Title: On the holonomicity of skein modules

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Collection/Series: Mathematical Physics

Subject: Mathematical physics

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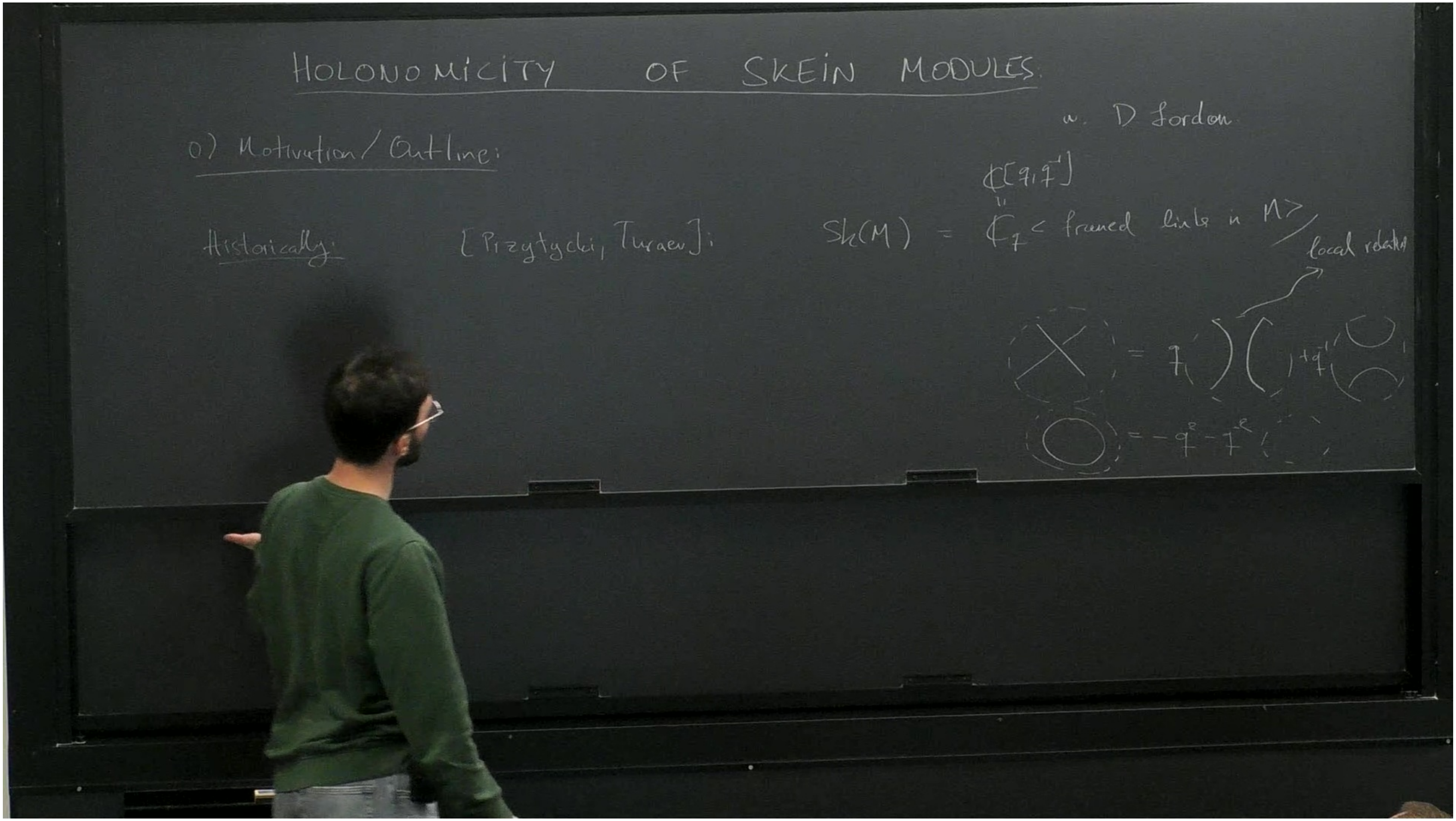
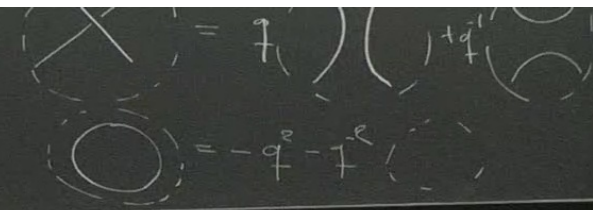
Abstract:

Skein theory forms a once-categorified 3d TQFT and assigns skein algebras to surfaces and skein modules to 3-manifolds. Motivated by physics, these modules are expected to satisfy a certain holonomicity property, generalizing Witten's finiteness conjecture of skein modules. In this talk, we will recall the basic notions of skein theory as a deformation quantization theory, and then state and discuss the generalized Witten's finiteness conjecture.


HOLONOMICITY OF SKEIN MODULES
w. D Jordan
0) Motivation/Outline:
Historically:
[Przytycki, Turaev]:
C[q, q^-1]
Sk(M) = C_q < framed links in M> / local relat
= q
+ q^-1
= -q^2 - q^-2



Extending in two ways

1) Algebraic Input: A ribbon tensor cat.

more colors + more complicated local relations

e.g. $Rep_q G$ ↖ alg red group

(SL_2 recover the above).

2) Lower dim: Skein Theory forms a 3d or. TQFT (once-categorified)

$\Sigma \longmapsto$ Skein Category / algebra

Thm (Witten's finiteness) [Gunningham, Jordan, Safronov]

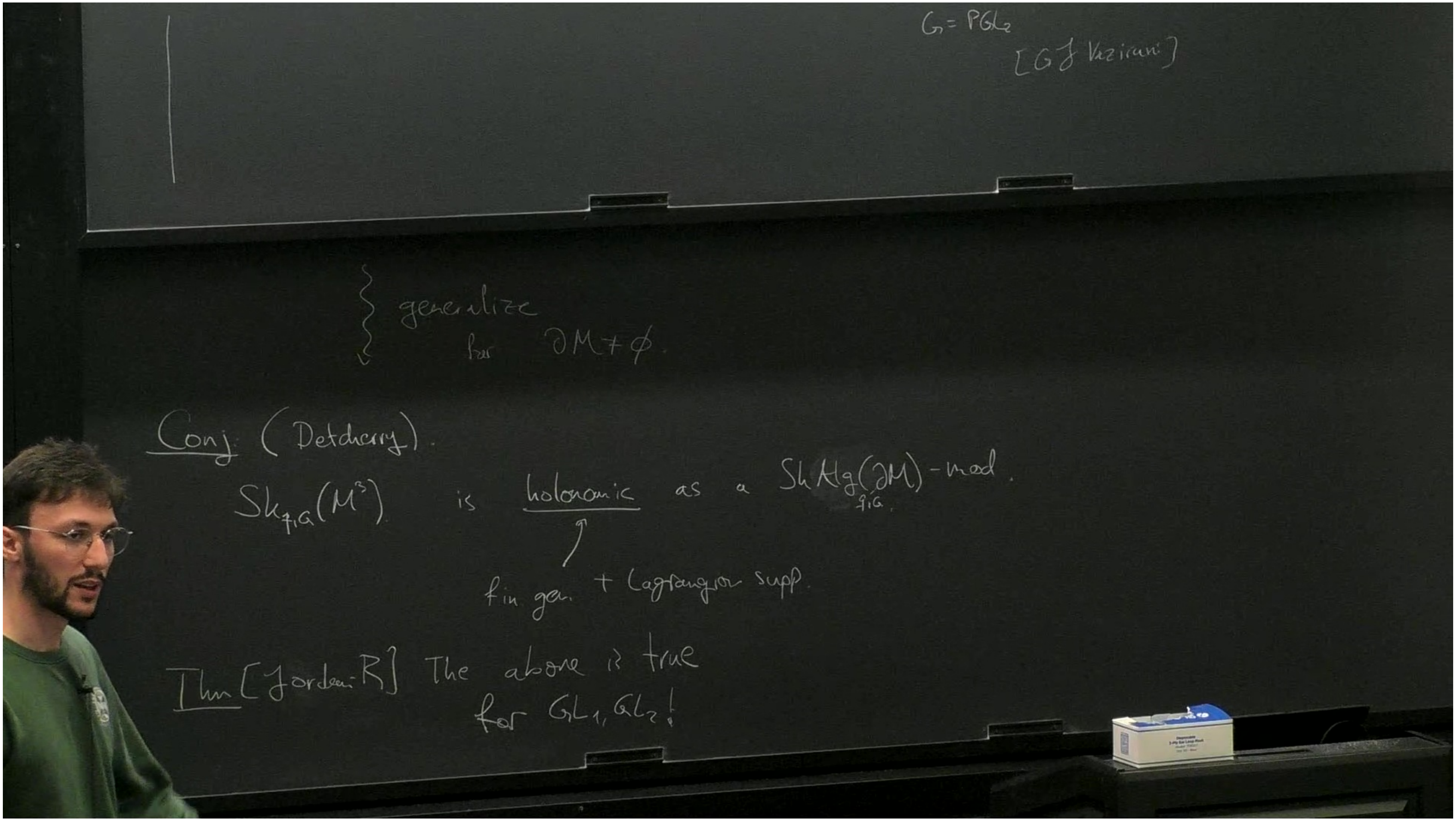
q - generic M^3: closed $Sk_{q,G}(M^3)$ is finite dimensional.

(not 1, not root of 1).

$G = SL_2$ [Carrega, Gilmer, Masbaum]
[Detch. Wolff]

Remark: Skein dim. computed for $\Sigma_g \times S^1$; $T^2 \times^\gamma S^1$ [Kinnear]

$G = PGL_2$
[GJ Vazirani]


G = PGL2
[GJ Vazirani]
generalize for ∂M ≠ ∅.
Conj (Detcherry).
Sk_{q,G}(M^3) is holonomic as a SkAlg_{q,G}(∂M)-mod.
fin gen. + Lagrangian supp.
Thm [Jordan-R] The above is true for GL1, GL2!

$G = PGL_2$

[G J Vazirani]

generalize for $\partial M \neq \emptyset$.

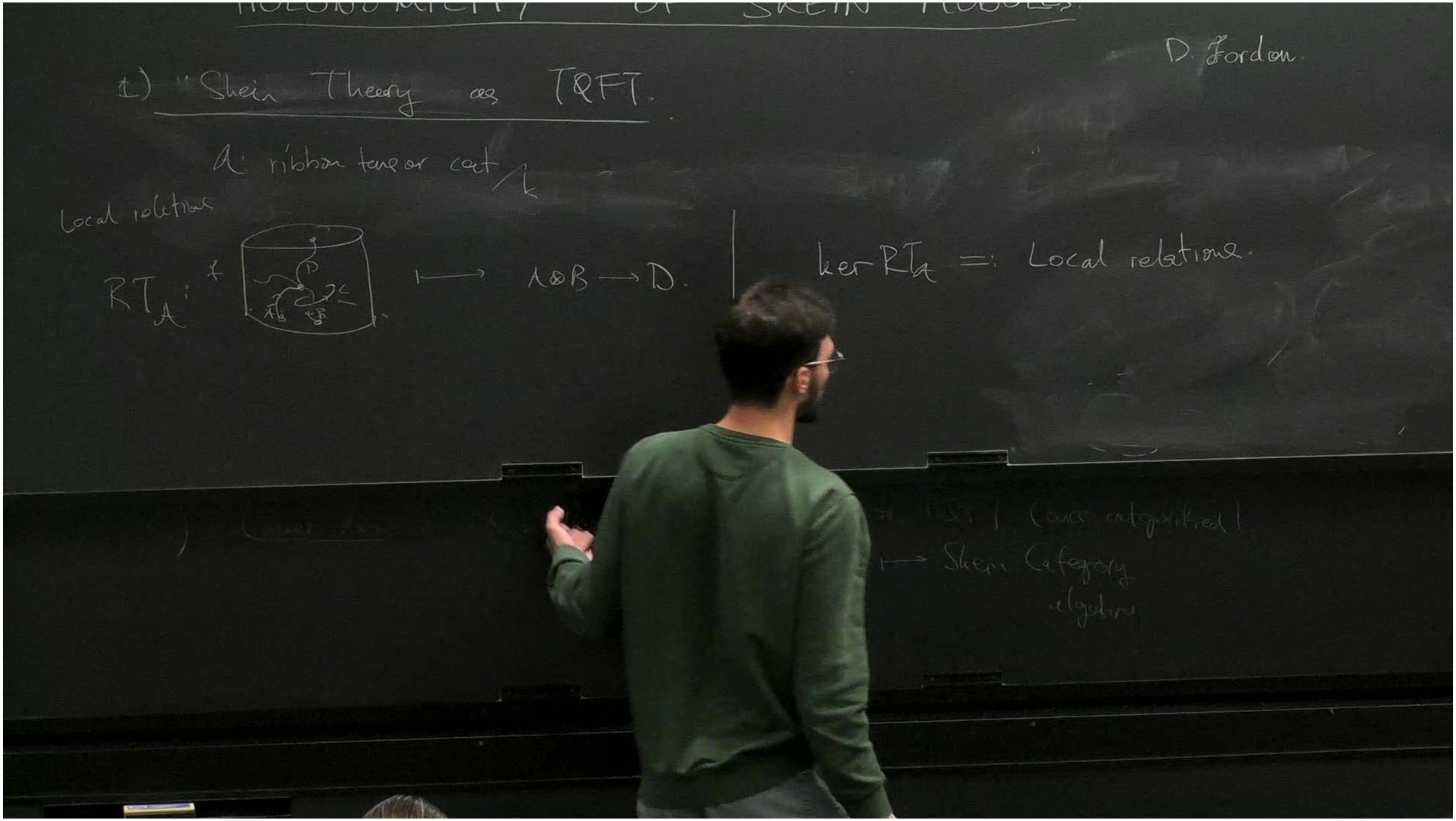
$\rightsquigarrow$ Include puncture Sk^{int}.

Conj (Detcherry).

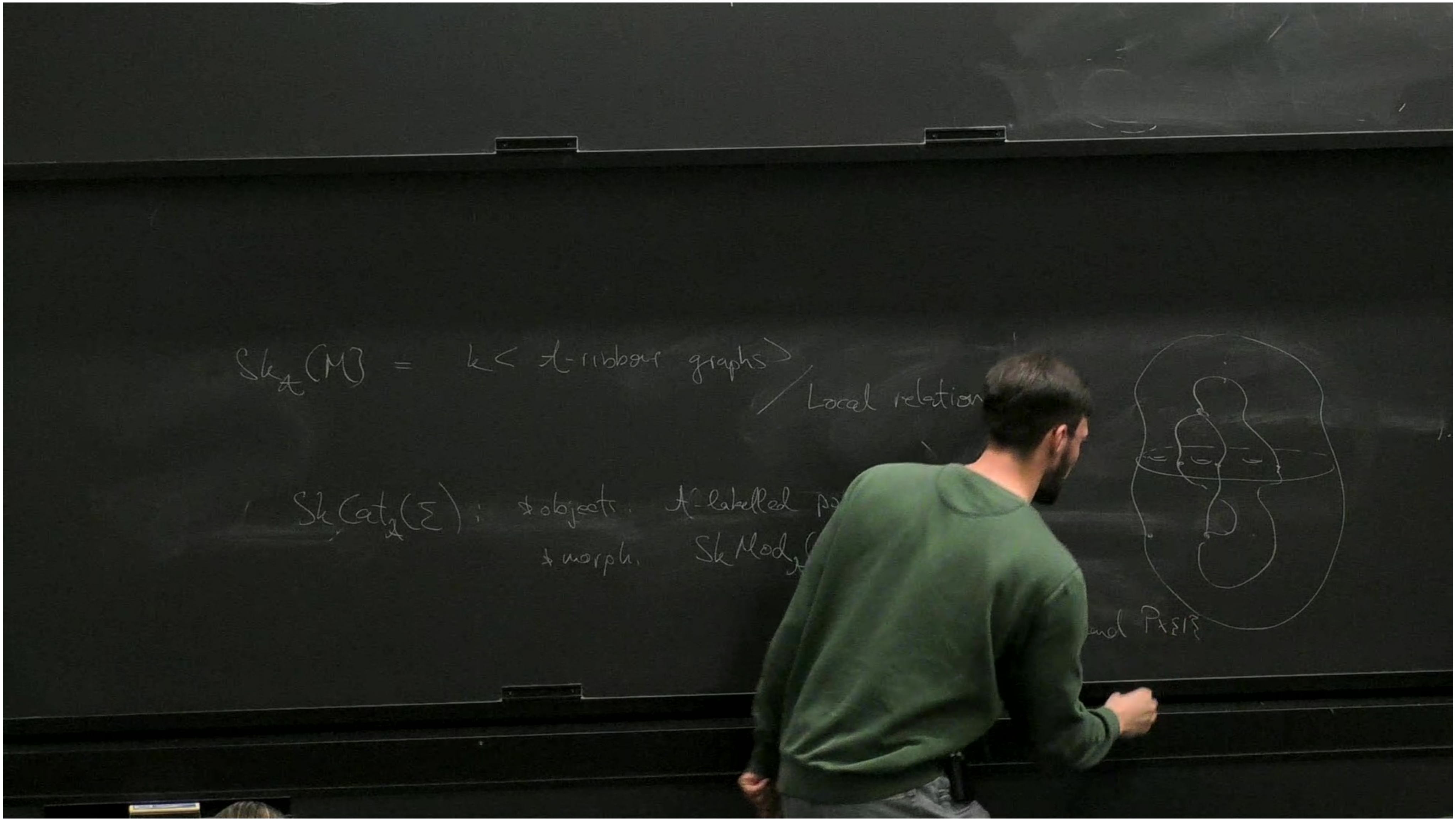
$Sk_{q,G}(M^3)$ is holonomic as a $SkAlg_{q,G}(\partial M)$-mod.

fin gen. + Lagrangian supp.

Thm [Jordan–R] The above is true for GL_1, GL_2!



D. Jordan
1) "Skein Theory as TQFT.
A: ribbon tensor cat
Local relations
ker RT_A =: Local relations.
Skein Category

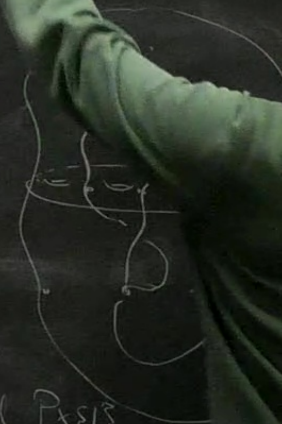

Sk_A(M) = k< A-ribbon graphs>
Local relation
SkCat_A(Σ): * objects. A-labelled p
* morph. SkMod_A
and P×{1}

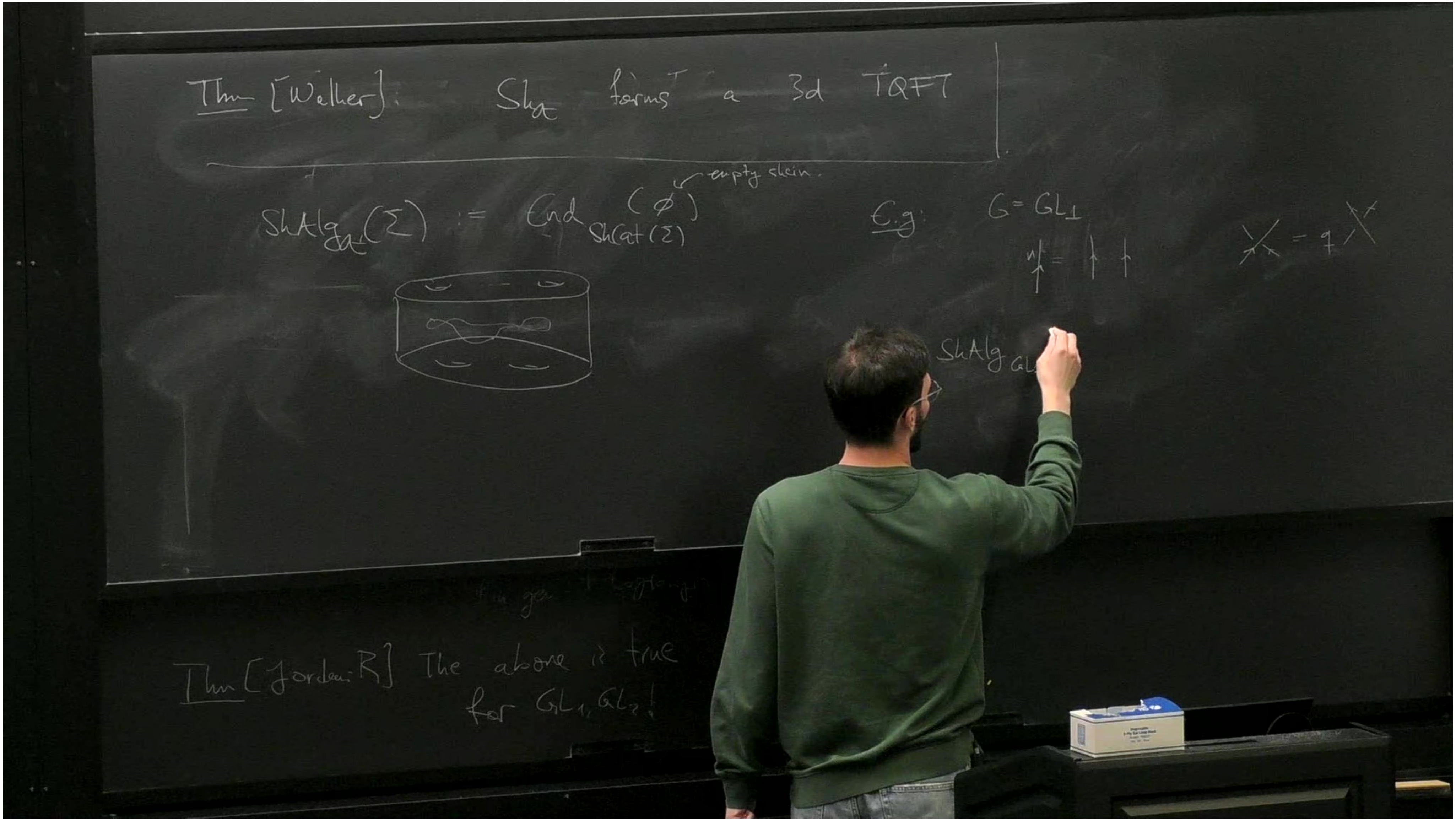
$Sk_{\mathcal{A}}(M) = k\langle \mathcal{A}\text{-ribbon graphs}\rangle \big/ \text{Local relations.}$

$SkCat_{\mathcal{A}}(\Sigma)$: * objects: $\mathcal{A}$-labelled points on Σ.

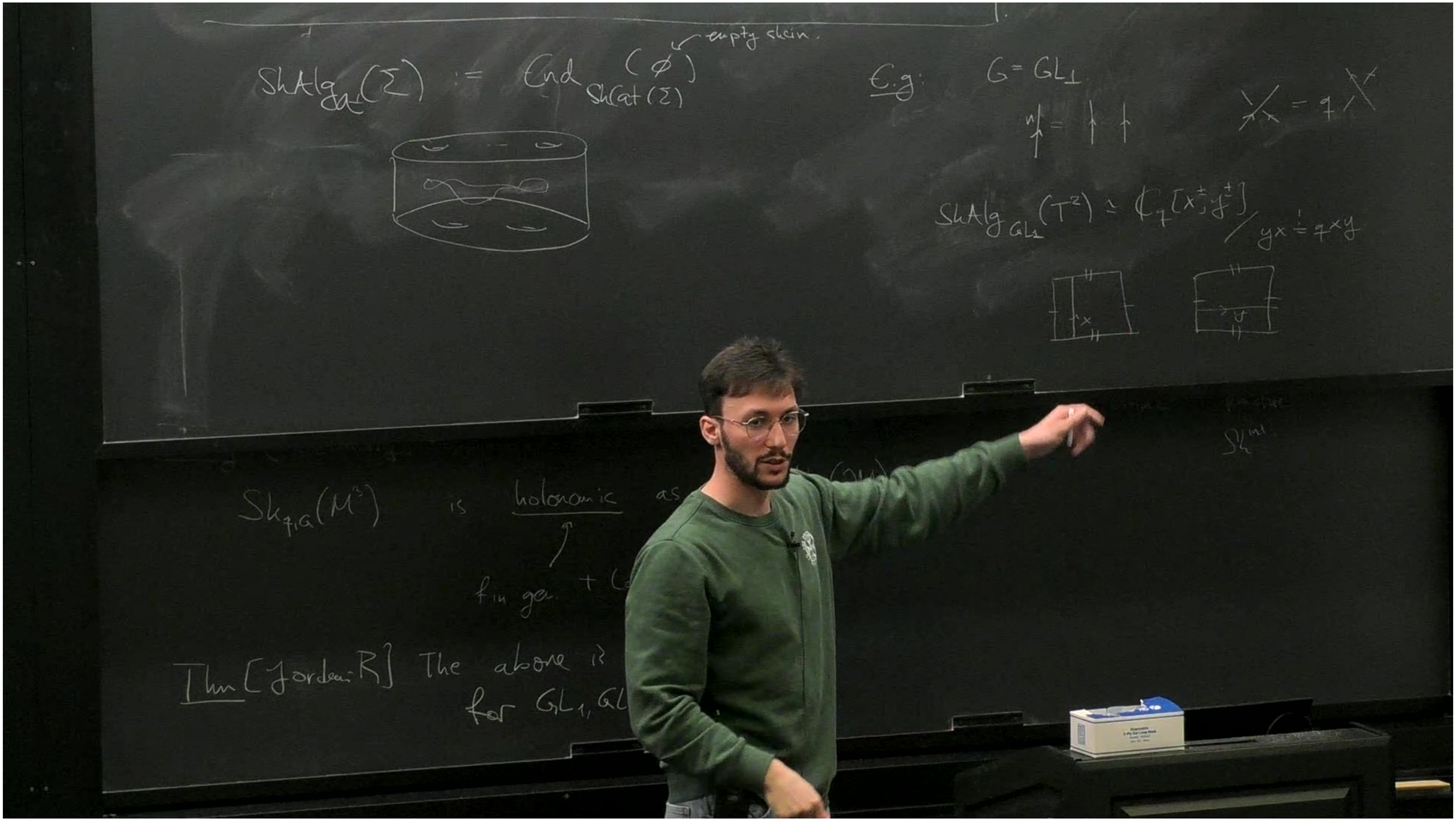
* morph. $SkMod_{\mathcal{A}}(\Sigma \times [0,1], P, P')$

skeins compatible w. $P \times \{0\}$ and $P \times \{1\}$





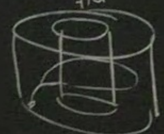
Thm [Walker]: Sk_G forms a 3d TQFT
empty skein.
SkAlg_G(Σ) := End_SkCat(Σ)(∅)
E.g: G = GL1
= q
SkAlg
Thm [Jordan-R] The above is true
for GL1, GL2!


empty skein.
ShAlg_G(Σ) := End_ShCat(Σ)(∅)
E.g: G = GL_1
ShAlg_GL1(T^2) ≃ C_q[x^±, y^±] / yx = qxy
Sk_q,G(M^3) is holonomic as
fin gen.
Thm [Jordan-R] The above is
for GL_1, GL

$\Sigma \rightsquigarrow \Sigma^{\flat}$: $\mathrm{SkAlg}^{\mathrm{int}}(\Sigma^{\flat}) :=$  $\mathcal{A}^{op} \to \mathrm{Mod}_k$

Related to Stated skeins, Alekseev-Schomerus-Grosse

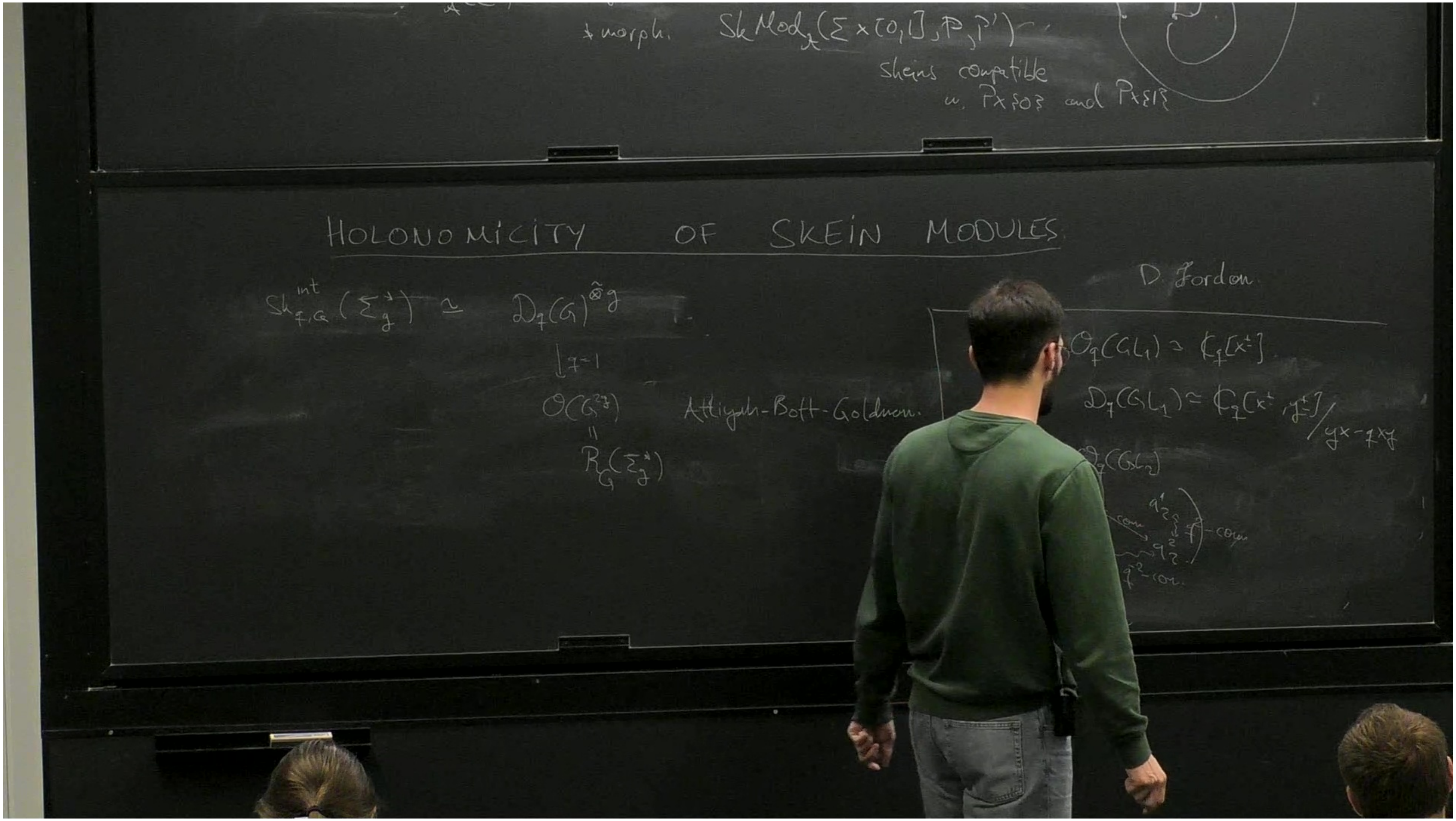
Example: $\mathrm{Sk}^{\mathrm{int}}_{q,G}(\mathrm{Ann}) \simeq \mathcal{O}_q(G)$ quantum on G

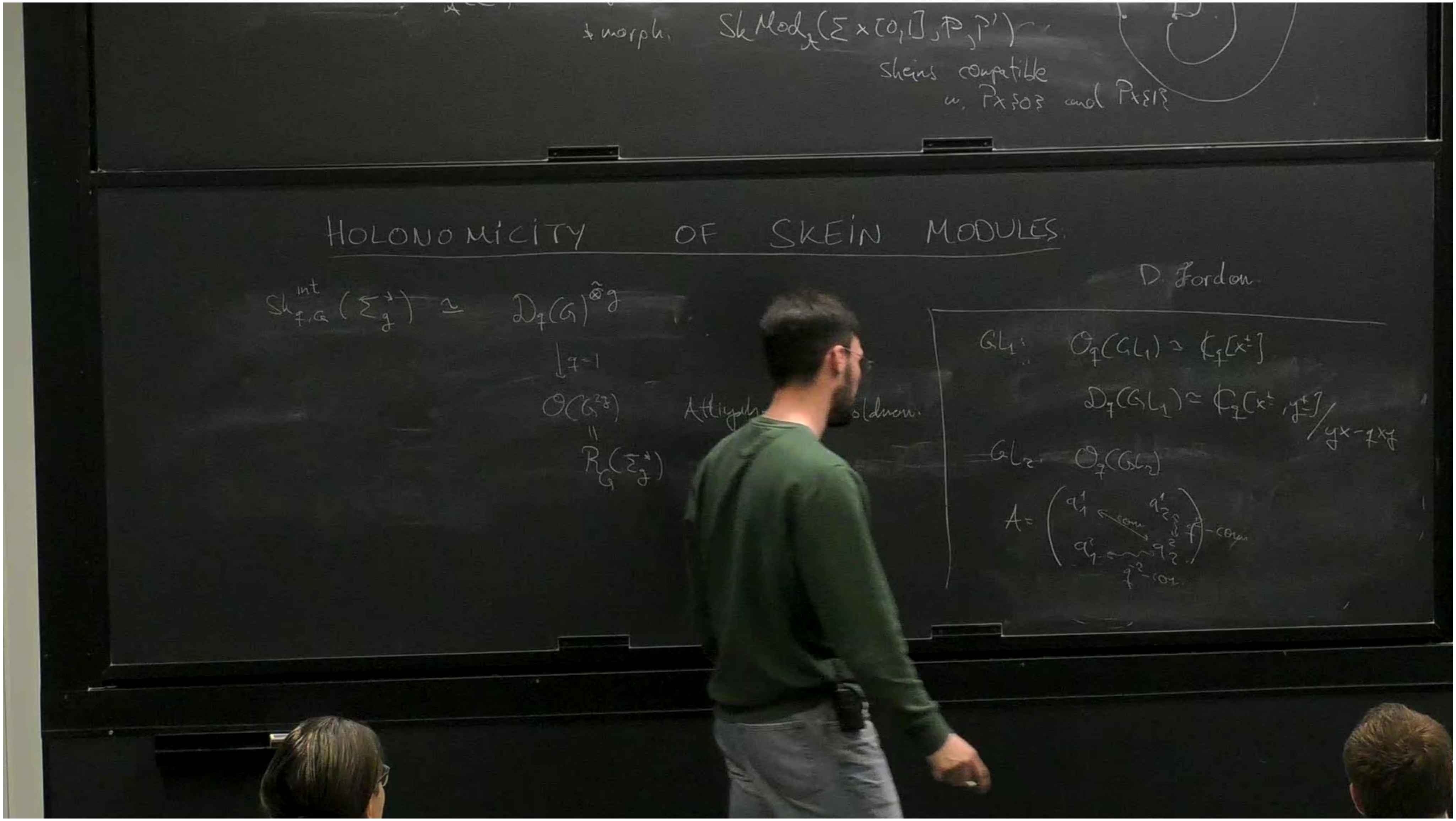


$$\mathcal{O}_q(G) \downarrow \mathcal{O}(G).$$

$\mathrm{Sk}^{\mathrm{int}}(T^*) \simeq \mathcal{D}_q(G)$ q-different. op.

$$\mathcal{D}_q(G) \xrightarrow{q=1} \mathcal{O}(G \times G)$$


morph. $SkMod_{\mathcal{A}}(\Sigma \times [0,1], P, P')$
Skeins compatible
w. $P \times \{0\}$ and $P \times \{1\}$
HOLONOMICITY OF SKEIN MODULES
D. Jordan
$Sk^{int}_{q,G}(\Sigma_g^*) \simeq \mathcal{D}_q(G)^{\hat{\otimes} g}$
$q=1$
$\mathcal{O}(G^{2g})$
Atiyah-Bott-Goldman
$R_G(\Sigma_g^*)$
$\mathcal{O}_q(GL_1) \simeq \mathbb{C}_q[x^{\pm}]$
$\mathcal{D}_q(GL_1) \simeq \mathbb{C}_q[x^{\pm}, y^{\pm}] / yx - qxy$



+ morph. SkMod(Σ × [0,1], P, P')
Skeins compatible
w. P×{0} and P×{1}
HOLONOMICITY OF SKEIN MODULES
D. Jordan
GL1: O_q(GL1) ≃ C_q[x±]
D_q(GL1) ≃ C_q[x±, y±] / yx − qxy
GL2: O_q(GL2)
A =

+ morph. $Sk\,Mod_A(\Sigma \times [0,1], P, P')$

Skeins compatible w. $P \times \{0\}$ and $P \times \{1\}$

HOLONOMICITY OF SKEIN MODULES

D. Jordan

$$Sk^{int}_{q,G}(\Sigma_g^{\circ}) \simeq D_q(G)^{\hat{\otimes} g}$$

$$\Big\downarrow {\scriptstyle q=1}$$

$$\mathcal{O}(G^{2g}) \quad \text{Atiyah-Bott-Goldman.}$$

$$\parallel$$

$$R_G(\Sigma_g^{\circ})$$

GL_1: $\mathcal{O}_q(GL_1) \simeq \mathbb{C}_q[x^{\pm}]$

$D_q(GL_1) \simeq \mathbb{C}_q[x^{\pm}, y^{\pm}] / yx - qxy$

GL_2: $\mathcal{O}_q(GL_2) \simeq \mathcal{O}_q(Mat_2)[det_q^{-1}]$

$$A = \begin{pmatrix} a^1_1 & a^1_2 \\ a^2_1 & a^2_2 \end{pmatrix}$$

com

q-com

q^{-2}-com

$A = \begin{pmatrix} q_1 \xleftrightarrow{\text{com}} q_2 & q^2 \\ q_1^2 \rightsquigarrow q_2^2 & \end{pmatrix}$ — com

q^{-2} — com

§) HOLONOMICITY

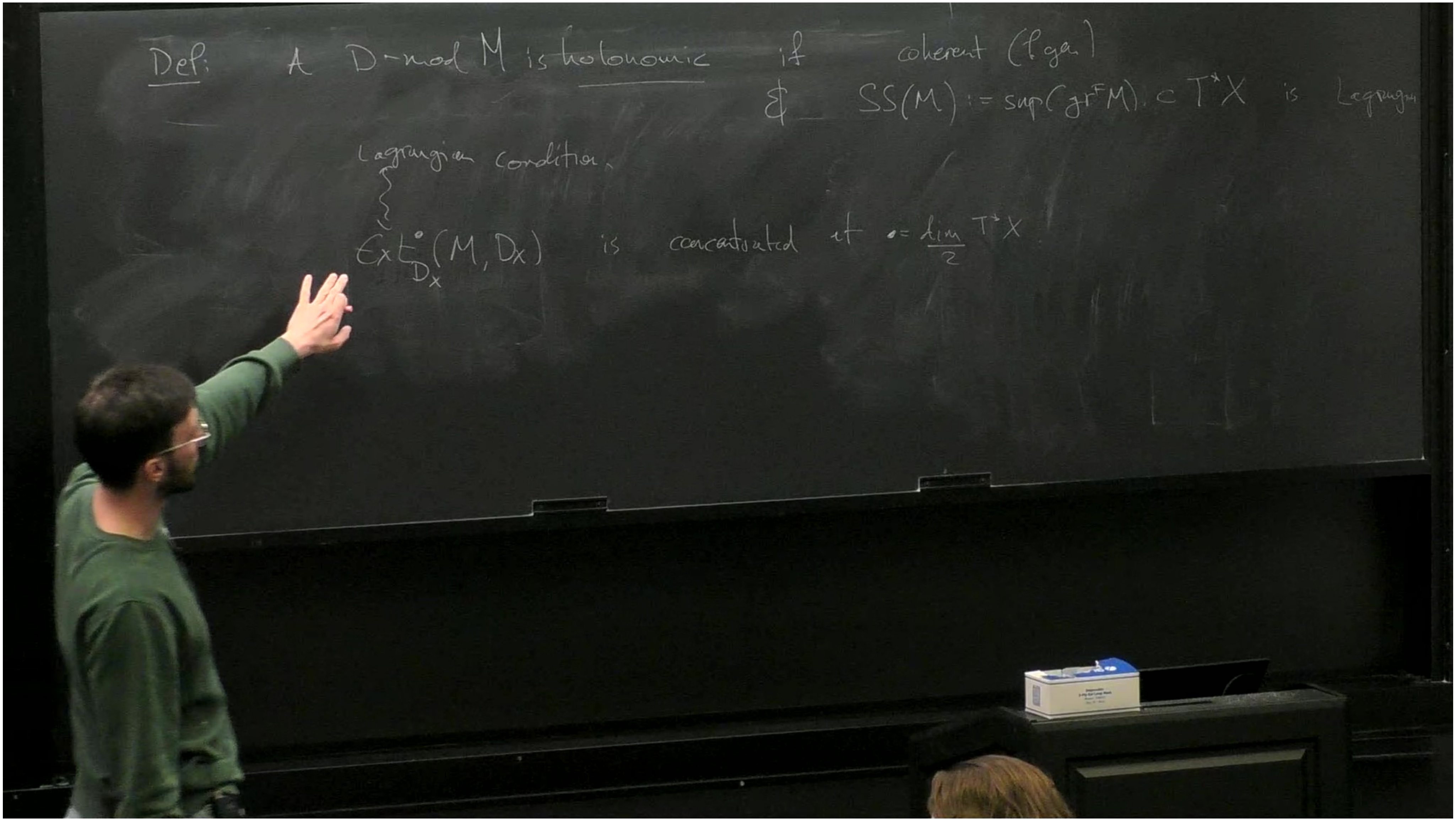
DETOUR ON D-MODS

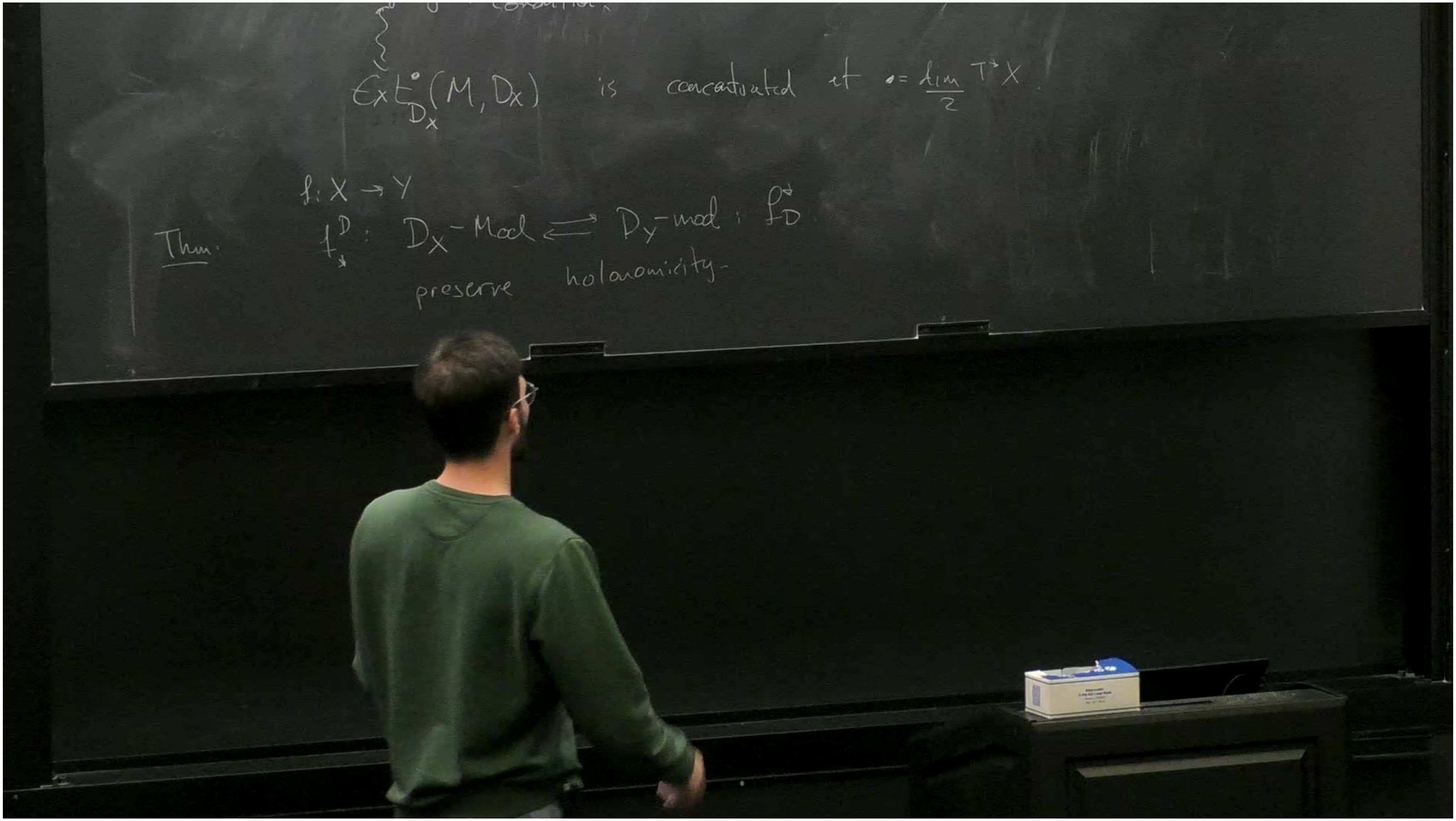
X var$/\mathbb{C}$ $\rightsquigarrow$ D_X (sheaf) alg of diff operators,

quantizes T^*X

$\mathcal{O}_{T^*X}$

as follows: $F_i D_X = \{$ op's of ∂-order $\leq i \}$ provides a filtration $\Rightarrow gr^F D_X \simeq \mathcal{O}_{T^*X}$


Def: A D-mod M is holonomic if coherent (f.gen)
& SS(M) := supp(gr^F M) ⊂ T*X is
Lagrangian condition.
Ext°_{D_X}(M, D_X) is concentrated at ° = dim T*X / 2


Ext$^\bullet_{D_X}$(M, D_X) is concentrated at $\bullet = \frac{\dim}{2} T^*X$.
f: X → Y
Thm. f_*^D : D_X-Mod ⇄ D_Y-mod : $f_D^!$
preserve holonomicity.

$$A = \begin{pmatrix} q_1^1 & q_2^1 \\ q_1^2 & q_2^2 \end{pmatrix}$$

com

q^2-com.

q^{-2}-com.

(Kashiwara–Schapira) for DQ-mods.

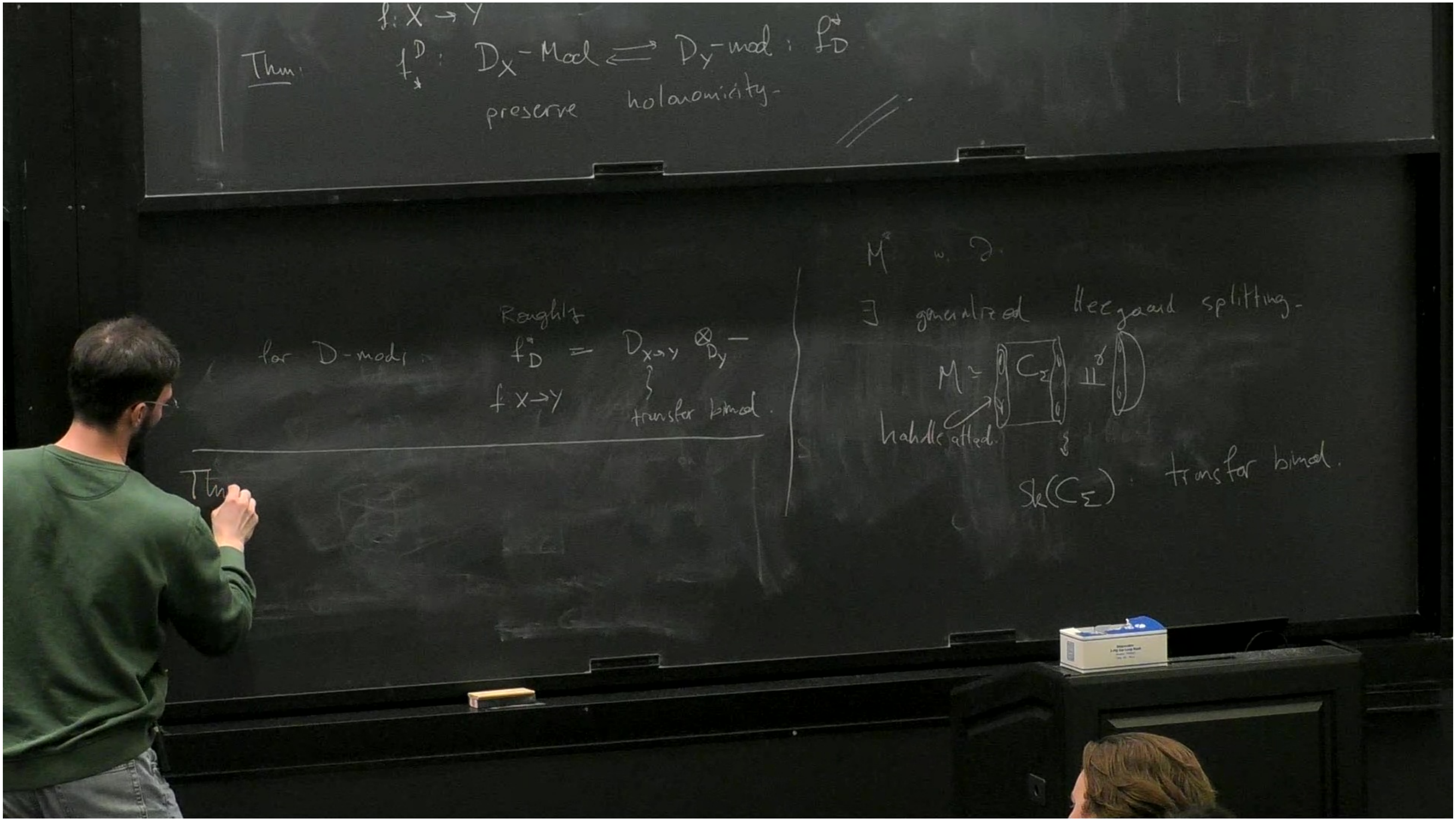
Def. A $Sh^{(int)}_{\hbar,G}(M)$ is holonomic $Sh^{(int)}_{\hbar,G}(\partial M)$ if fin gen

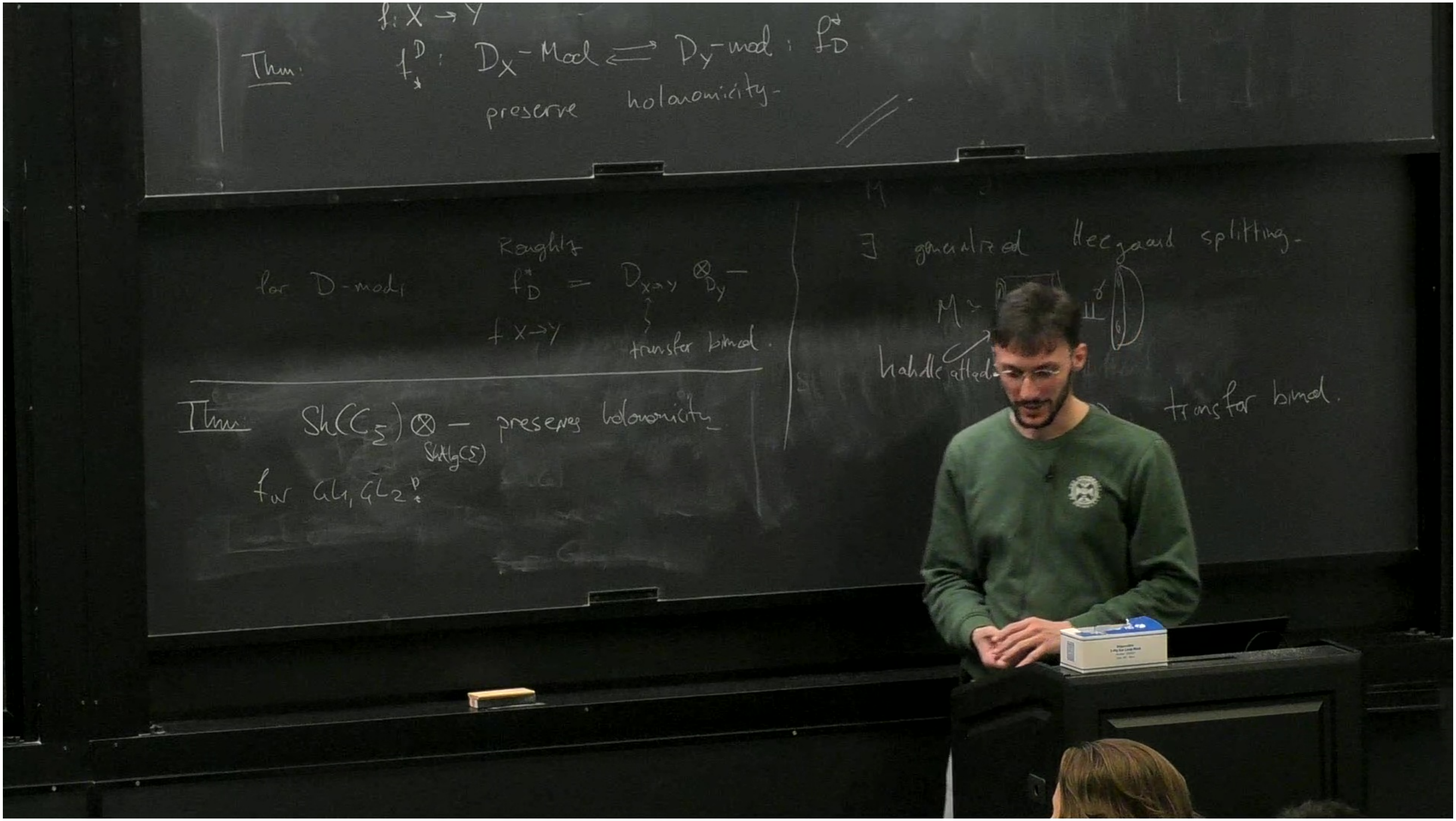
$+ \; SS(Sh^{int}_{\hbar,G}(M)) \subset R_G(\partial M)$

is Lagrangian

$q = \exp(\hbar)$

$\mathbb{C}[q, q^{-1}] \rightsquigarrow \mathbb{C}[[\hbar]]$


f: X → Y
Thm: f_*^D : D_X-Mod ⇄ D_Y-mod : f_D^*
preserve holonomicity.
for D-mod: Roughly f_D^* = D_{X→Y} ⊗_{D_Y} —
f X→Y
transfer bimod.
Thm
M^3 w. ∂.
∃ generalized Heegaard splitting.
M ≅ C_Σ ∐
handle attd.
Sk(C_Σ): transfer bimod.


$f: X \to Y$
Thm: $f_*^D : D_X\text{-Mod} \rightleftarrows D_Y\text{-mod} : f_D^*$
preserve holonomicity.
Roughly
for D-mod $f_D^* = D_{X \to Y} \otimes_{D_Y} -$
$f: X \to Y$
transfer bimod.
Thm $Sh(C_\Sigma) \otimes_{ShAlg(\Sigma)} -$ preserves holonomicity
for GL_1, GL_2 !
∃ generalized Heegaard splitting.
handle attached
trans for bimod.