Title: Physics-Informed Renormalization Group Flows

Speakers: Friederike Ihssen

Collection/Series: Machine Learning Initiative

Date: November 22, 2024 - 2:30 PM

URL: https://pirsa.org/24110078

Abstract:

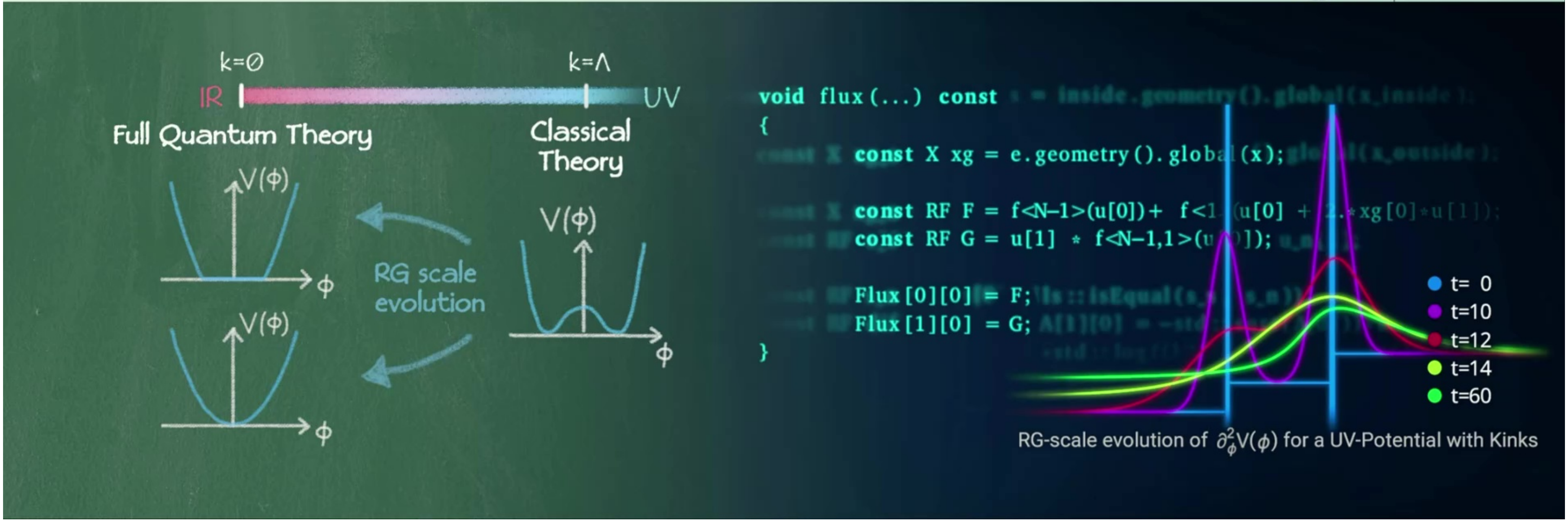
The physics of strongly correlated systems offers some of the most intriguing physics challenges such as competing orders or the emergence of dynamical composite degrees of freedom. Often, the resolution of these physics challenges is computationally hard, but can be simplified by a formulation in terms of the appropriate dynamical degrees of freedom.

Physics-informed RG flows

arXiv:2409.13679

Friederike Ihssen
Heidelberg University/ETH Zürich

STRUCTURES
CLUSTER OF
EXCELLENCE



```
void flux (...) const
{
    const X xg = e.geometry().global(x);

    const RF F = f<N-1>(u[0])+ f<1 (u[0] + 2.*xg[0]*u[1]);
    const RF G = u[1] * f<N-1,1>(u ]);

    Flux[0][0] = F;
    Flux[1][0] = G;
}
```

RG-scale evolution of $\partial_{\phi}^{2}V(\phi)$ for a UV-Potential with Kinks

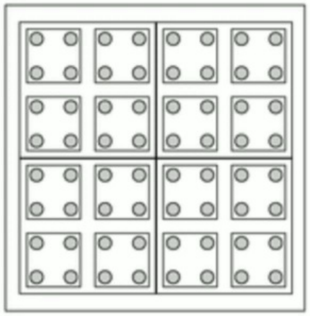
RG (real-space/momentum space)

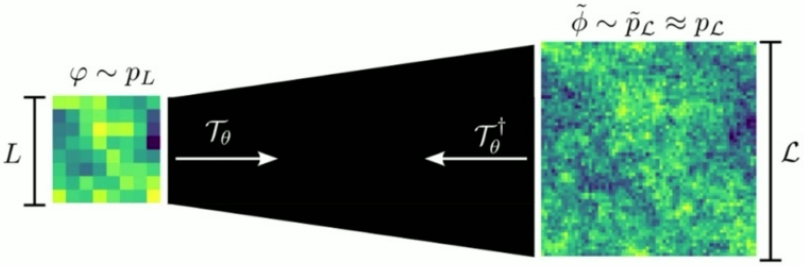
- Transition from **microscopic** to **macroscopic** behaviour
- **Emergence** of stable configurations
- Fixed-points and **phase structures**

ML task:

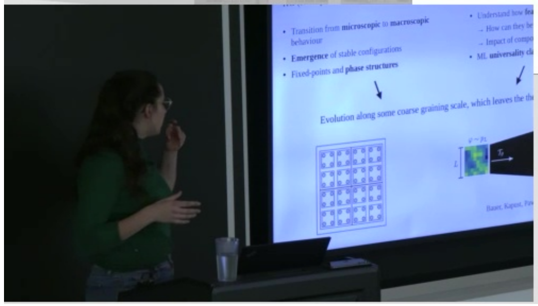
- Understand how **features emerge** in the NN
  - → How can they be captured efficiently?
  - → Impact of components within the architecture?
- ML **universality classes**?

Evolution along some coarse graining scale, which leaves the theory unchanged



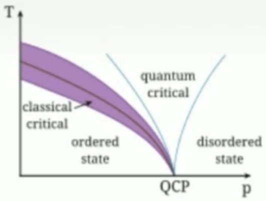


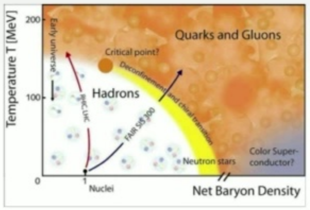
Bauer, Kapust, Pawlowski, Temmen in Prep.

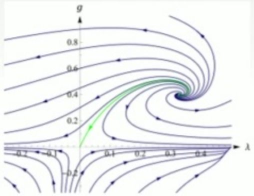


Contents: Generalised field transformations

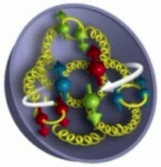
- What is the RG and why do we use it?

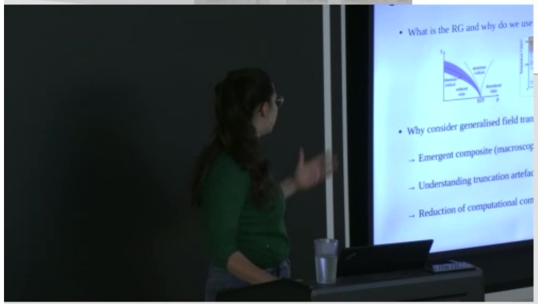


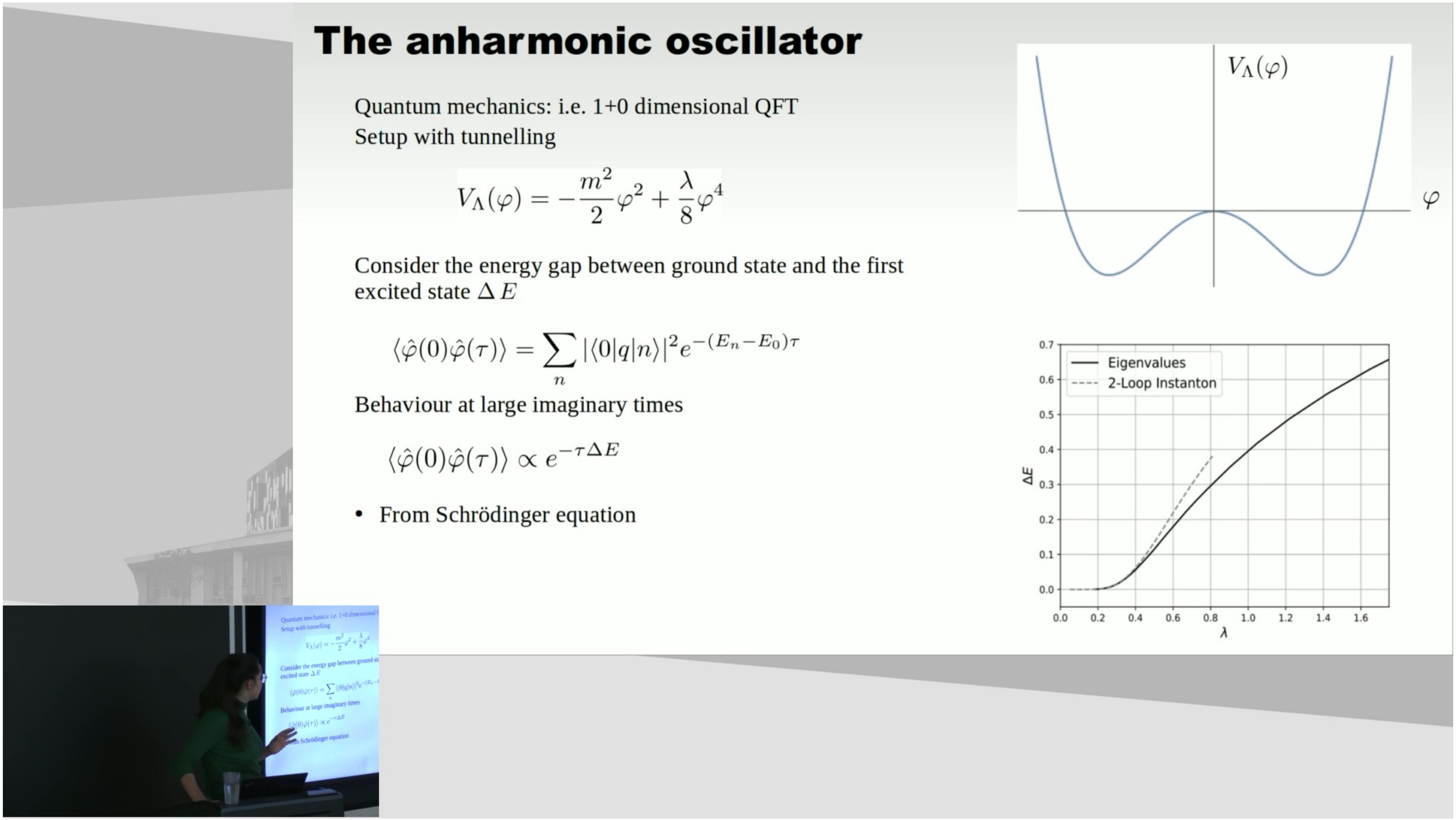




- Why consider generalised field transformations in the RG?
  - → Emergent composite (macroscopic) operators
  - → Understanding truncation artefacts (architectures)
  - → Reduction of computational complexity







The anharmonic oscillator

Quantum mechanics: i.e. 1+0 dimensional QFT

Setup with tunnelling

$$V_\Lambda(\varphi) = -\frac{m^2}{2}\varphi^2 + \frac{\lambda}{8}\varphi^4$$

Consider the energy gap between ground state and the first excited state ΔE

$$\langle\hat{\varphi}(0)\hat{\varphi}(\tau)\rangle = \sum_n |\langle 0|q|n\rangle|^2 e^{-(E_n - E_0)\tau}$$

Behaviour at large imaginary times

$$\langle\hat{\varphi}(0)\hat{\varphi}(\tau)\rangle \propto e^{-\tau\Delta E}$$

- From Schrödinger equation

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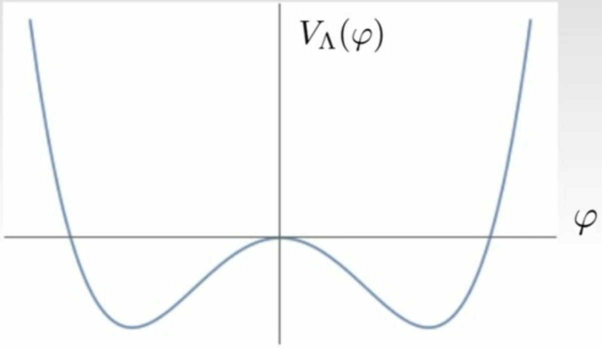

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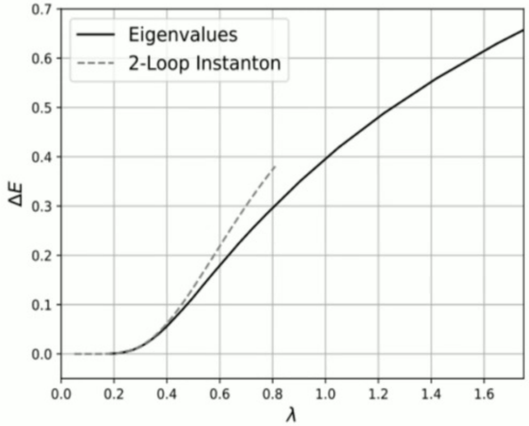
- From perturbation theory for $m^2/\lambda \ll 1$



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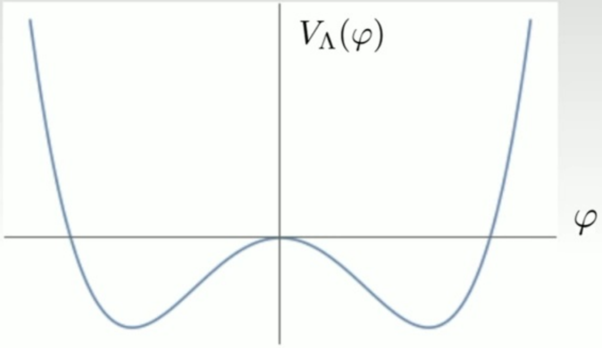

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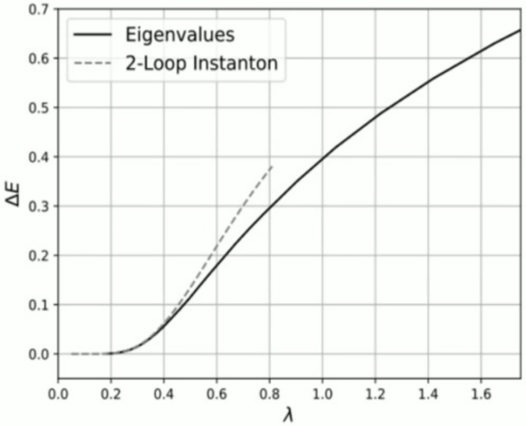
Behaviour at large imaginary times

$$\langle \hat{\varphi}(0)\hat{\varphi}(\tau)\rangle \propto e^{-\tau\Delta E}$$

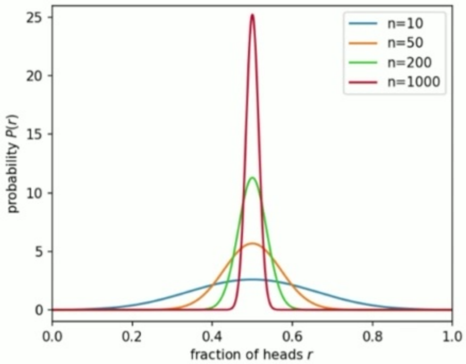
- From Schrödinger equation
- From perturbation theory for $m^2/\lambda \ll 1$
- Expansion about Instanton solution (non-perturbative)

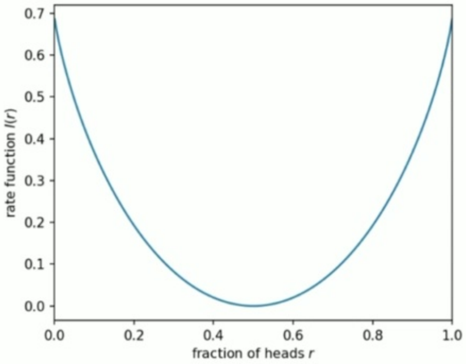


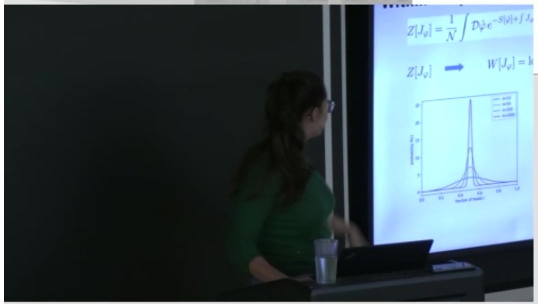
Within the path-integral formalism

$$Z[J_\varphi] = \frac{1}{\mathcal{N}} \int \mathcal{D}\hat{\varphi}\, e^{-S[\hat{\varphi}] + \int J_\varphi \hat{\varphi}} \quad \Rightarrow \quad \langle \hat{\varphi}(0)\hat{\varphi}(\tau)\rangle = \frac{\delta^2}{\delta J(0)\delta J(\tau)} Z[J_\varphi]$$

$$Z[J_\varphi] \quad \Rightarrow \quad W[J_\varphi] = \log(Z[J_\varphi]) \quad \Rightarrow \quad \Gamma[\varphi] = \sup_J \int_x J\varphi - W[J]$$







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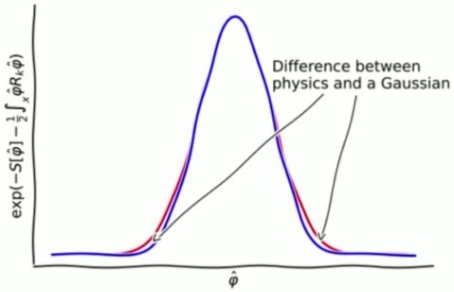

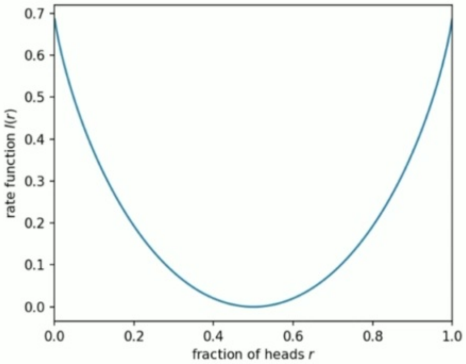

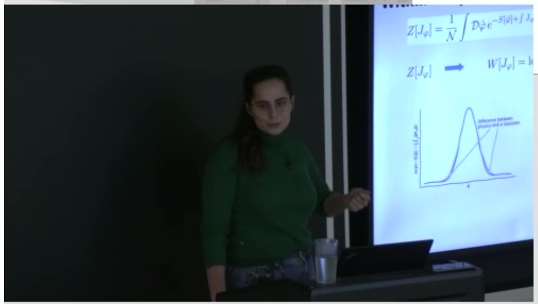

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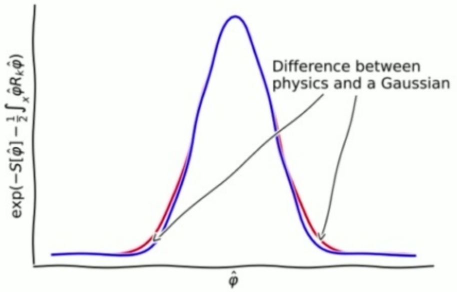
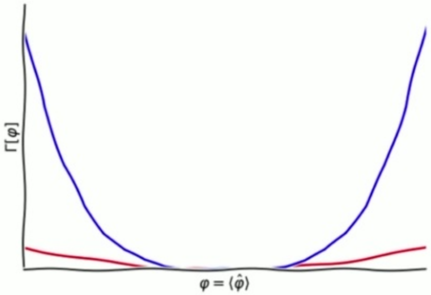
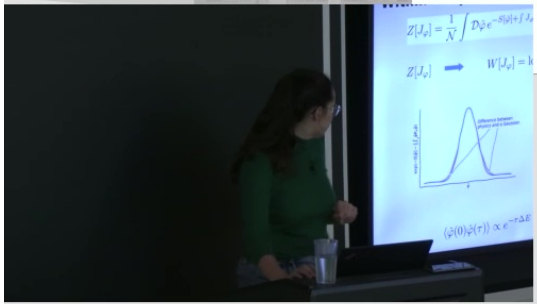




$$\langle \hat{\varphi}(0)\hat{\varphi}(\tau)\rangle \propto e^{-\tau \Delta E} \quad \longrightarrow \quad \Delta E = \sqrt{\frac{\Gamma^{(2)}[\varphi_{\mathrm{EoM}}]|_{p=0}}{\tau}}$$



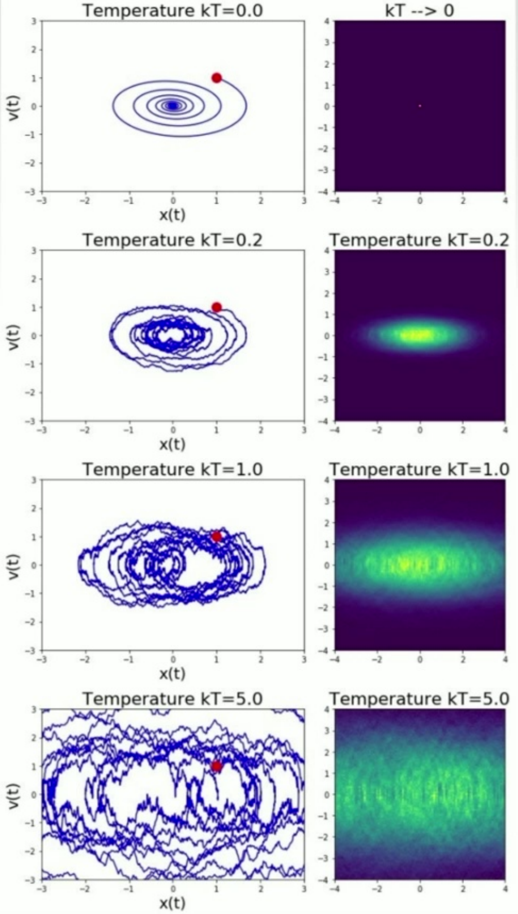
The Renormalisation Group

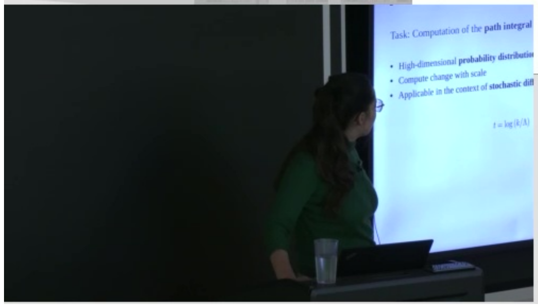
Task: Computation of the **path integral**

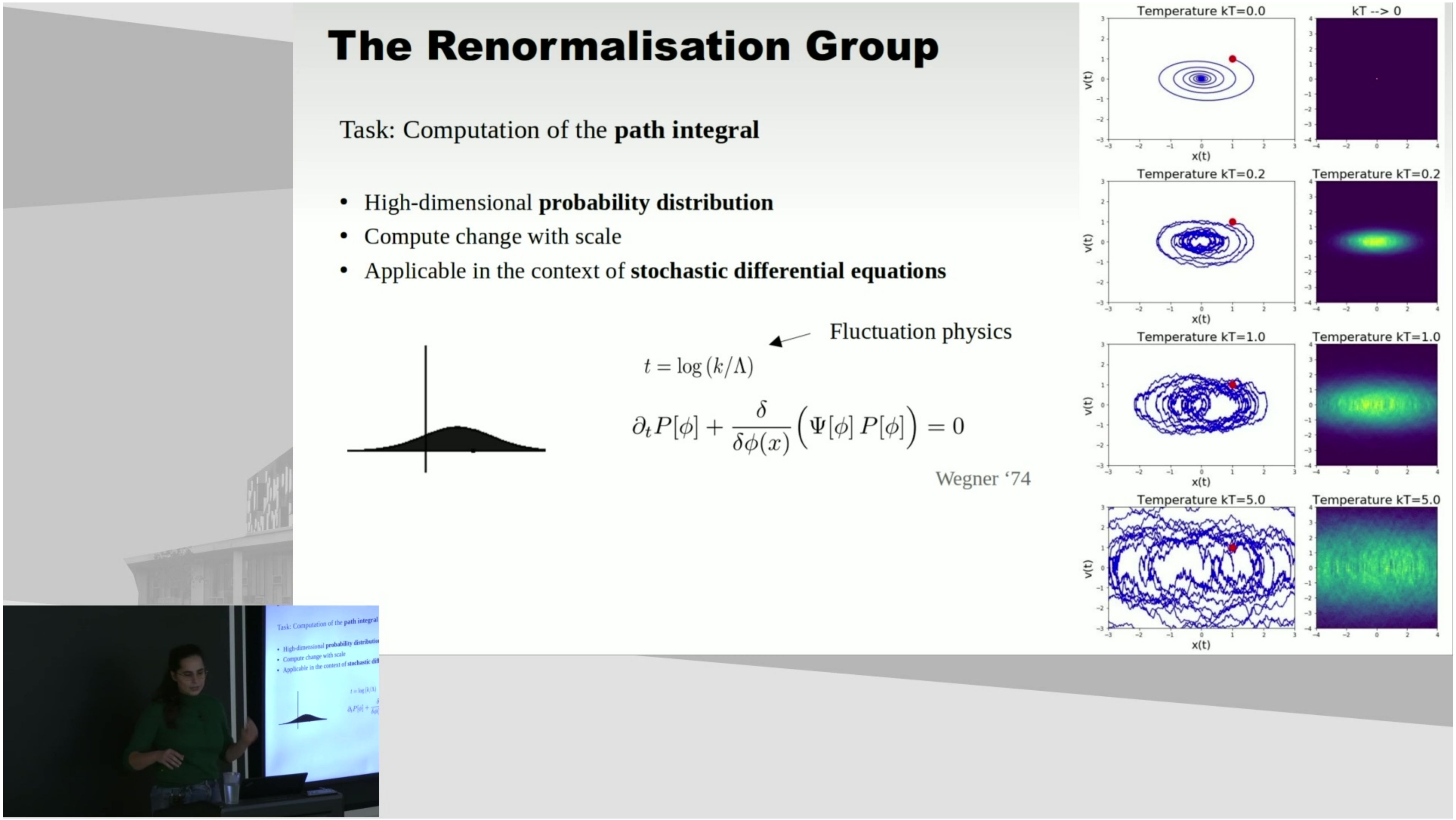
- High-dimensional **probability distribution**
- Compute change with scale
- Applicable in the context of **stochastic differential equations**

Fluctuation physics

$$t = \log\left(k/\Lambda\right)$$







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Wegner '74

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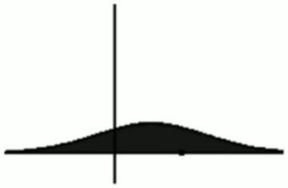

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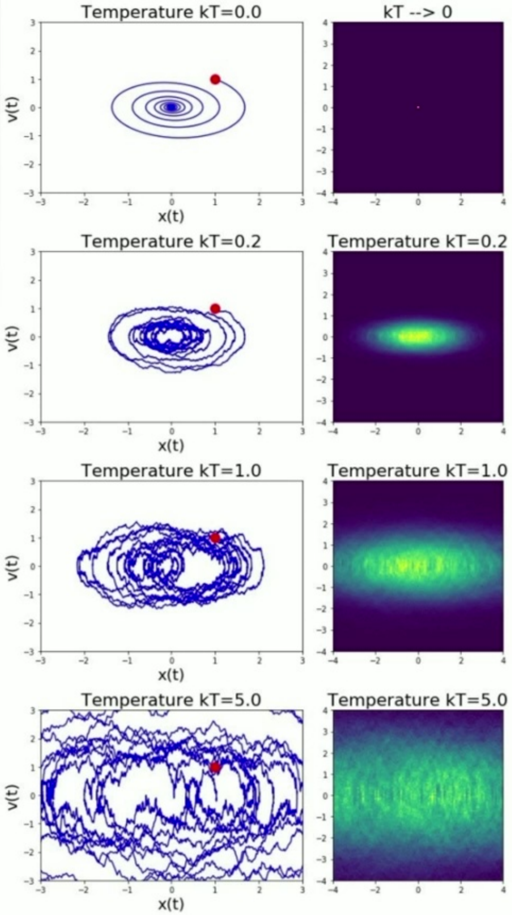
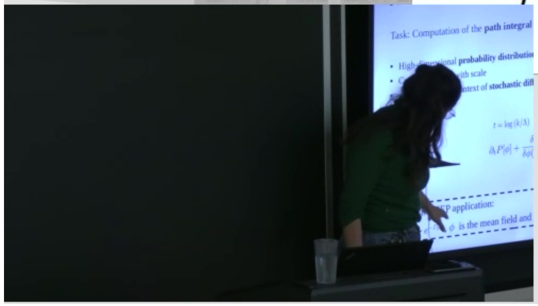
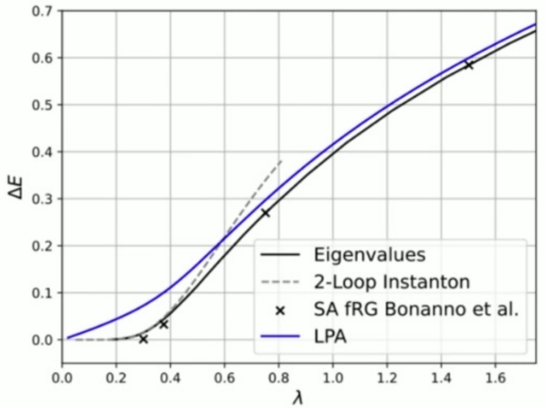
Standard HEP application:

$P[\phi] \sim e^{-\Gamma[\phi]}$, ϕ is the mean field and $\Psi[\phi]$ regulates momenta





Application to the anharmonic oscillator



Chose **approximation** for the effective action (0th order derivative expansion)

$$\Gamma_k[\varphi] = \int_x \left[\frac{1}{2}(\partial_\mu\varphi)^2 + V_k(\varphi)\right]$$

Solve **PDE** for V_k resulting from inserting the approximation in the evolution equation, with **initial condition**

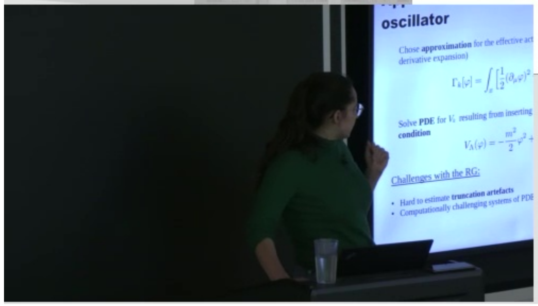
$$V_\Lambda(\varphi) = -\frac{m^2}{2}\varphi^2 + \frac{\lambda}{8}\varphi^4$$

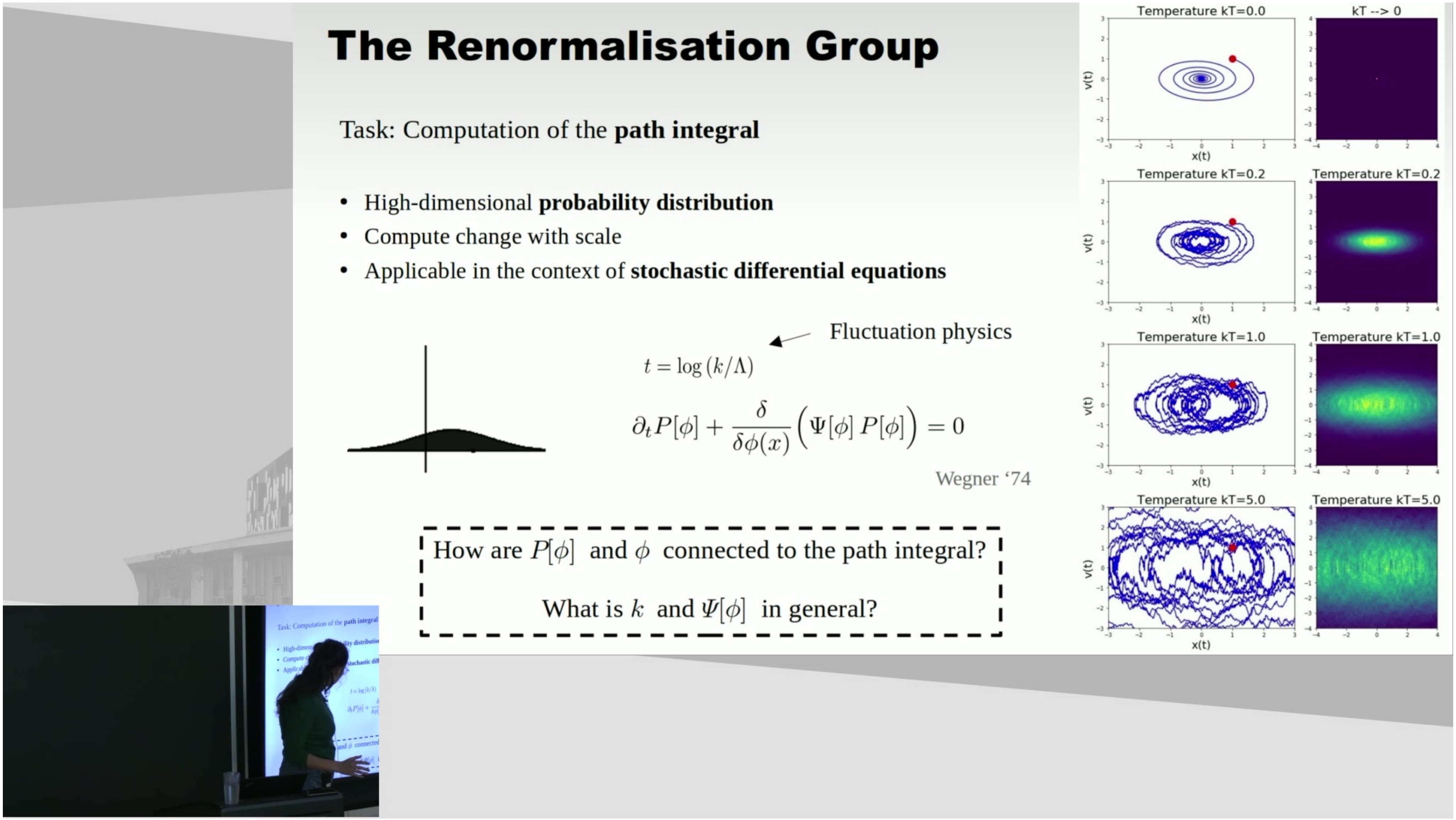
Challenges with the RG:

- Hard to estimate **truncation artefacts**
- Computationally challenging systems of PDE

Physics-informed RG flows:

- Improve expansion schemes with knowledge of the system/physics at hand
- Reduce computational cost
- Application to a wider set of problems





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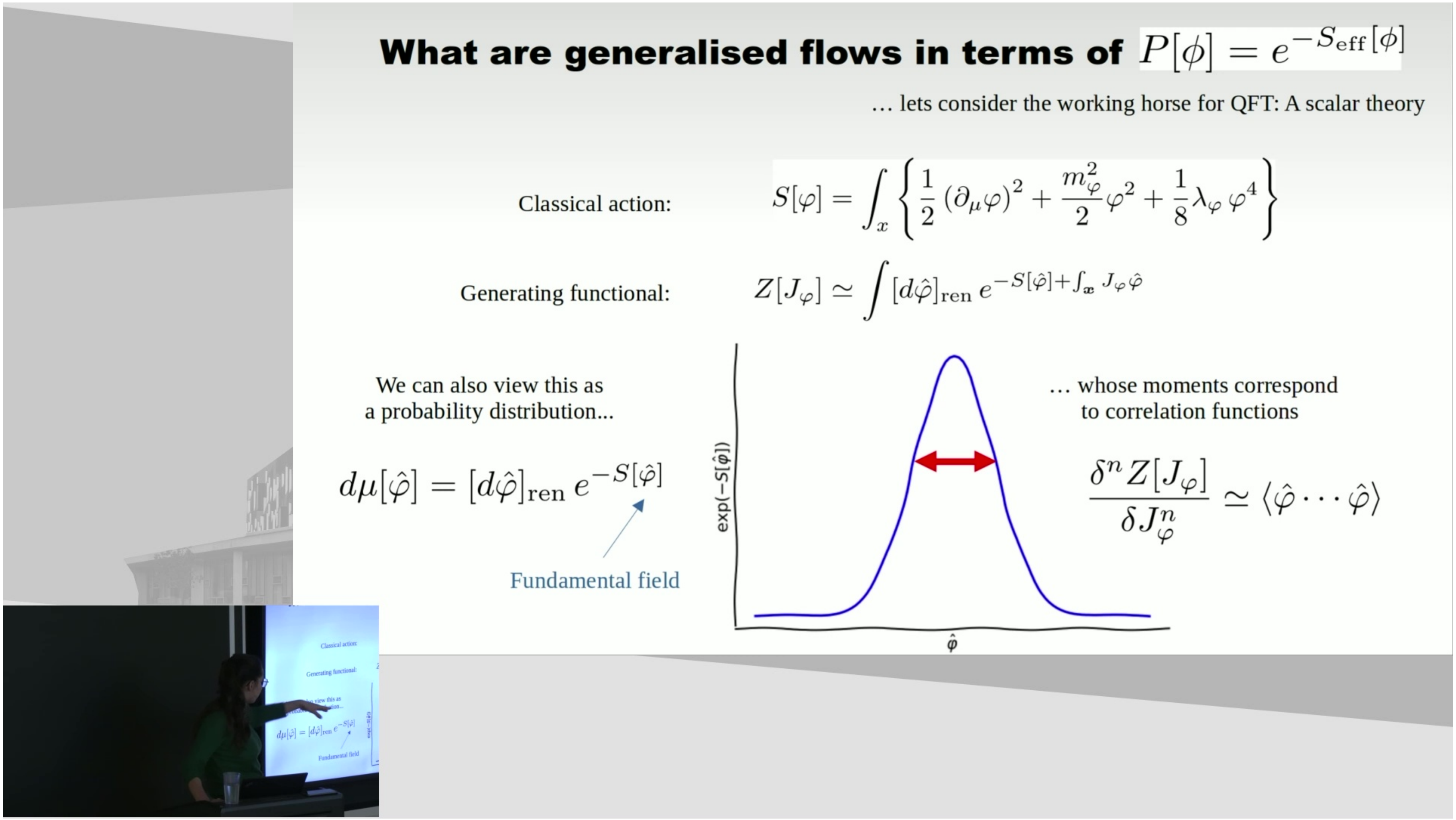
Wegner '74

How are $P[\phi]$ and ϕ connected to the path integral?

What is k and $\Psi[\phi]$ in general?



What are generalised flows in terms of $P[\phi] = e^{-S_{\rm eff}[\phi]}$

… lets consider the working horse for QFT: A scalar theory

Classical action: $$S[\varphi] = \int_x \left\{ \frac{1}{2}\left(\partial_\mu \varphi\right)^2 + \frac{m_\varphi^2}{2}\varphi^2 + \frac{1}{8}\lambda_\varphi\, \varphi^4 \right\}$$

Generating functional: $$Z[J_\varphi] \simeq \int [d\hat{\varphi}]_{\rm ren}\, e^{-S[\hat{\varphi}]+\int_{\boldsymbol{x}} J_\varphi \hat{\varphi}}$$

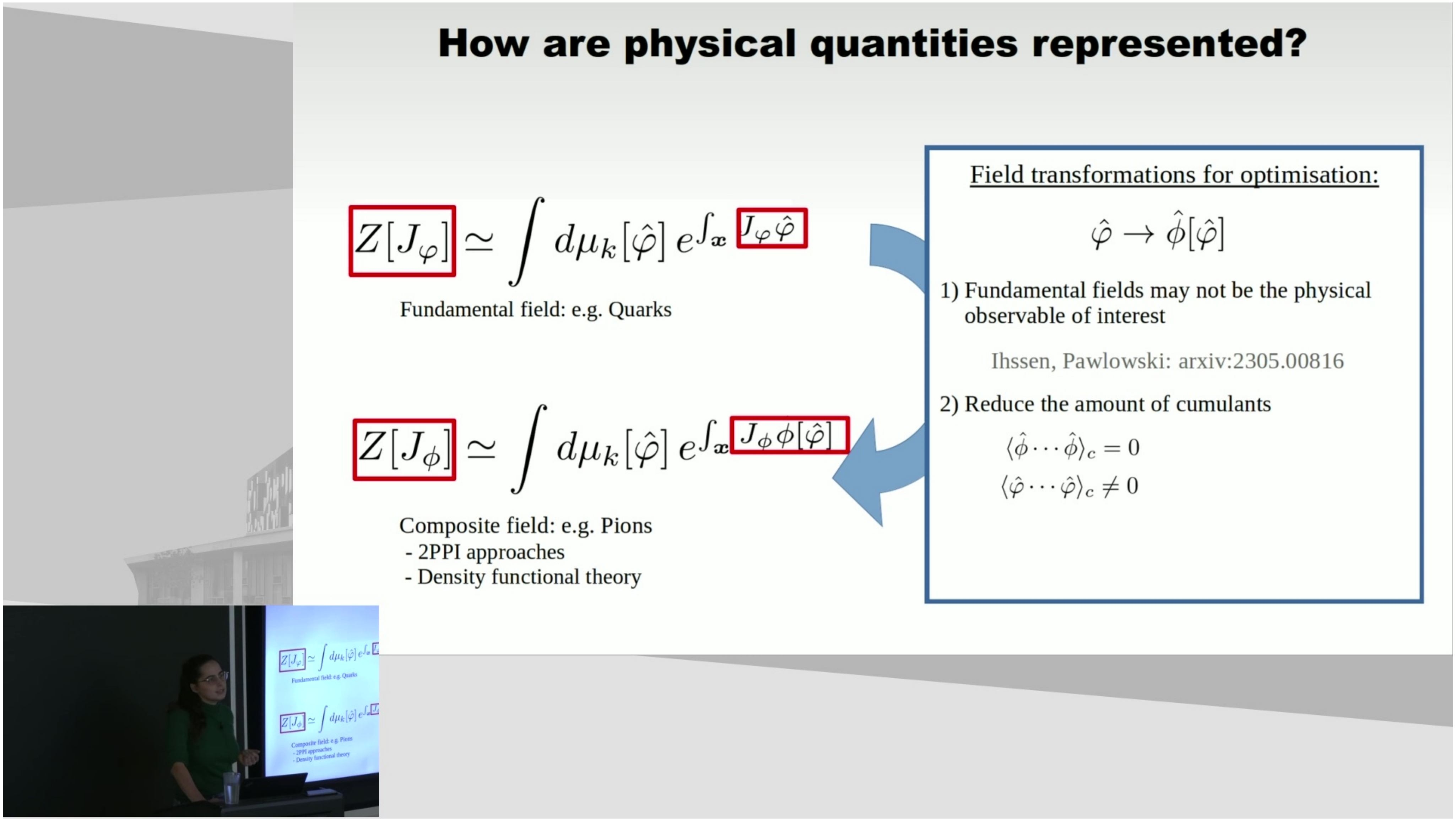
We can also view this as
a probability distribution...

$$d\mu[\hat{\varphi}] = [d\hat{\varphi}]_{\rm ren}\, e^{-S[\hat{\varphi}]}$$

Fundamental field

… whose moments correspond
to correlation functions

$$\frac{\delta^n Z[J_\varphi]}{\delta J_\varphi^n} \simeq \langle \hat{\varphi} \cdots \hat{\varphi} \rangle$$



How are physical quantities represented?

$$\boxed{Z[J_\varphi]} \simeq \int d\mu_k[\hat{\varphi}]\, e^{\int_{\boldsymbol{x}} \boxed{J_\varphi \hat{\varphi}}}$$

Fundamental field: e.g. Quarks

$$\boxed{Z[J_\phi]} \simeq \int d\mu_k[\hat{\varphi}]\, e^{\int_{\boldsymbol{x}} \boxed{J_\phi \phi[\hat{\varphi}]}}$$

Composite field: e.g. Pions
- 2PPI approaches
- Density functional theory

Field transformations for optimisation:

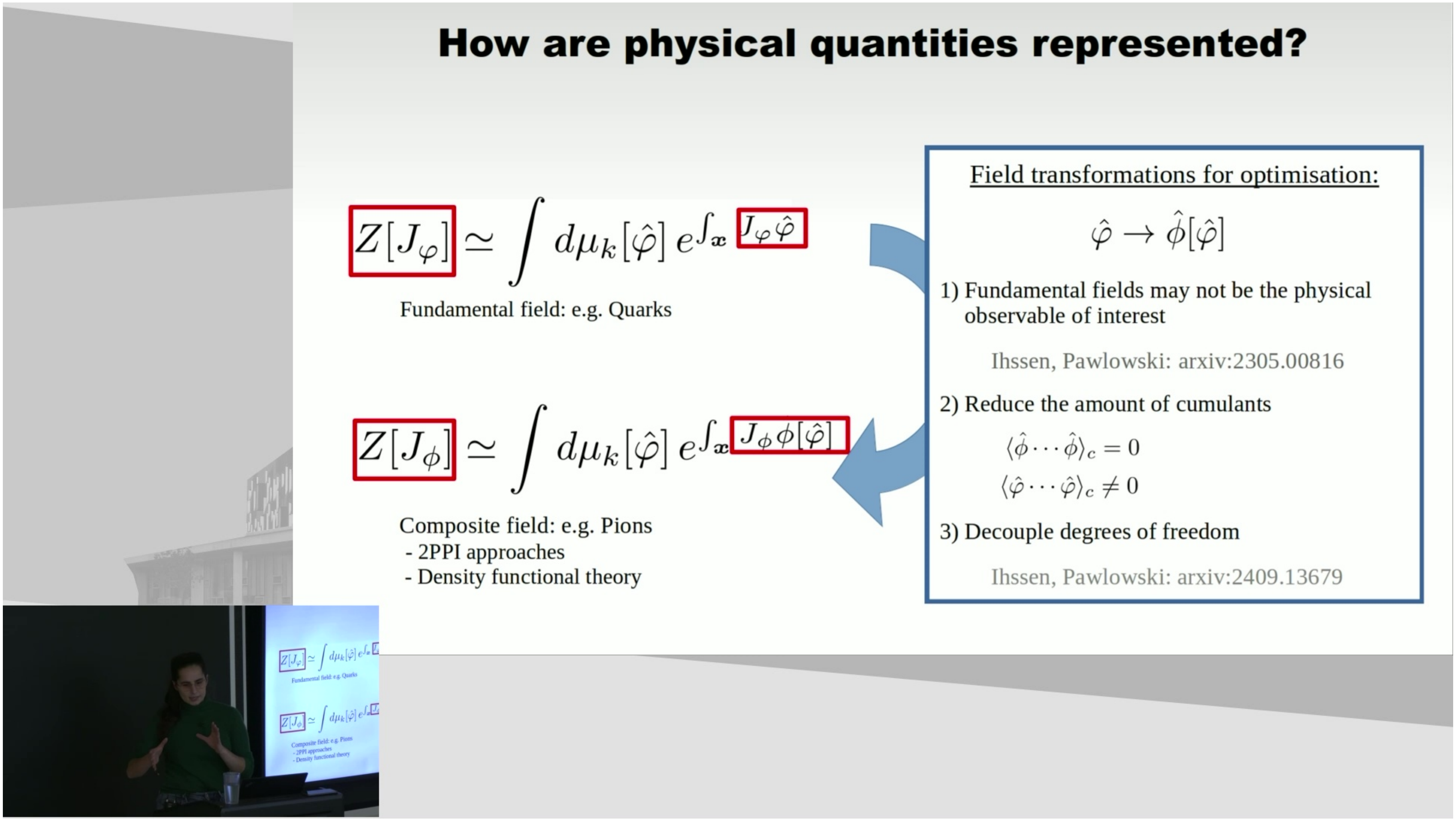
$$\hat{\varphi} \to \hat{\phi}[\hat{\varphi}]$$

1) Fundamental fields may not be the physical observable of interest

Ihssen, Pawlowski: arxiv:2305.00816

2) Reduce the amount of cumulants

$$\langle \hat{\phi} \cdots \hat{\phi} \rangle_c = 0$$
$$\langle \hat{\varphi} \cdots \hat{\varphi} \rangle_c \neq 0$$



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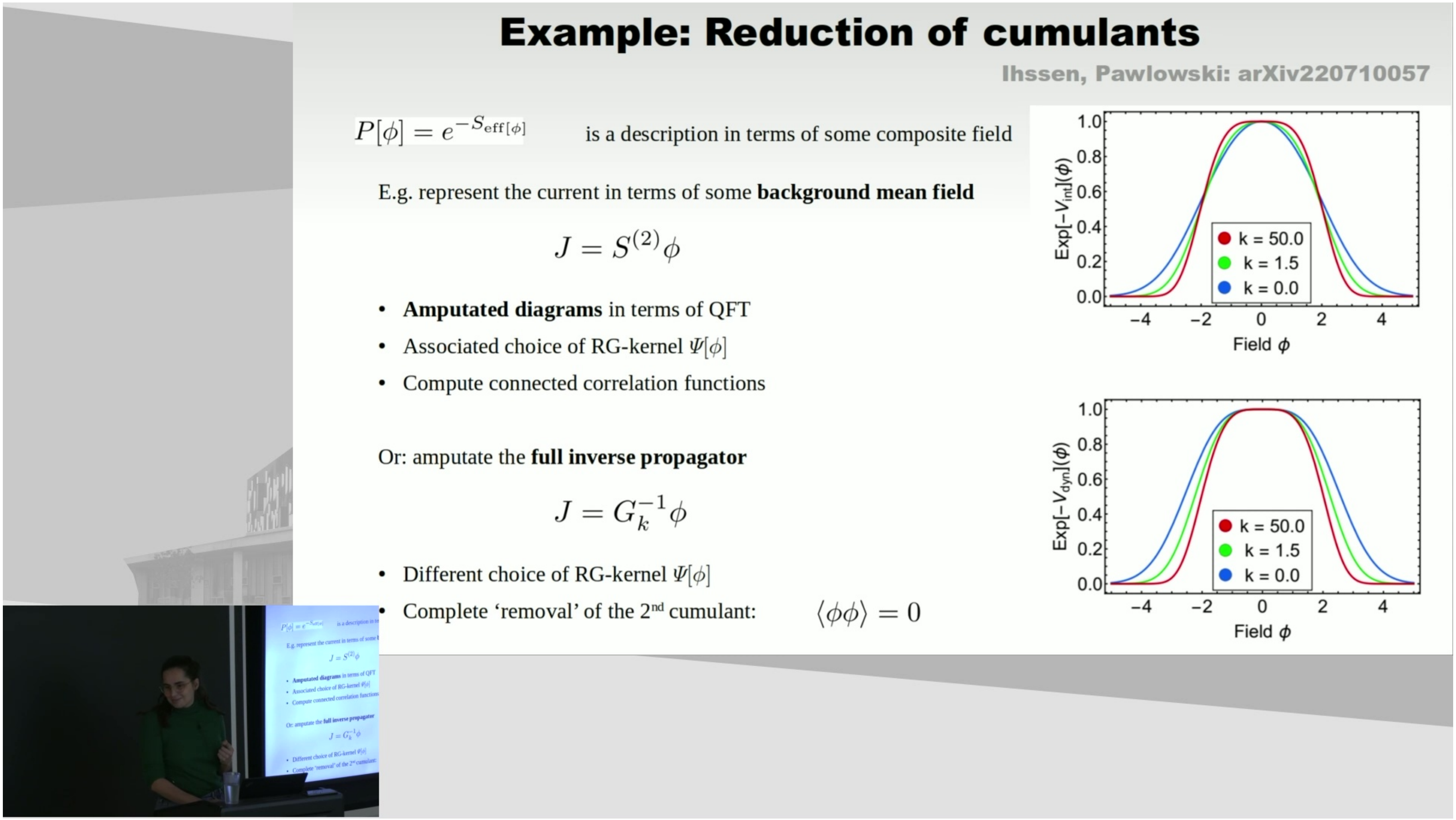
3) Decouple degrees of freedom

Ihssen, Pawlowski: arxiv:2409.13679



Example: Reduction of cumulants

Ihssen, Pawlowski: arXiv220710057

$P[\phi] = e^{-S_{\mathrm{eff}[\phi]}}$ is a description in terms of some composite field

E.g. represent the current in terms of some **background mean field**

$$J = S^{(2)}\phi$$

- **Amputated diagrams** in terms of QFT
- Associated choice of RG-kernel $\Psi[\phi]$
- Compute connected correlation functions

Or: amputate the **full inverse propagator**

$$J = G_k^{-1}\phi$$

- Different choice of RG-kernel $\Psi[\phi]$
- Complete 'removal' of the 2nd cumulant: $\langle\phi\phi\rangle = 0$

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Simpler representation of the same physical object:

Better representation of relevant DoFs

Or: amputate the **full inverse propagator**

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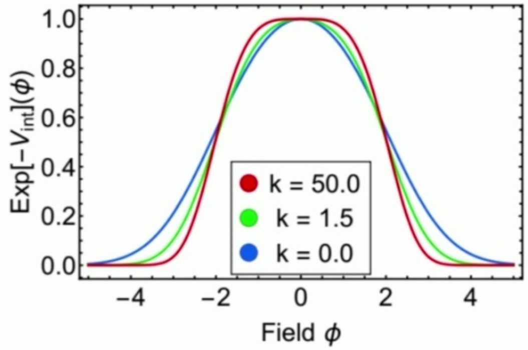
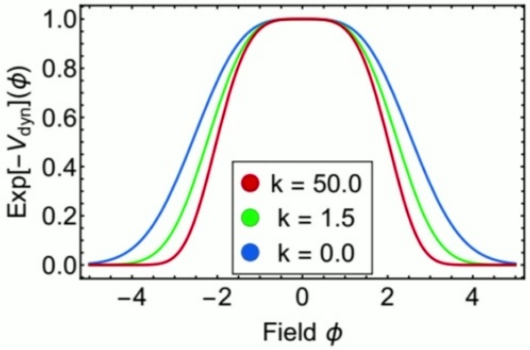
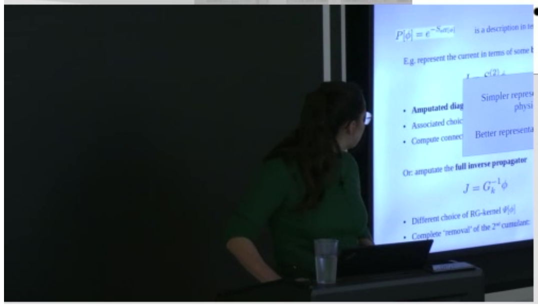
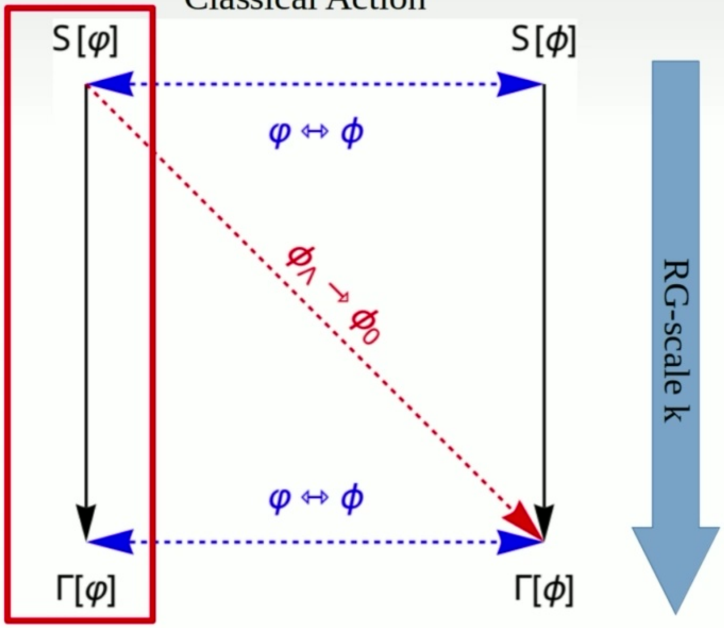






General field transformations in the (f)RG

Classical Action



$$\begin{tikzcd}
S[\varphi] \arrow[rr, leftrightarrow, dashed, "\varphi \leftrightarrow \phi"'] \arrow[d] \arrow[drr, dashed, "\phi_\Lambda \to \phi_0"] & & S[\phi] \arrow[d] \\
\Gamma[\varphi] \arrow[rr, leftrightarrow, dashed, "\varphi \leftrightarrow \phi"] & & \Gamma[\phi]
\end{tikzcd}$$

RG-scale k

Quantum effective Action

- Infrared regularised theory ↔ classical theory

$$\Gamma_\Lambda[\varphi] = S[\varphi]$$

- Solve RG-flow by integrating over RG-time/RG-scale

Regulator

$$\partial_t \Gamma_k[\varphi] = \frac{1}{2}\, G_k[\varphi]\, \partial_t R_k$$

RG-time

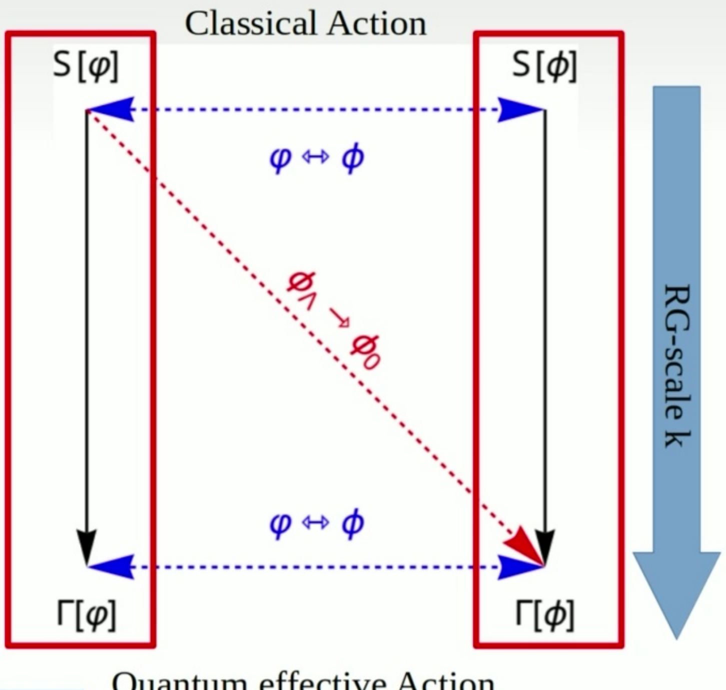
$$t = \log\left(\frac{k}{\Lambda}\right)$$

Mean-field

$$\varphi = \langle \hat{\varphi} \rangle$$

Propagator

General field transformations in the (f)RG



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- Solve RG-flow by integrating over RG-time/RG-scale

Regulator

$$\partial_t \Gamma_k[\varphi] = \frac{1}{2}\, G_k[\varphi]\, \partial_t R_k$$

RG-time

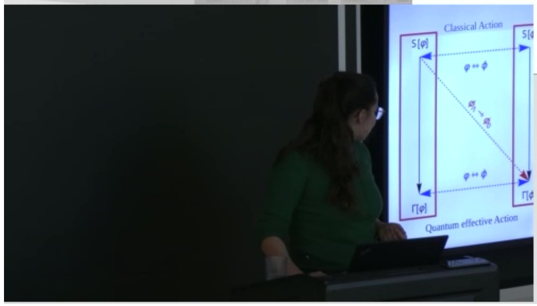
$$t = \log\left(\frac{k}{\Lambda}\right)$$

Mean-field

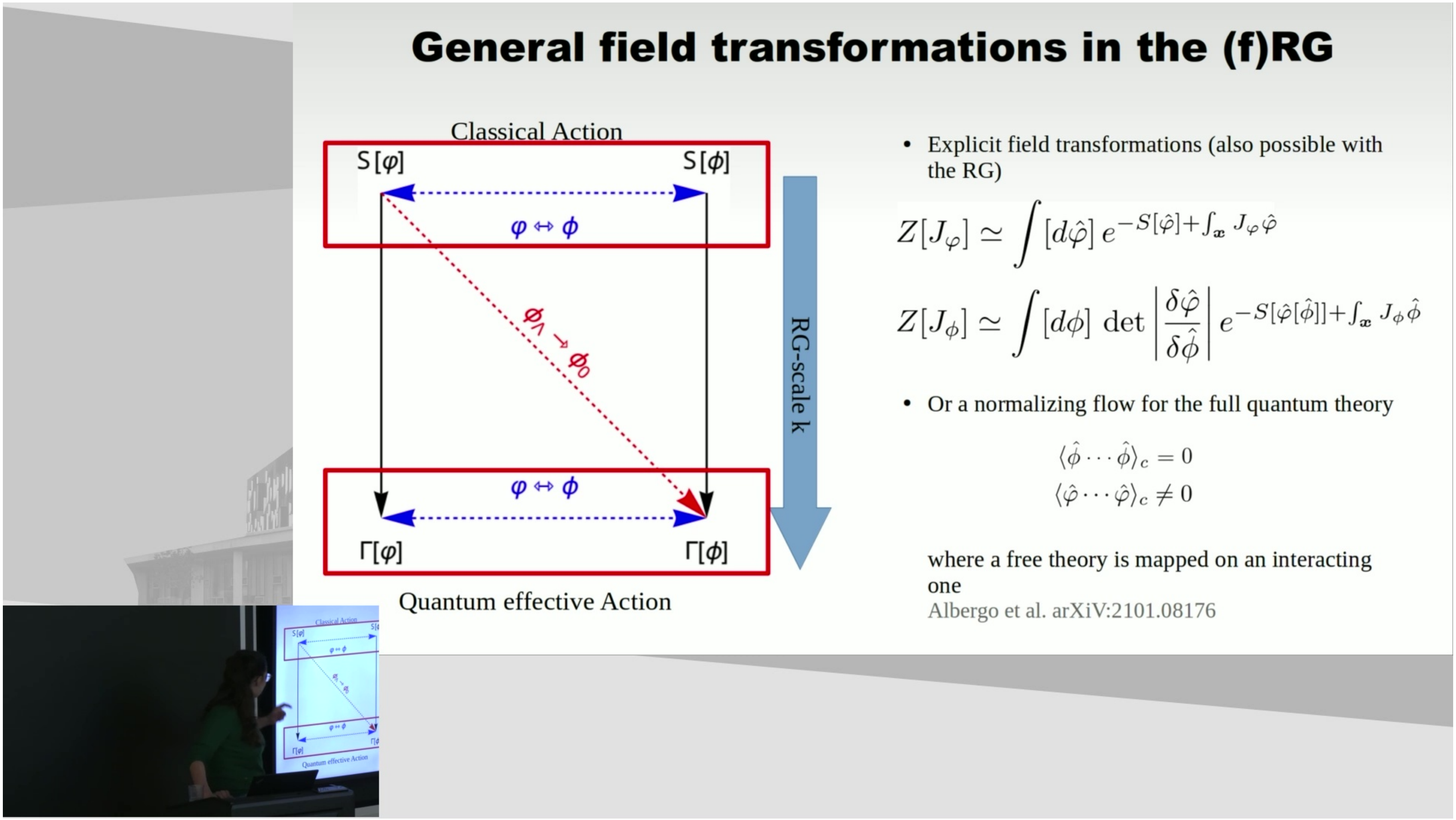
$$\varphi = \langle \hat{\varphi} \rangle$$

Propagator

- Full quantum effective action $\Gamma[\varphi]$
  Wetterich'92
  - Solve coupled PDEs for all generated couplings in the effective action



General field transformations in the (f)RG



- Explicit field transformations (also possible with the RG)

$$Z[J_\varphi] \simeq \int [d\hat{\varphi}]\, e^{-S[\hat{\varphi}]+\int_{\boldsymbol{x}} J_\varphi \hat{\varphi}}$$

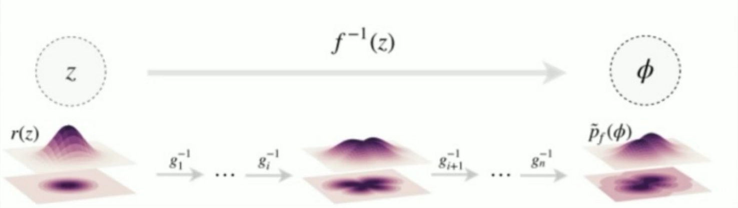
$$Z[J_\phi] \simeq \int [d\phi]\, \det \left| \frac{\delta\hat{\varphi}}{\delta\hat{\phi}} \right| e^{-S[\hat{\varphi}[\hat{\phi}]]+\int_{\boldsymbol{x}} J_\phi \hat{\phi}}$$

- Or a normalizing flow for the full quantum theory

$$\langle \hat{\phi} \cdots \hat{\phi} \rangle_c = 0$$
$$\langle \hat{\varphi} \cdots \hat{\varphi} \rangle_c \neq 0$$

where a free theory is mapped on an interacting one

Albergo et al. arXiV:2101.08176

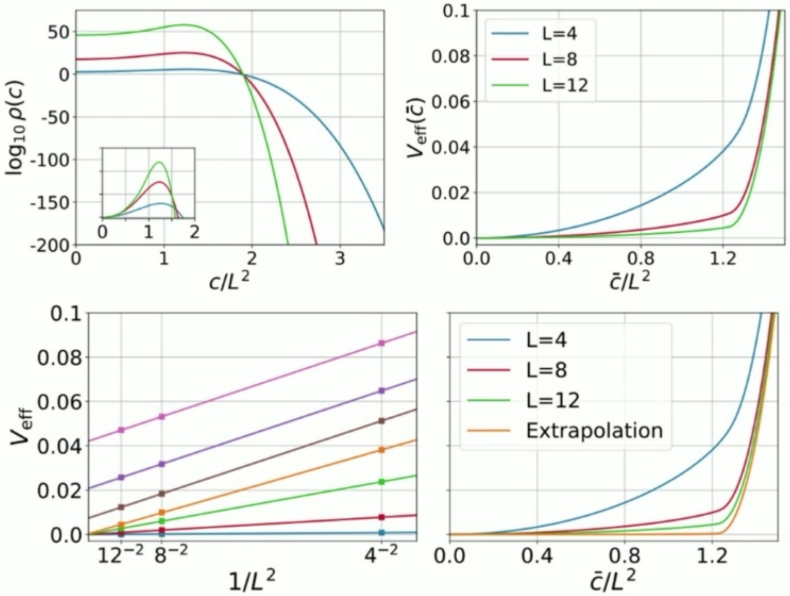


Generative AI for producing lattice configurations

- "Learn" field configuration for some action
  - → Invertible field transformation
- Limited lattice size

Normalising flows and the effective action

- Resolution of the effective action requires a **cheap sampling method**
  Attanasio, Bauer, Pawlowski, Temmen in Prep.
- Field transformation between theories with different UV-cutoffs
  Bauer, Kapust, Pawlowski, Temmen in Prep.
- Approximation artifacts: field transformation + Lattice discretisation



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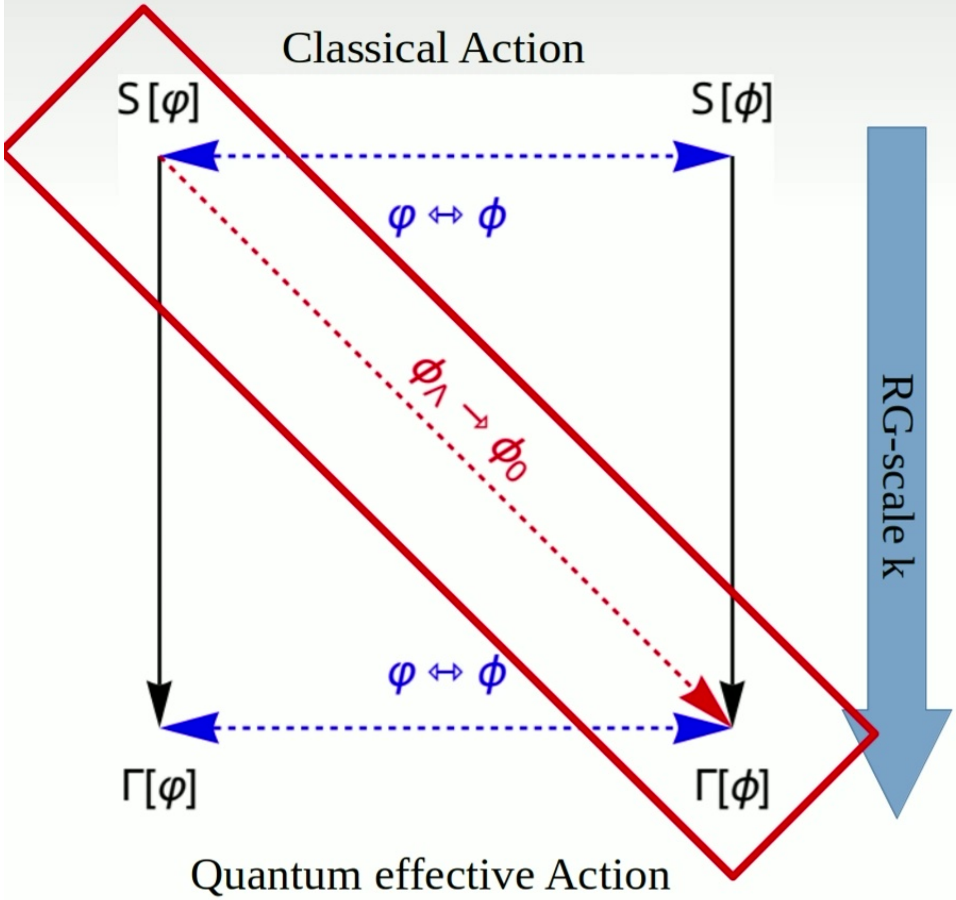

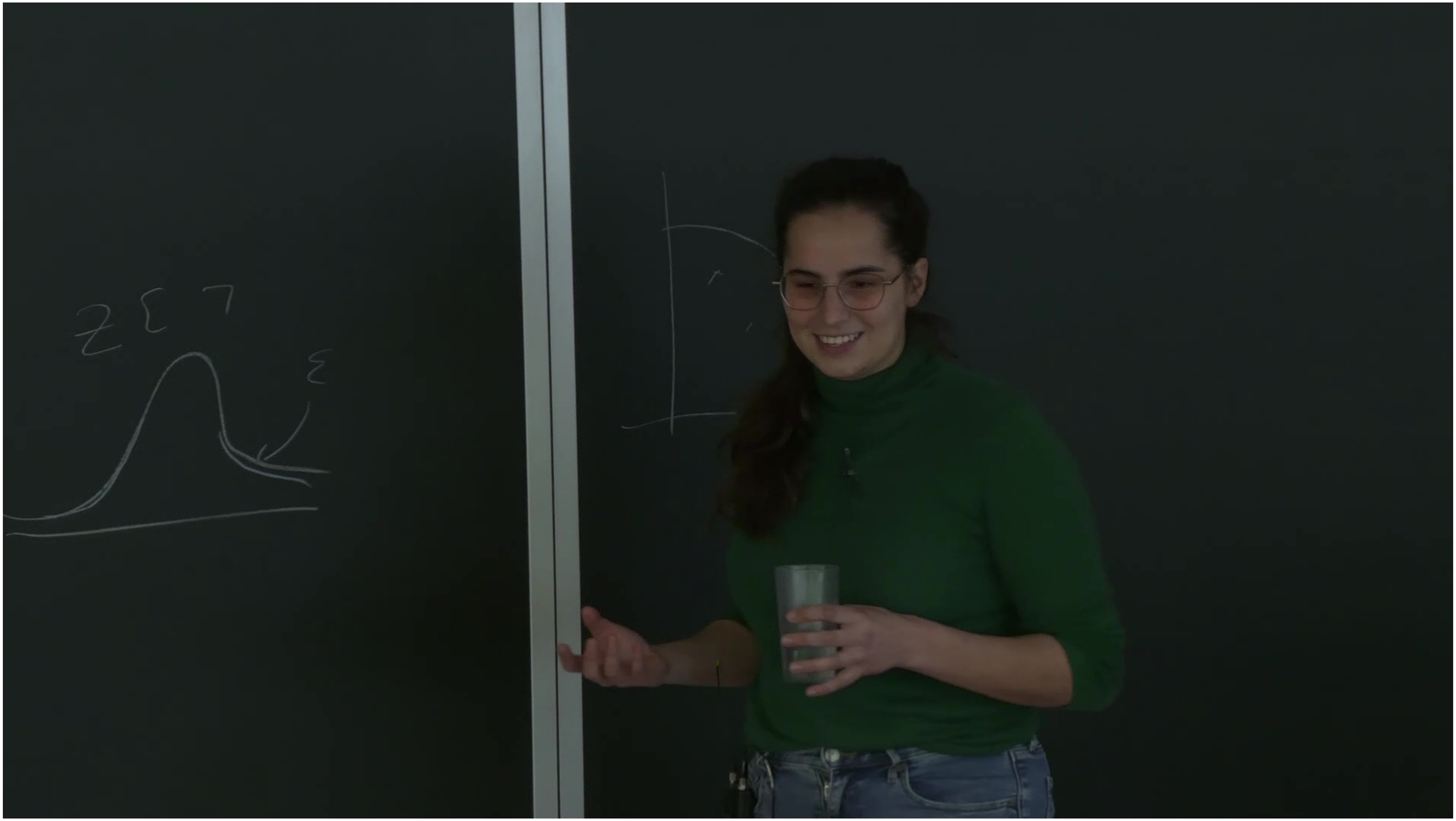

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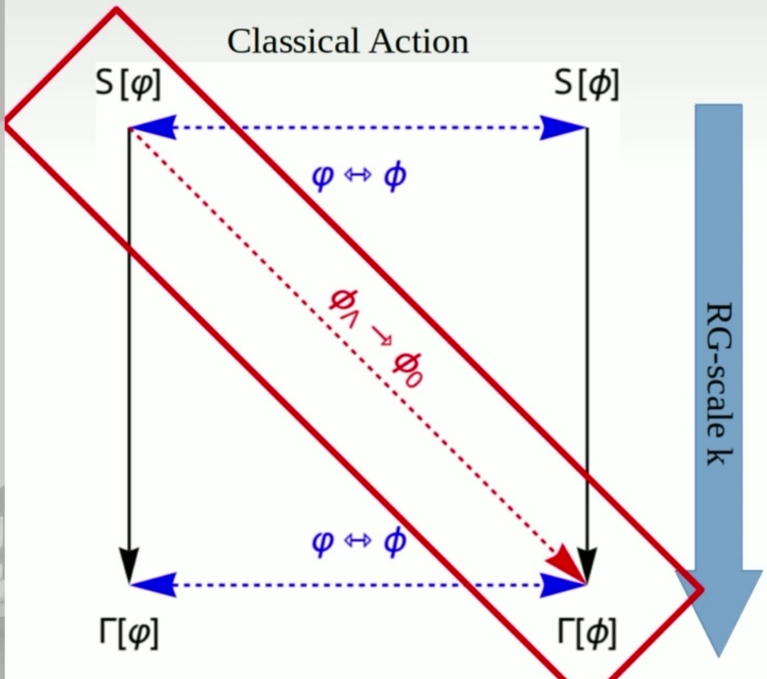
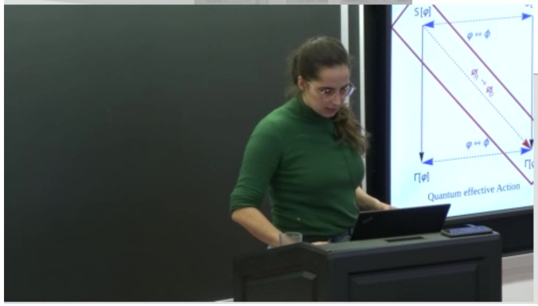


Classical Action

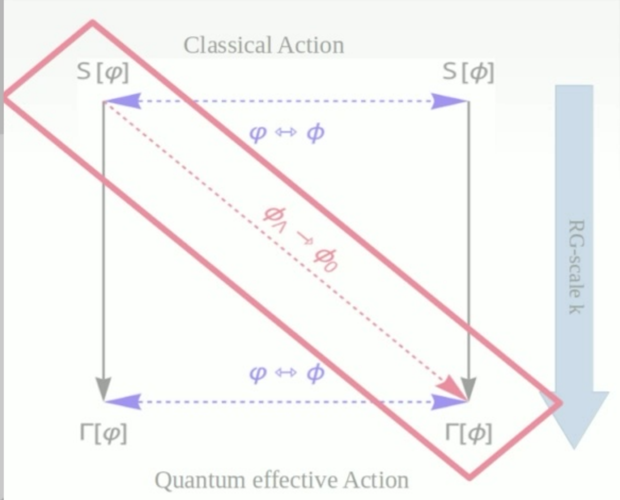
$$\begin{tikzcd}
S[\varphi] \arrow[rr, leftrightarrow, dashed, "\varphi \leftrightarrow \phi"] \arrow[d] \arrow[rrd, dashed, "\phi_\Lambda \to \phi_0"] & & S[\phi] \arrow[d] \\
\Gamma[\varphi] \arrow[rr, leftrightarrow, dashed, "\varphi \leftrightarrow \phi"] & & \Gamma[\phi]
\end{tikzcd}$$

Quantum effective Action

RG-scale k



General field transformations in the fRG



- Generalised functional Flows

RG-time

1PI gen. funct.

Propagator

Regulator

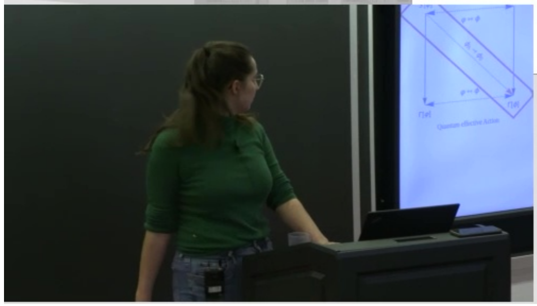
$t = \log\left(\frac{k}{\Lambda}\right)$

$$\left(\partial_t + \int_x \underline{\dot{\phi}\frac{\delta}{\delta\phi}}\right)\Gamma_k[\phi] = \frac{1}{2}\left[G[\phi]\left(\partial_t + 2\underline{\frac{\delta\dot{\phi}}{\delta\phi}}\right)R_k\right]$$

Pawlowski '05

- RG-scale dependent composite

$$\phi = \langle\hat{\phi}[\hat{\varphi}]\rangle \qquad \dot{\phi}[\phi] = \langle\partial_t\hat{\phi}_k\rangle[\phi]$$



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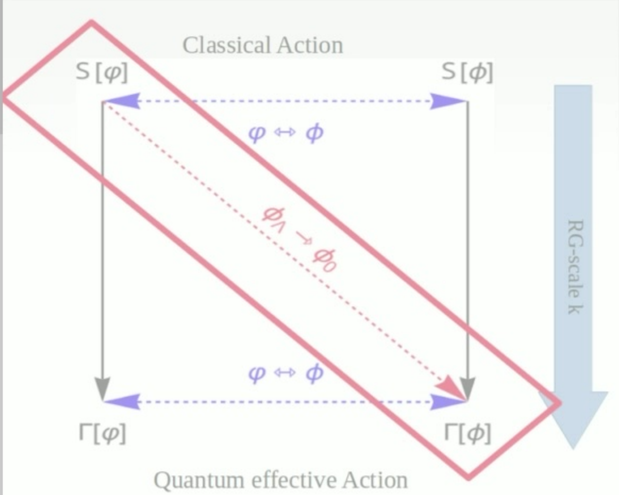


- Generalised functional Flows

RG-time 1PI gen. funct. Propagator Regulator

$t = \log\left(\frac{k}{\Lambda}\right)$

$$\left(\partial_t + \int_x \underline{\dot{\phi}\frac{\delta}{\delta\phi}}\right)\Gamma_k[\phi] = \frac{1}{2}\left[G[\phi]\left(\partial_t + 2\underline{\frac{\delta\dot{\phi}}{\delta\phi}}\right)R_k\right]$$

Pawlowski ‘05

- RG-scale dependent composite

$$\phi = \langle\hat{\phi}[\hat{\varphi}]\rangle \qquad \dot{\phi}[\phi] = \langle\partial_t\hat{\phi}_k\rangle[\phi]$$

- At the level of the effective action, physics is stored in the pair

$$(\Gamma_\phi, \phi[\varphi])$$

he field transformation allows the definition of an additional constraint → **Physics-informed**

Application to the anharmonic oscillator

In Prep. Bonanno, FI, Pawlowski

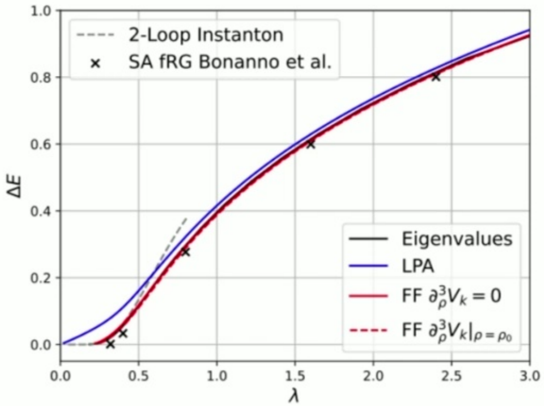
$$\boxed{\partial_t \Gamma_k[\varphi] = \frac{1}{2}\, G_k[\varphi]\, \partial_t R_k}$$

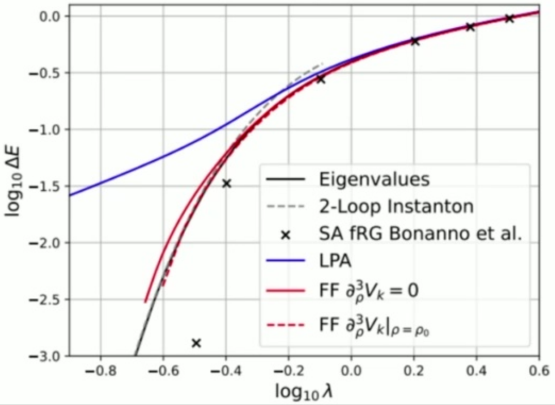
$$\boxed{\left(\partial_t + \int_x \dot{\phi} \frac{\delta}{\delta\phi}\right) \Gamma_k[\phi] = \frac{1}{2}\left[G[\phi] \left(\partial_t + 2\frac{\delta\dot{\phi}}{\delta\phi}\right) R_k\right]}$$

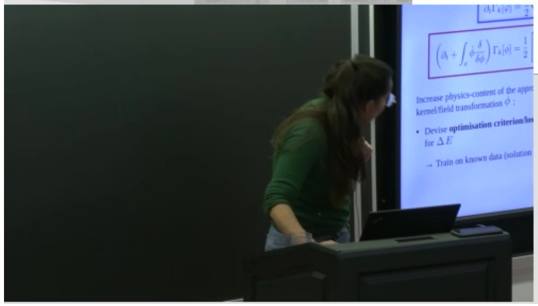
Increase physics-content of the approximation by optimising the RG-kernel/field transformation $\dot{\phi}$:

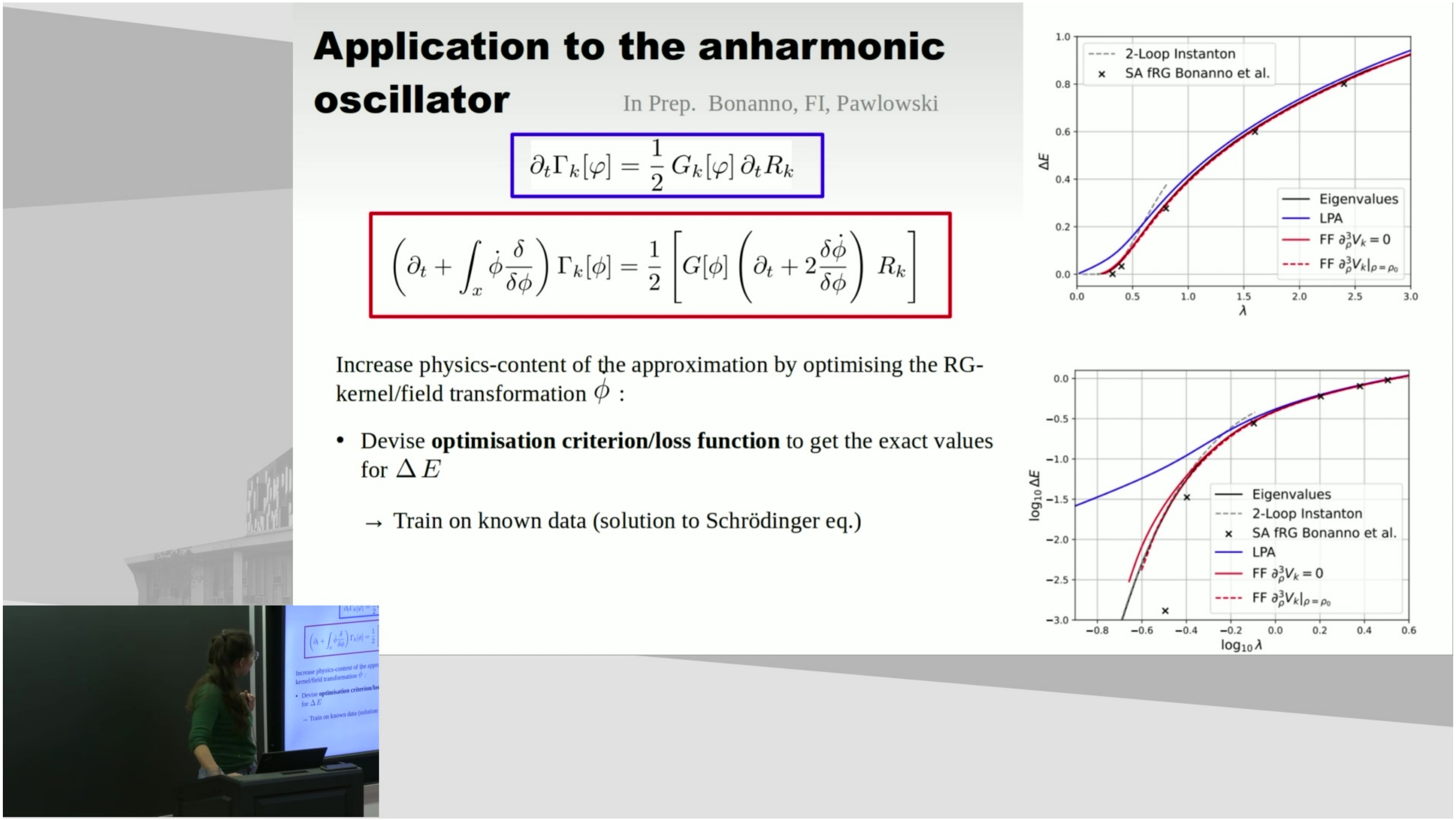
- Devise **optimisation criterion/loss function** to get the exact values for ΔE

  → Train on known data (solution to Schrödinger eq.)









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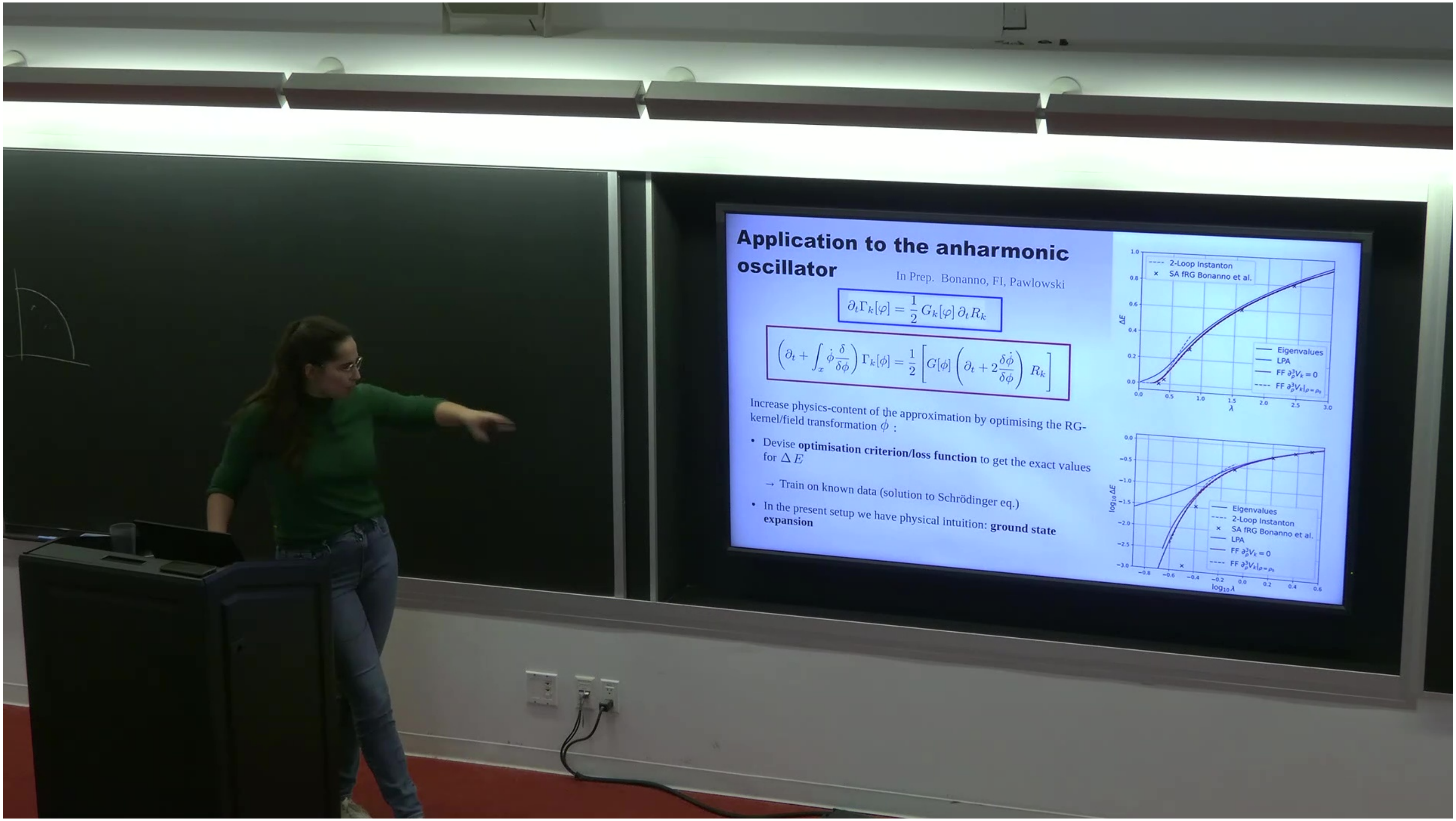

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Increase physics-content of the approximation by optimising the RG-kernel/field transformation $\dot{\phi}$:

- Devise **optimisation criterion/loss function** to get the exact values for ΔE

  → Train on known data (solution to Schrödinger eq.)

- In the present setup we have physical intuition: **ground state expansion**

An expansion about the ground state

FI, Pawlowski ‘23 : arXiv:2305.00816

O(N) model: $\varphi^t = (\varphi_1, \ldots, \varphi_N)$ vs. $\phi^t = (\phi_1, \ldots, \phi_N)$

$$\rho_\varphi = \frac{\varphi^2}{2} \qquad \rho = \frac{\phi^2}{2}$$

$$\Gamma_k[\varphi] = \int_x \left[\frac{1}{2} Z_{\varphi,k}\, (\partial_\mu \varphi)^2 + V_k(\rho_\varphi)\right]$$

$$\Gamma_k[\phi] = \int_x \left[\frac{1}{2} Z_{\phi,k}\, (\partial_\mu \phi)^2 + V_k(\rho)\right]$$

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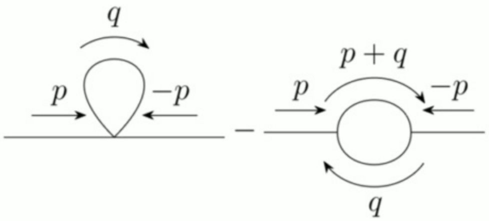
$$\Gamma^{(2)}_{\varphi_i\varphi_i}[\varphi](p) = \boxed{Z_\varphi(\rho_\varphi, p)}\Big(p^2 + m^2_{\varphi_i}(\rho_\varphi)\Big)$$

Field dependent wave function renormalisation and its derivatives



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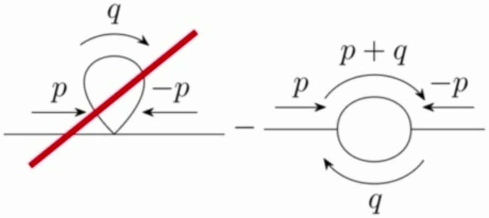

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Field dependent wave
function renormalisation
and its derivatives

$$\Gamma^{(2)}_{\phi_i \phi_i}[\phi](p) = \left[p^2 + m^2_{\phi_i}(\rho)\right] \longrightarrow$$

Expand about ground state using the flowing fields:

$$\dot{\phi}_k(\phi, k) \;\to\; Z_{\phi,k}(\phi, p) \equiv 1$$

$$\uparrow$$

$$\Gamma_k[\phi] = \int_x \left[\frac{1}{2} Z_{\phi,k}\, (\partial_\mu \phi)^2 + V_k(\rho)\right] \longleftarrow$$

$$\partial_t Z_\phi(\phi, p) \equiv 0$$

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Field dependent wave function renormalisation and its derivatives

Take away message:

- Expansion about **classical dispersion**:
  - → **Optimised expansion** (quicker convergence)
- Technical simplification with improved truncation

$$\Gamma^{(2)}_{\phi_i \phi_i}[\phi](p) = \left[p^2 + m^2_{\phi_i}(\rho)\right] \longrightarrow$$

Expand about ground state using the flowing fields:

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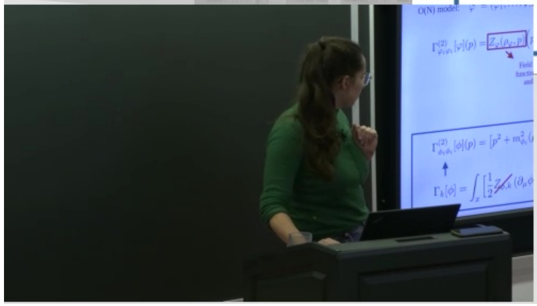


- Application: $Z_\phi(\rho, p) \approx Z_\phi(\rho)$ (1st order deriv. exp.)
- Task: Solve two equations

1) $\boxed{\partial_t Z_\phi} = 0$: determines $\eta_\phi(\rho)$
2) $\boxed{\partial_t V_k} = \dots$: PDE, integrate $k \to k - \Delta k$

Parametrisation:

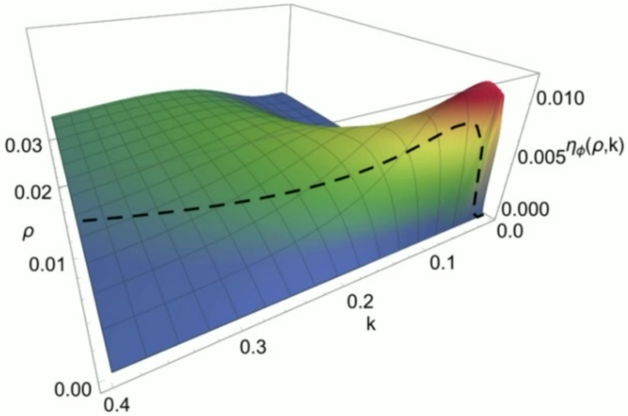
$$\dot{\phi} = -\frac{1}{2}\eta_\phi(\rho)\,\phi$$

And accordingly:

$$\eta_\phi(\rho) = -\frac{\partial_t Z_\varphi(\rho)}{Z_\varphi(\rho)}$$

Take away message:

- With (1) we implement a dynamical field, which is the **correctly renormalised** one
  - → Improved approximation (**momentum dependences**)
- Gain intuition about the system
  - → Maximise efficiency



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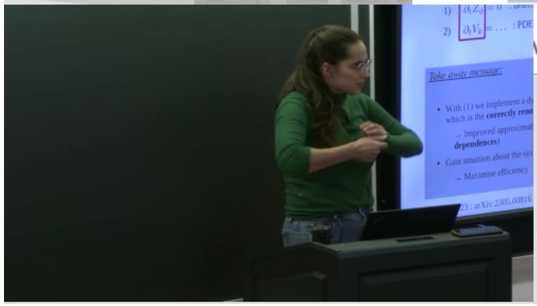

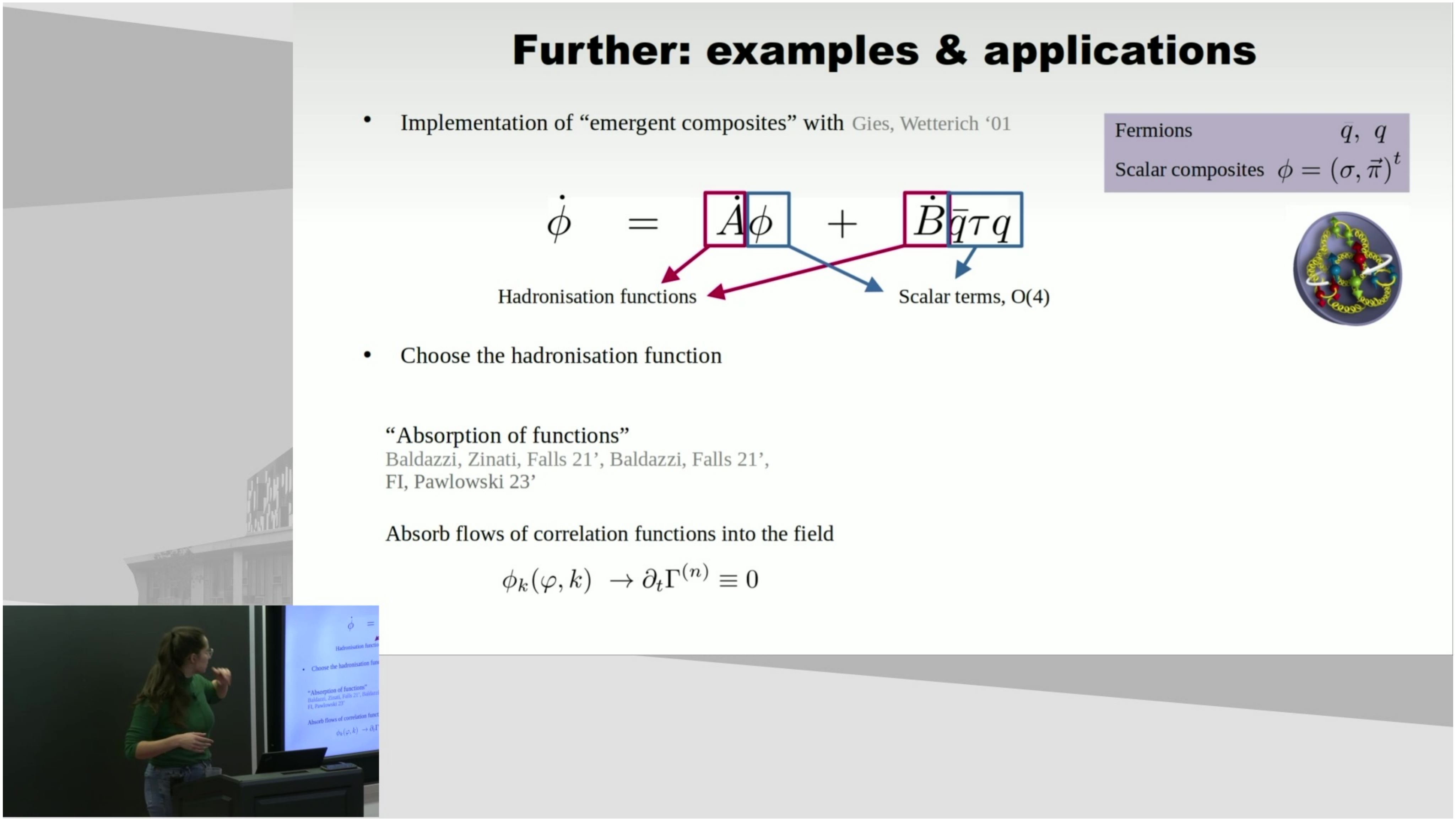

Further: examples & applications

- Implementation of "emergent composites" with Gies, Wetterich '01

Fermions	$\bar{q},\ q$
Scalar composites	$\phi = (\sigma, \vec{\pi})^t$

$$\dot{\phi} = \boxed{\dot{A}}\boxed{\phi} + \boxed{\dot{B}}\boxed{\bar{q}\tau q}$$

Hadronisation functions

Scalar terms, O(4)

- Choose the hadronisation function

"Absorption of functions"
Baldazzi, Zinati, Falls 21', Baldazzi, Falls 21',
FI, Pawlowski 23'

Absorb flows of correlation functions into the field

$$\phi_k(\varphi, k) \ \rightarrow \partial_t \Gamma^{(n)} \equiv 0$$

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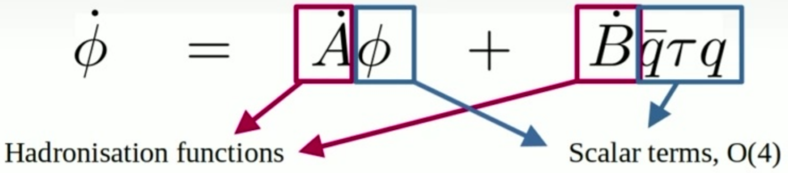

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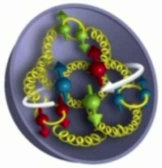

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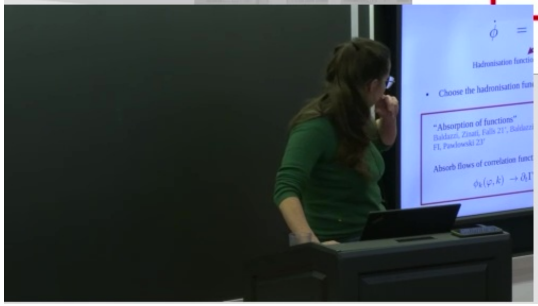
"Geometric transformations"
Flow from a Cartesian to a polar basis
Lamprecht '07, Isaule, Birse, Walet '18, Isaule, Birse, Walet'19, Daviet, Dupuis '21

$$\phi^t = (\rho, \theta)$$

$$\varphi = \sqrt{2\rho}\, e^{\theta^a t^a}\, (1,\, 0,\, \ldots,\, 0)^t$$



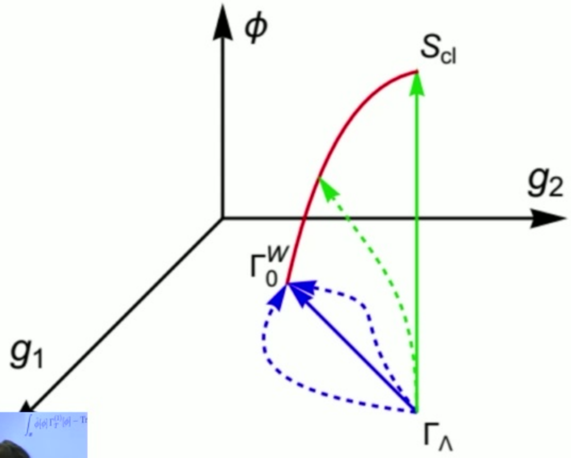
Target Actions: How general is it really?

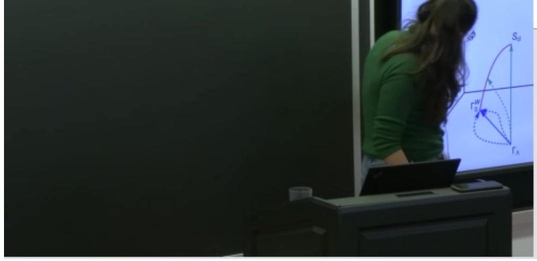
FI, Pawlowski 24' ,
arXiv:2409.13679

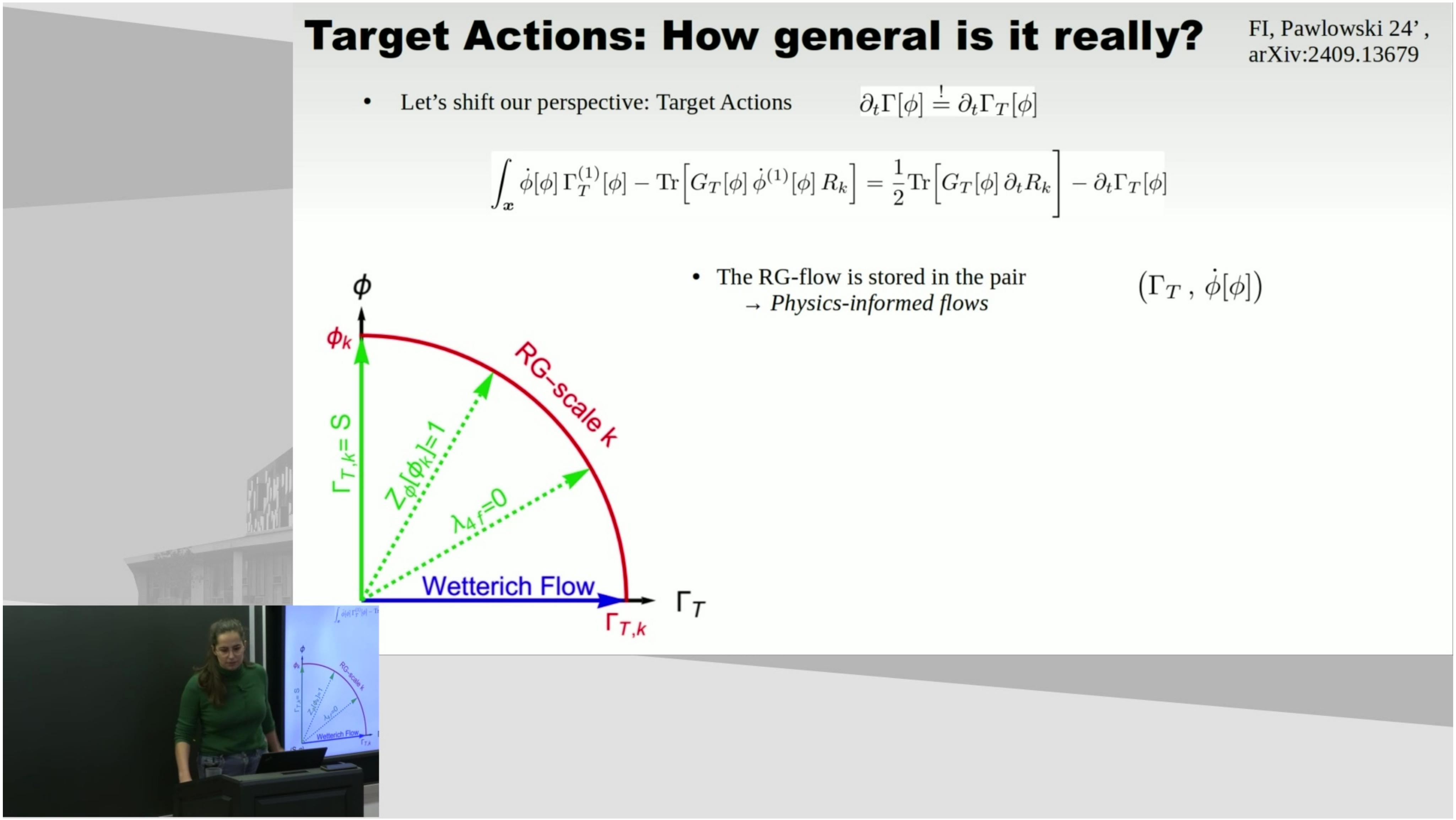
- Let's shift our perspective: Target Actions $\partial_t \Gamma[\phi] \overset{!}{=} \partial_t \Gamma_T[\phi]$

$$\int_{\boldsymbol{x}} \dot{\phi}[\phi]\,\Gamma_T^{(1)}[\phi] - \mathrm{Tr}\Big[G_T[\phi]\,\dot{\phi}^{(1)}[\phi]\,R_k\Big] = \frac{1}{2}\mathrm{Tr}\Big[G_T[\phi]\,\partial_t R_k\Big] - \partial_t \Gamma_T[\phi]$$

- The RG-flow is stored in the pair $\big(\Gamma_T\,,\,\dot{\phi}[\phi]\big)$
  → *Physics-informed flows*







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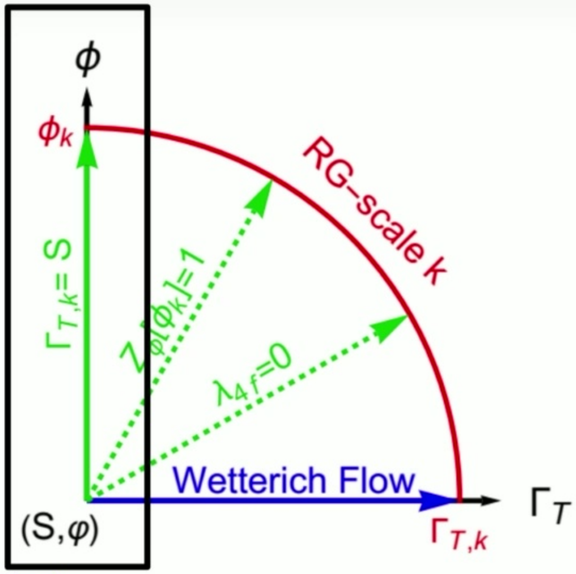
- The RG-flow is stored in the pair $\big(\Gamma_T\,,\ \dot{\phi}[\phi]\big)$
  → *Physics-informed flows*

Example: Classical Target Actions

...most extreme example

$$\partial_t \Gamma_T[\phi] \overset{!}{=} 0 \quad \longrightarrow \quad \dot{\phi}_k[\phi]$$



- The potential does NOT flow

$$\Gamma_T[\phi] = S[\phi] + \mathcal{C}_k$$

=> Entirety of physics in the field transformation

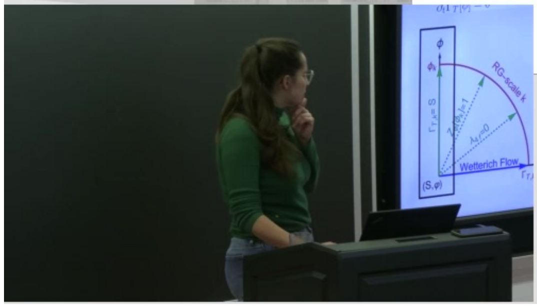
- Benchmark in d=0

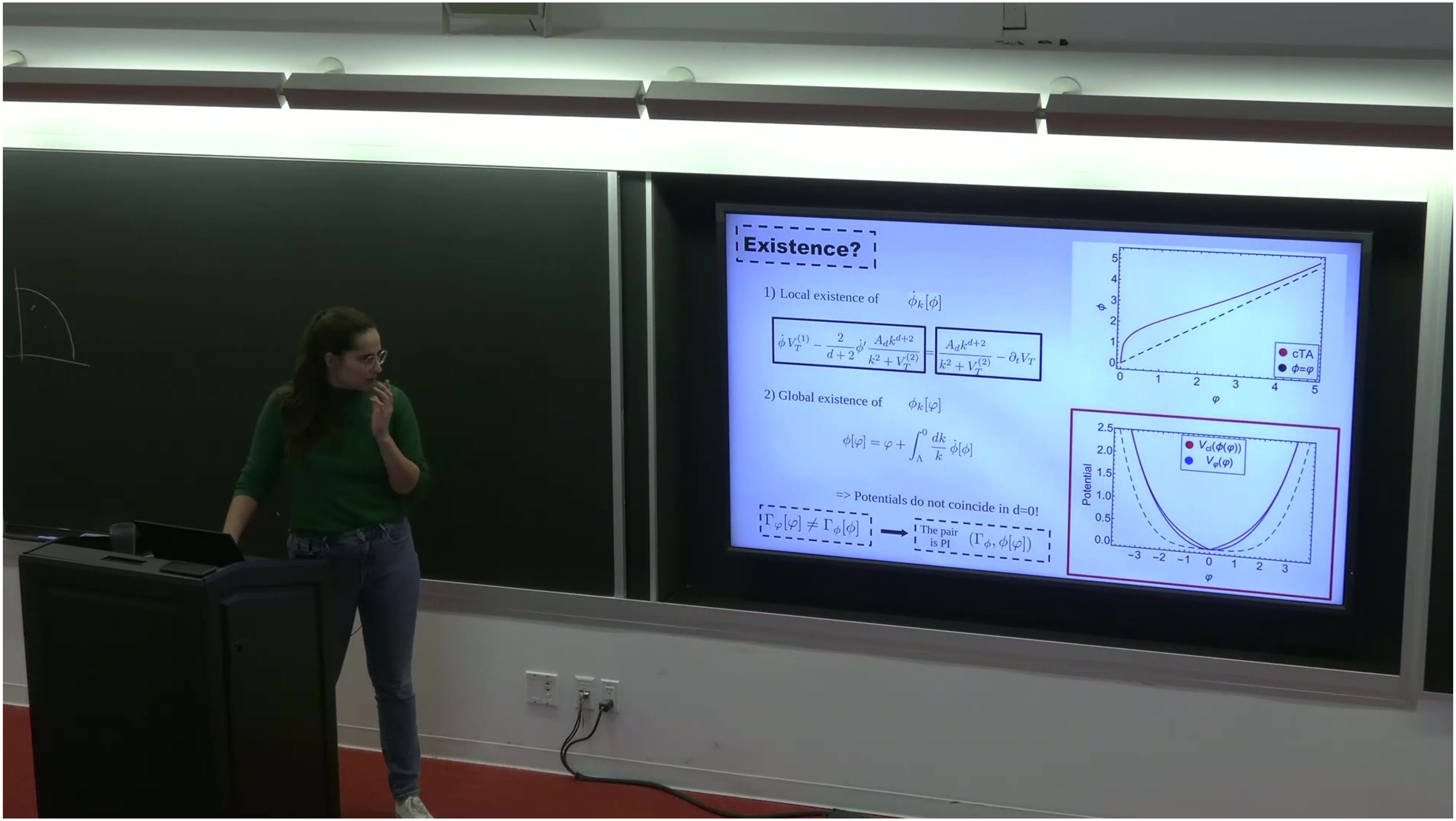
$$V_{\rm cl}(\phi) = \frac{\mu_\varphi}{2}\phi^2 + \frac{\lambda_\varphi}{8}\phi^4$$

(+) Comparison to exact solution

(-) No estimate of ‘truncation artefacts’

FI, Pawlowski 24’ , arXiv:2409.13679





Existence?

1) Local existence of $\dot{\phi}_k[\phi]$

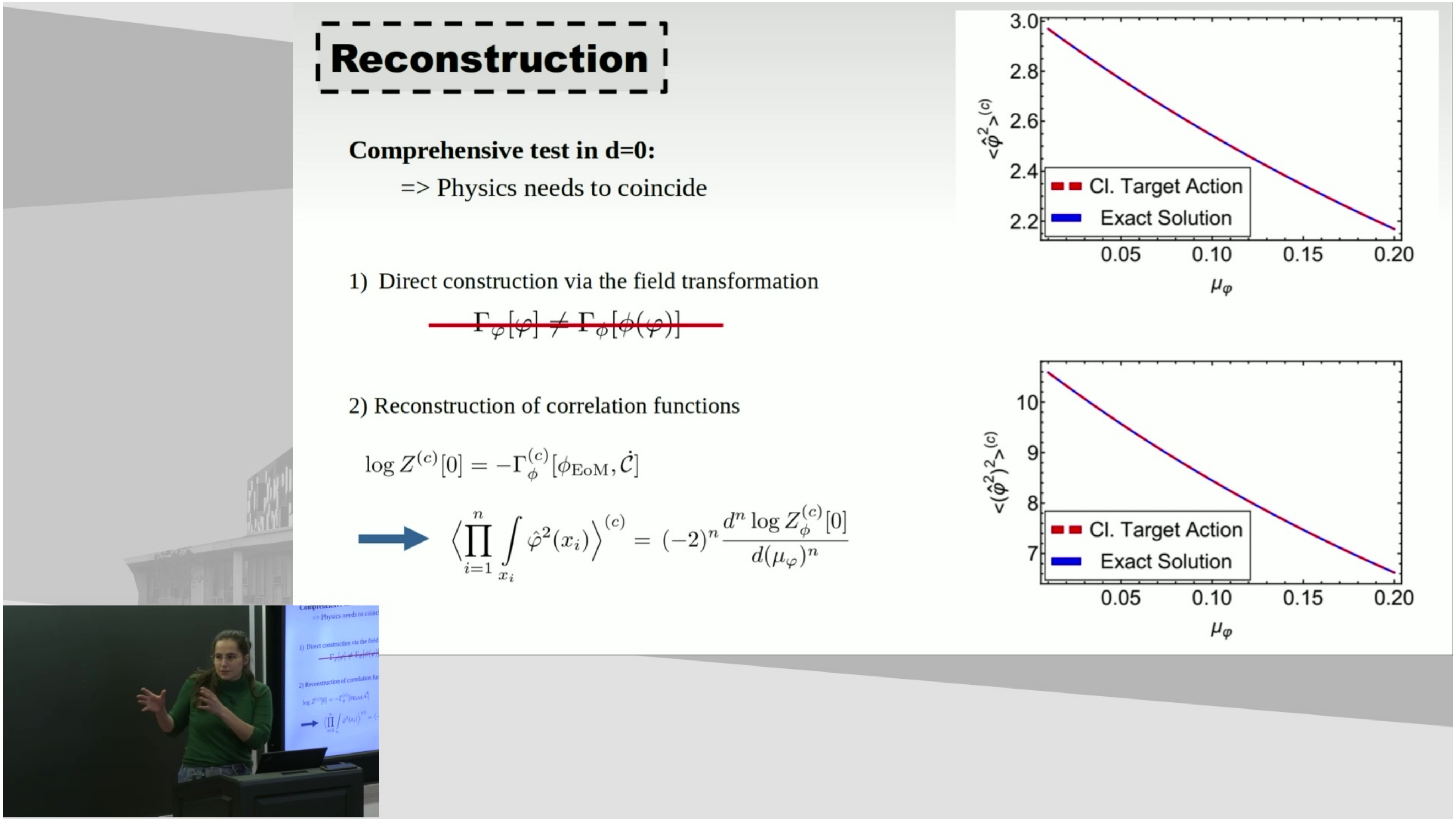
$$\boxed{\dot{\phi}\, V_T^{(1)} - \frac{2}{d+2}\dot{\phi}' \frac{A_d k^{d+2}}{k^2 + V_T^{(2)}}} = \boxed{\frac{A_d k^{d+2}}{k^2 + V_T^{(2)}} - \partial_t V_T}$$

2) Global existence of $\phi_k[\varphi]$

$$\phi[\varphi] = \varphi + \int_\Lambda^0 \frac{dk}{k}\, \dot{\phi}[\phi]$$

=> Potentials do not coincide in d=0!

$\Gamma_\varphi[\varphi] \neq \Gamma_\phi[\phi]$ ⟶ The pair is PI $(\Gamma_\phi, \phi[\varphi])$



Reconstruction

Comprehensive test in d=0:

=> Physics needs to coincide

1) Direct construction via the field transformation

~~$\Gamma_\varphi[\varphi] \neq \Gamma_\phi[\phi(\varphi)]$~~

2) Reconstruction of correlation functions

$$\log Z^{(c)}[0] = -\Gamma_\phi^{(c)}[\phi_{\mathrm{EoM}}, \dot{\mathcal{C}}]$$

$$\rightarrow \left\langle \prod_{i=1}^{n} \int_{x_i} \hat{\varphi}^2(x_i) \right\rangle^{(c)} = (-2)^n \frac{d^n \log Z_\phi^{(c)}[0]}{d(\mu_\varphi)^n}$$

Directions for Optimisation

Convergence of expansion

Example: 1st order derivative expansion

$$\partial_t Z_\phi[\phi](p) \overset{!}{=} 0 \quad \longrightarrow \quad \dot{\phi}_k[\phi]$$

Ground-state expansion:

$$\dot{\phi} = -\frac{1}{2}\eta_\phi(\rho)\,\phi$$

Composite
↔
Physical field

$$\Gamma^{(2)}_{\phi\phi} = p^2 + m^2_{\text{phys}}(\rho)$$

No further Reconstruction

Computational complexity

$$\Gamma_T[\phi] = \int_x \left\{ \frac{1}{2}\left(\partial_\mu \phi^a\right)^2 + V_T(\phi) \right\}$$

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Feed-down flow:

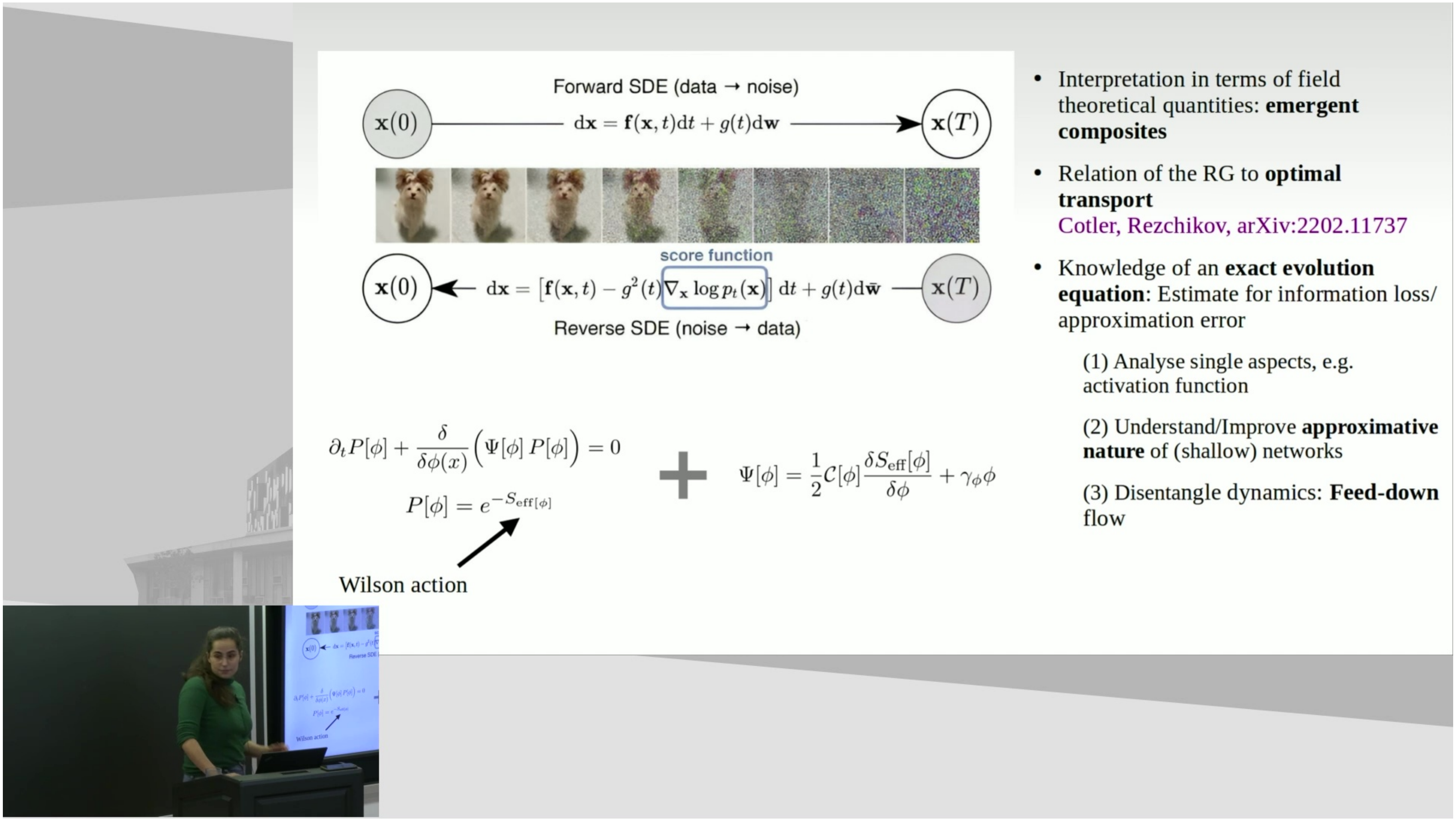
First order

$$\dot{\phi} = \boxed{\phi\left(\partial_\mu \phi\right)^2 f(\rho)}$$

Composite
↔
Computational tool

$$\Gamma^{(2)}_{\phi\phi} = p^2 + m^2_{\mathrm{LPA}}(\rho)$$

Reconstruction



Forward SDE (data → noise)

$$\mathbf{x}(0) \longrightarrow \mathrm{d}\mathbf{x} = \mathbf{f}(\mathbf{x},t)\mathrm{d}t + g(t)\mathrm{d}\mathbf{w} \longrightarrow \mathbf{x}(T)$$

score function

$$\mathbf{x}(0) \longleftarrow \mathrm{d}\mathbf{x} = \left[\mathbf{f}(\mathbf{x},t) - g^2(t)\boxed{\nabla_{\mathbf{x}} \log p_t(\mathbf{x})}\right]\mathrm{d}t + g(t)\mathrm{d}\bar{\mathbf{w}} \longleftarrow \mathbf{x}(T)$$

Reverse SDE (noise → data)

$$\partial_t P[\phi] + \frac{\delta}{\delta\phi(x)}\Big(\Psi[\phi]\, P[\phi]\Big) = 0$$

$$P[\phi] = e^{-S_{\mathrm{eff}}[\phi]}$$

Wilson action

$$+$$

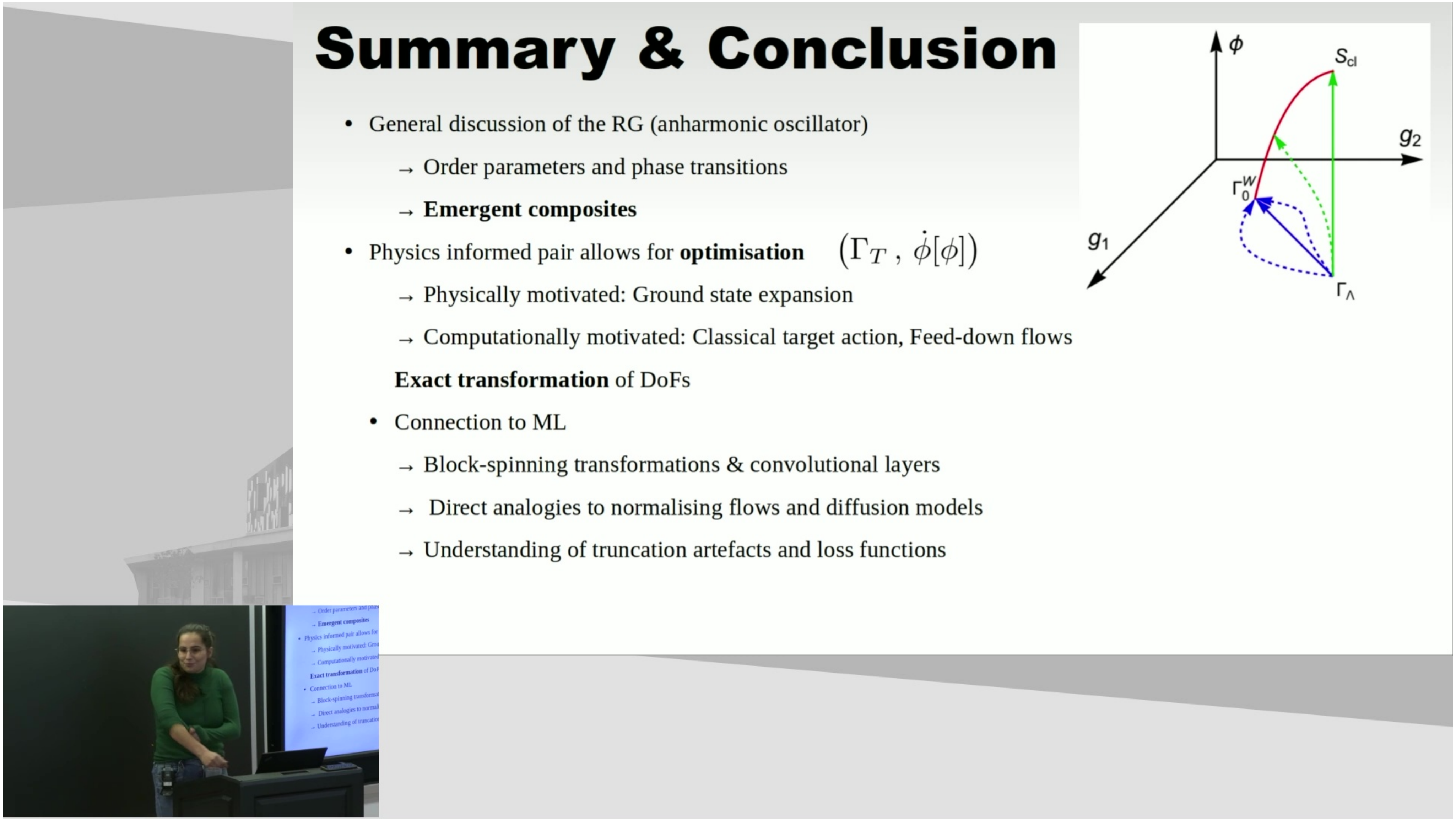
$$\Psi[\phi] = \frac{1}{2}\mathcal{C}[\phi]\frac{\delta S_{\mathrm{eff}}[\phi]}{\delta\phi} + \gamma_\phi \phi$$

- Interpretation in terms of field theoretical quantities: **emergent composites**
- Relation of the RG to **optimal transport**
  Cotler, Rezchikov, arXiv:2202.11737
- Knowledge of an **exact evolution equation**: Estimate for information loss/ approximation error

  (1) Analyse single aspects, e.g. activation function

  (2) Understand/Improve **approximative nature** of (shallow) networks

  (3) Disentangle dynamics: **Feed-down** flow

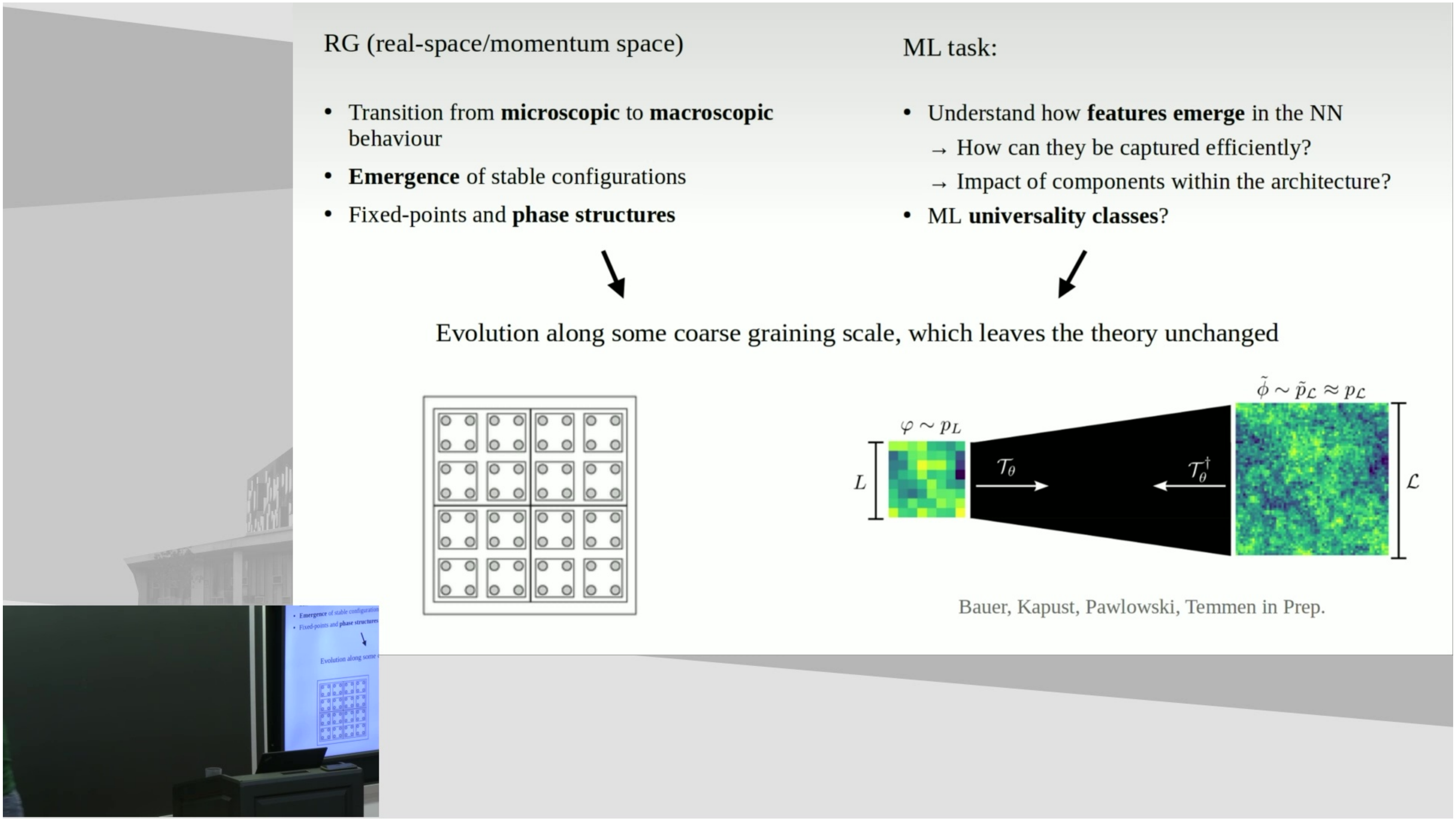


Summary & Conclusion

- General discussion of the RG (anharmonic oscillator)
  - → Order parameters and phase transitions
  - → **Emergent composites**
- Physics informed pair allows for **optimisation** $\left(\Gamma_T\, ,\ \dot{\phi}[\phi]\right)$
  - → Physically motivated: Ground state expansion
  - → Computationally motivated: Classical target action, Feed-down flows

Exact transformation of DoFs

- Connection to ML
  - → Block-spinning transformations & convolutional layers
  - → Direct analogies to normalising flows and diffusion models
  - → Understanding of truncation artefacts and loss functions



RG (real-space/momentum space)

- Transition from **microscopic** to **macroscopic** behaviour
- **Emergence** of stable configurations
- Fixed-points and **phase structures**

ML task:

- Understand how **features emerge** in the NN
  - → How can they be captured efficiently?
  - → Impact of components within the architecture?
- ML **universality classes**?

Evolution along some coarse graining scale, which leaves the theory unchanged

Bauer, Kapust, Pawlowski, Temmen in Prep.