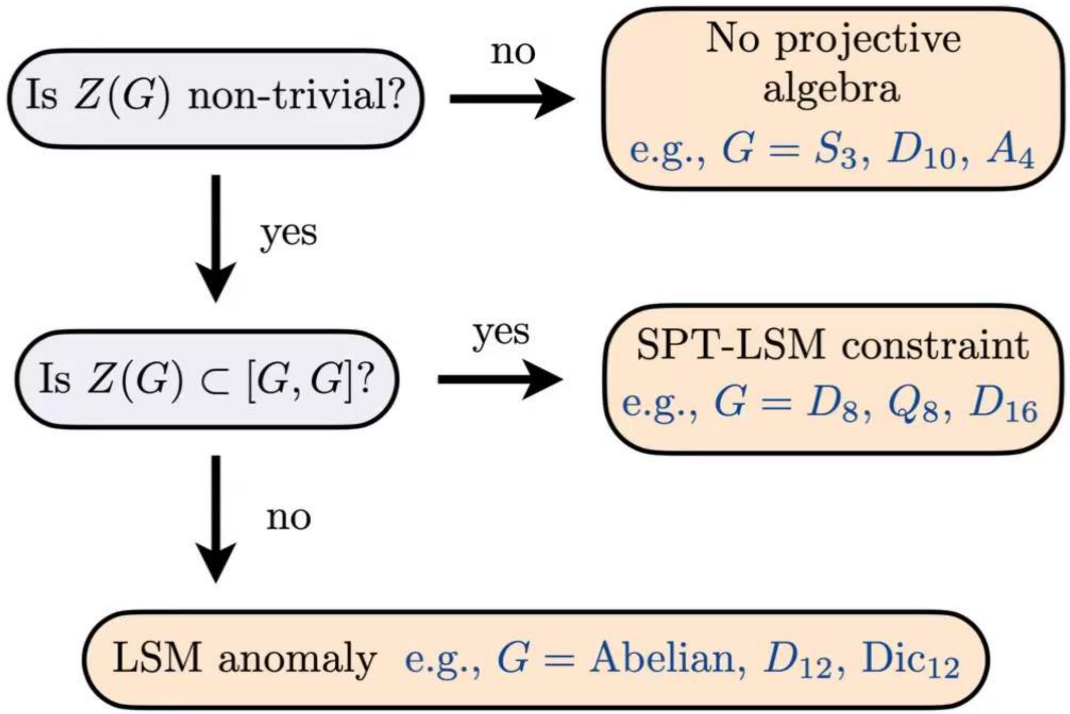
When there is no LSM theorem, the projective algebra gives rise to an SPT-LSM theorem

- Any SPT state must have non-zero entanglement

[Lu 2017; Yang, Jiang, Vishwanath, Ran 2017; Lu, Ran, Oshikawa 2017; ⋯]

(SPT)-LSM theorems

Whether there is an (SPT)-LSM theorem depends on G:



Non-invertible weak SPT

If there is an SPT phase, $R_\Gamma U_z = (\mathrm{e}^{\mathrm{i}\phi_\Gamma(z)})^L\, U_z R_\Gamma$ forces its ground state to satisfy $R_\Gamma |\mathrm{GS}\rangle = 0$ for nontrivial $(\mathrm{e}^{\mathrm{i}\phi_\Gamma(z)})^L$

Two possibilities:

1. An SPT state satisfies $R_\Gamma |\mathrm{GS}\rangle = 0$ for all system sizes L
2. For $L = L^*$ where all $(\mathrm{e}^{\mathrm{i}\phi_\Gamma(z)})^{L^*} = 1$, an SPT state satisfies $R_\Gamma |\mathrm{GS}\rangle = \lambda_\Gamma |\mathrm{GS}\rangle$, but $R_\Gamma |\mathrm{GS}\rangle = 0$ for $L \neq L^*$

The first is incompatible with $1+1$D TQFT, where $\langle R_\Gamma \rangle = d_\Gamma$

[Chang, Lin, Shao, Wang, Yin 2018]

➤ Reasonable to assume that this SPT state at some $L = L^*$ is described by a TQFT in the IR

Non-invertible weak SPT

If there is an SPT phase, $R_\Gamma U_z = (\mathrm{e}^{\mathrm{i}\phi_\Gamma(z)})^L U_z R_\Gamma$ forces its

ground state to satisfy $R_\Gamma|\mathrm{GS}\rangle = 0$ for nontrivial $(\mathrm{e}^{\mathrm{i}\phi_\Gamma(z)})^L$

Tw

1.

2.

The $= d_\Gamma$

[Chang, Lin, Shao, Wang, Yin 2018]

At $L = L^*$, SPTs satisfy $R_\Gamma|\mathrm{GS}\rangle = \lambda_\Gamma|\mathrm{GS}\rangle$

At $L = L^* + 1$, SPTs satisfy $R_\Gamma|\mathrm{GS}\rangle = 0$

- All SPT states have translation defects dressed by non-trivial $\mathsf{Rep}(G)$ symmetry charge
- $\nexists$ a trivial SPT $\Longrightarrow$ SPT-LSM theorem

- Reasonable to assume that this SPT state at some $L = L^*$ is described by a TQFT in the IR

SPT-LSM theorem

To prove this SPT-LSM theorem, we

1. Use that the $Z(G)$ symmetry is on-site:

$$U_z = \prod_j U_j^{(z)} \quad \text{which satisfies} \quad R_\Gamma U_j^{(z)} = \mathrm{e}^{\mathrm{i}\phi_\Gamma(z)} U_j^{(z)} R_\Gamma$$

2. Assume that any unique gapped ground state $|\mathrm{GS}\rangle$ satisfies $R_\Gamma|\mathrm{GS}\rangle \neq 0$ for some system size $L = L^*$

We prove this assumption for product states in $\otimes_j \mathbb{C}^{|G|}$, *where*

$$R_\Gamma = \sum_{\{g_j\}} \mathrm{Tr}[\Gamma(g_1 \cdots g_L)]\,|g_1, \cdots, g_L\rangle\langle g_1, \cdots, g_L|$$

but it is true as long as there is an IR TQFT description

SPT-LSM theorem

If there is a unique gapped $|\mathrm{GS}\rangle$ that is a product state:

➤ $U_z|\mathrm{GS}\rangle = |\mathrm{GS}\rangle \implies U_j^{(z)}|\mathrm{GS}\rangle = |\mathrm{GS}\rangle$

Using the assumption, $R_\Gamma|\mathrm{GS}\rangle = \lambda_\Gamma|\mathrm{GS}\rangle$ at $L = L^*$:

1. $R_\Gamma U_j^{(z)}|\mathrm{GS}\rangle = R_\Gamma|\mathrm{GS}\rangle = \lambda_\Gamma|\mathrm{GS}\rangle$
2. $R_\Gamma U_j^{(z)}|\mathrm{GS}\rangle = \mathrm{e}^{\mathrm{i}\phi_\Gamma(z)}U_j^{(z)}R_\Gamma|\mathrm{GS}\rangle = \lambda_\Gamma \mathrm{e}^{\mathrm{i}\phi_\Gamma(z)}|\mathrm{GS}\rangle$

} *Contradiction*

$\implies$ Cannot be an SPT state that is a product state at $L = L^*$

$\implies$ By locality, there cannot be an SPT state that is a product state for any L

Outlook

We found a new class of entangled weak SPTs characterized by a projective $Z(G) \times \mathsf{Rep}(G)$ non-invertible symmetry

1. An exactly solvable model in a weak SPT phase characterized by a projective $\mathbb{Z}_2 \times \mathsf{Rep}(D_8)$ symmetry
2. General discussion on decorated domain wall pattern of these $Z(G) \times \mathsf{Rep}(G)$ weak SPTs $\Longrightarrow$ an SPT-LSM theorem

New quantum phases and models can be discovered using generalized symmetries as a guide!

SP, Lam, Aksoy arXiv:2409.18113