Title: Askey-Wilson algebra, Chern-Simons theory and link invariants

Speakers: Meri Zaimi

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Abstract:

Chern-Simons theory is a topological quantum field theory which leads to link invariants, such as the Jones polynomial, through the expectation values of Wilson loops. The same link invariants also appear in a mathematical construction of Reshetikhin and Turaev which uses a trace on Yang-Baxter operators. Several algebraic structures are involved in these frameworks for computing link invariants, including the braid group, quantum algebras and centralizer algebras (such as the Temperley-Lieb algebra). In this talk, I will explain how the Askey-Wilson algebra, originally introduced in the context of orthogonal polynomials, can also be understood within the Chern-Simons theory and the Reshetikhin-Turaev link invariant construction.

Askey–Wilson algebra, Chern–Simons theory and link invariants

Meri Zaimi

Perimeter Institute for Theoretical Physcis

based on joint work with

Nicolas Crampé (CNRS, Université de Tours, LAPTh)

Luc Vinet (Université de Montréal)

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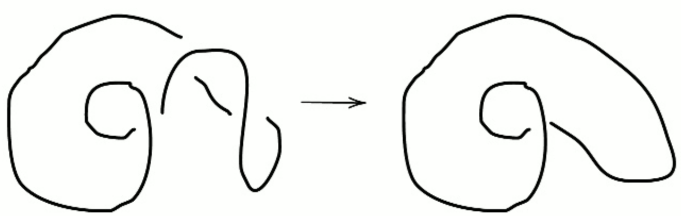
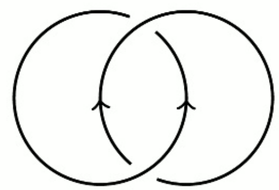
Knot theory

Knot: smooth embedding of S^1 in $\mathbb{R}^3$.

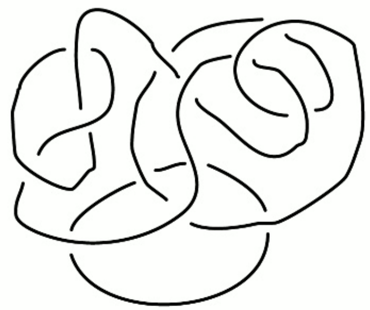
Equivalent if related by smooth deformations in $\mathbb{R}^3$. (ambiant isotopy)



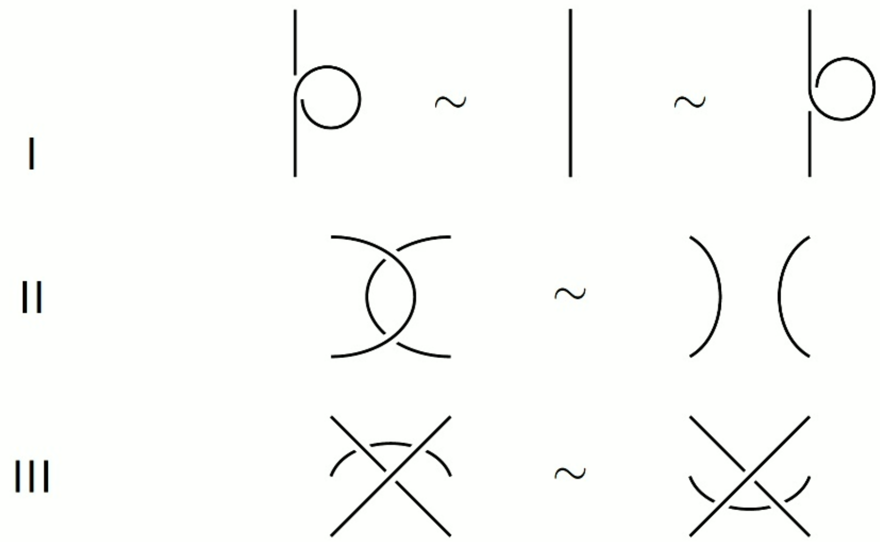
Link: finite union of non-intersecting knots.



Hopf link



Reidemeister moves (RM):



Ambiant isotopy: Two link diagrams represent equivalent links iff they are related by a finite sequence of planar isotopies and RMs.
Regular isotopy: use only RM II and III. (Adapted for *framed* links.)

In practice, define a **link invariant**.
A mapping $L \mapsto I(L)$ such that if $L_1 \sim L_2$ then $I(L_1) = I(L_2)$.

Jones polynomial $V(L; q^{\frac{1}{2}})$ is a link invariant, Laurent polynomial in $q^{\frac{1}{2}}$.

- Originally discovered using the Temperley–Lieb algebra.
- Three-dimensional interpretation through Chern–Simons theory with gauge group $SU(2)$ in fundamental irrep.
- Reshetikhin–Turaev link invariant construction with quantum algebra $U_q(\mathfrak{sl}_2)$ (Yang–Baxter representations of the braid group).

Other interesting polynomial link invariants:

- HOMFLYPT polynomial
  (fundamental irrep $SU(N)$, Hecke algebra)
- Kauffman polynomial
  (fundamental irrep $SO(N)$, Birman–Murakami–Wenzl algebra)

TL, Hecke, BMW: centralizers of quantum algebras $U_q(\mathfrak{g})$.

Askey–Wilson algebra

- Askey–Wilson polynomials : family of q-hypergeometric orthogonal polynomials, on top of the q-Askey scheme.
- Askey–Wilson algebra encodes bispectrality of these polynomials:

$$\mu(x)P_n(\mu(x)) = a_n P_{n+1}(\mu(x)) + b_n P_n(\mu(x)) + c_n P_{n-1}(\mu(x)),$$
$$\lambda_n P_n(\mu(x)) = A(x)P_n(\mu(x+1)) + B(x)P_n(\mu(x)) + C(x)P_n(\mu(x-1)).$$

- Recoupling of three spin representations for $U_q(\mathfrak{sl}_2)$

$$j_1 \otimes j_2 \otimes j_3 = (j_1 \otimes j_2) \otimes j_3 = j_1 \otimes (j_2 \otimes j_3)$$

  $6j$-symbols given in terms of q-Racah polynomials.
- Realization in diagonal centralizer of $U_q(\mathfrak{sl}_2)$ in $U_q(\mathfrak{sl}_2)^{\otimes 3}$.
- Quotients of braid group (e.g. TL, BMW) as quotients of AW.
- Connected to Kauffman skein algebra of links in punctured surfaces.

Objective: Interpretation of AW algebra in CS and RT frameworks.

The Askey–Wilson algebra is the unital associative algebra generated by C_{12}, C_{23}, C_{13} and central elements C_1, C_2, C_3, C_{123} subject to the defining relations

$$[C_{12}, C_{23}]_q + (q^2 - q^{-2})C_{13} = (q - q^{-1})(C_1 C_3 + C_2 C_{123}),$$
$$[C_{23}, C_{13}]_q + (q^2 - q^{-2})C_{12} = (q - q^{-1})(C_1 C_2 + C_3 C_{123}),$$
$$[C_{13}, C_{12}]_q + (q^2 - q^{-2})C_{23} = (q - q^{-1})(C_2 C_3 + C_1 C_{123}),$$

where $q \in \mathbb{C}$ is not root of unity and $[X, Y]_q = qXY - q^{-1}YX$.

There is a Casimir element:

$$\Omega = qC_{12}C_{23}C_{13} + q^2 C_{12}^2 + q^{-2} C_{23}^2 + q^2 C_{13}^2 - qC_{12}(C_1 C_2 + C_3 C_{123}) - q^{-1} C_{23}(C_2 C_3 + C_1 C_{123}) - qC_{13}(C_1 C_3 + C_2 C_{123}).$$

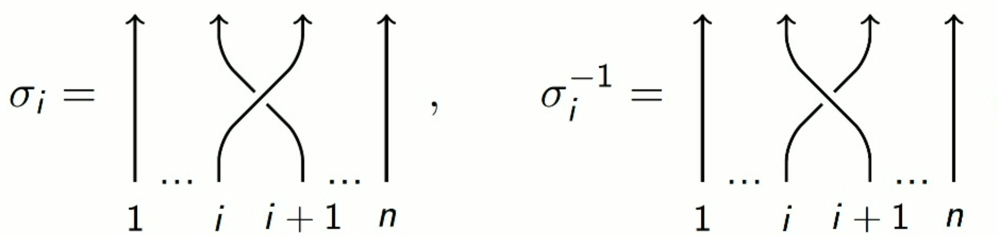
The special Askey–Wilson **saw**(3) is the quotient by the relation

$$\Omega = (q + q^{-1})^2 - C_{123}^2 - C_1^2 - C_2^2 - C_3^2 - C_1 C_2 C_3 C_{123}.$$

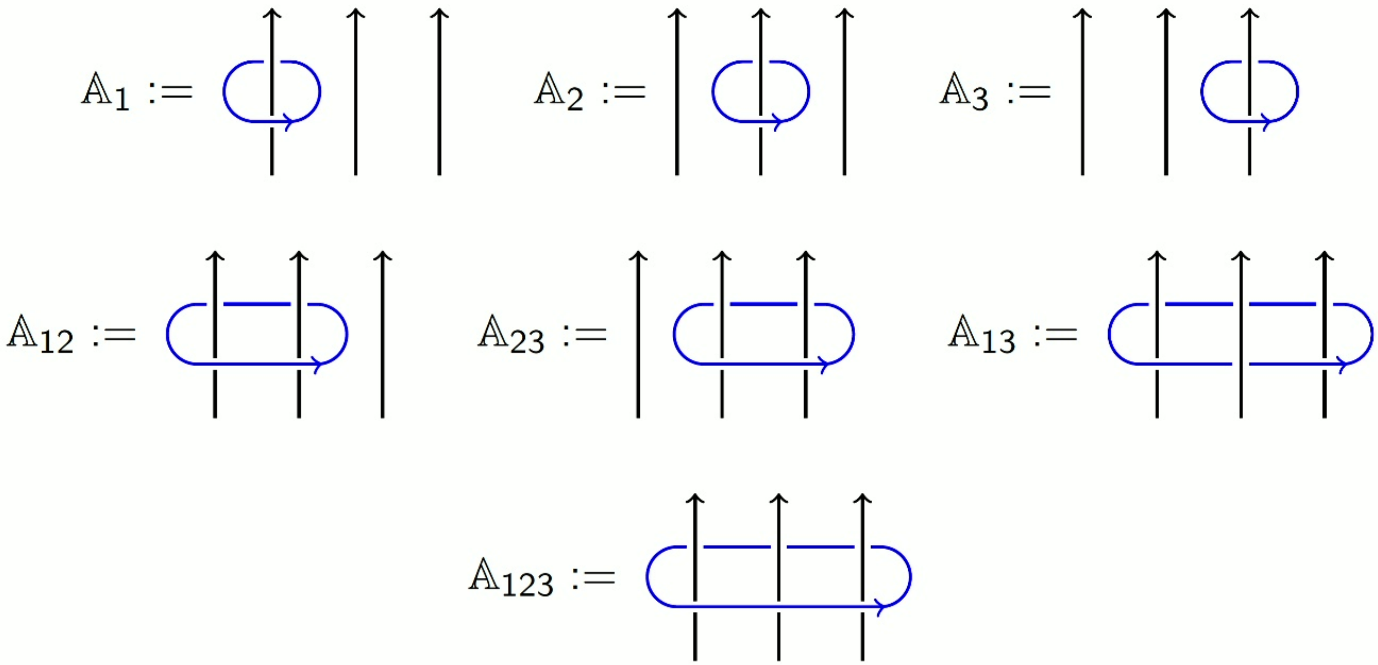
Tangle diagrams

Oriented tangle diagrams of type $(3,3)$, coloured with elements of $\frac{1}{2}\mathbb{Z}_{\geq 0}$.
$X = Y$ if the tangle diagrams X and Y are regular isotopic.
XY is the vertical concatenation of the diagram X on top of Y.

Braid diagrams:



We will consider $n = 3$.

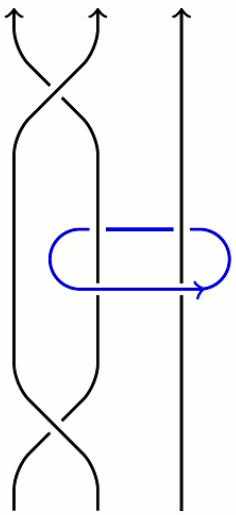
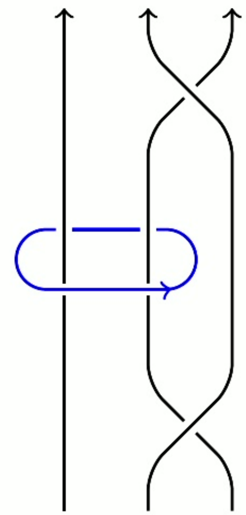


Three strands coloured by j_1, j_2, j_3, loop coloured by $1/2$.

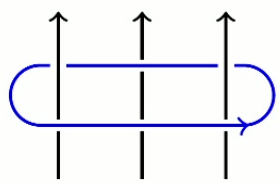
We will argue with two different approaches that there is a correspondence

$$C_I \mapsto \mathbb{A}_I, \qquad \forall I \in \{1, 2, 3, 12, 23, 13, 123\}.$$

Note that:

$\mathbb{A}_{13} = \sigma_1 \mathbb{A}_{23} \sigma_1^{-1} =$  $=$  $= \sigma_2^{-1} \mathbb{A}_{12} \sigma_2.$

Also note:

$\widetilde{\mathbb{A}}_{13} :=$  $= \sigma_2 \mathbb{A}_{12} \sigma_2^{-1} = \sigma_1^{-1} \mathbb{A}_{23} \sigma_1.$

Plan

1. Chern–Simons theory

2. Reshetikhin–Turaev construction

Chern–Simons action

Spacetime $\mathcal{M}$ of dimension $1+2$.

Lie algebra $\mathfrak{g}$ with generators T^a, associated to Lie group G.

Gauge potential $A = \sum_\mu A_\mu dx^\mu$ with $A_\mu(x) = \sum_a A_\mu^a(x) T^a$.

Action:

$$S_{CS} = \frac{\kappa}{4\pi} \int_{\mathcal{M}} \mathrm{Tr}\left(A \wedge dA + \frac{2i}{3} A \wedge A \wedge A \right).$$

In this talk, $\mathcal{M} = \mathbb{R}^3$.

Wilson loops

Appropriate observables for quantum CS are Wilson loops:

$$W(K,\rho) = \mathrm{Tr}\left[P\exp\left(i\oint_K A^a_\mu T^a_{(\rho)} dx^\mu\right)\right],$$

K oriented knot in $\mathbb{R}^3$ associated to (coloured by) an irrep ρ of $\mathfrak{g}$.

Product of Wilson loops:

$$W(L) = \prod_i W(K_i, \rho_i),$$

L is a link with each component K_i associated to an irrep ρ_i.

Vacuum expectation value:

$$\langle W(L)\rangle = \frac{\int \mathcal{D}A\ W(L) e^{iS_{CS}}}{\int \mathcal{D}A\ e^{iS_{CS}}}.$$

Abelian case, gauge group $U(1)$:

$$\langle W(L)\rangle = \exp\left(-i\frac{2\pi}{\kappa}\sum_{i,j} n_i n_j \chi(K_i, K_j)\right),$$

$\chi(K_i, K_j)$ is the linking number $=$ number of times K_i winds around K_j.

Framing of knot K: continuous and nowhere vanishing vector field which is normal to K.

Vertical framing (VF): vector field is perpendicular to projection plane.

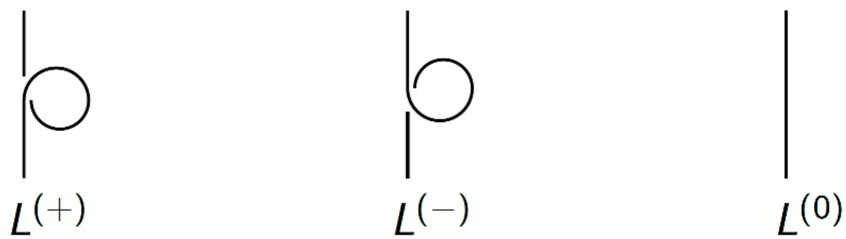
Of interest for us: gauge group $SU(2)$, spin irreps $\rho_i \to j_i \in \{0, \frac{1}{2}, 1, \dots\}$.
Define $I_{CS}(L) := \langle W(L) \rangle_{VF}$.

- I_{CS} is an ambiant isotopy invariant for oriented, coloured, framed links in $\mathbb{R}^3$, and a regular isotopy invariant of oriented and coloured link diagrams.
- Can be expressed in terms of the deformation parameter

$$q = \exp\left(-\frac{i\pi}{\kappa}\right).$$

- Change of framing:

$$I_{CS}(L^{(\pm)}; j) = q^{\pm 2j(j+1)} I_{CS}(L^{(0)}; j).$$

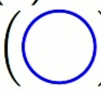
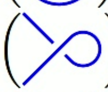
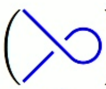
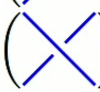
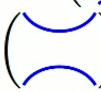
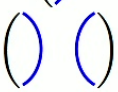


- When all components of L carry spin $1/2$,

$$I_{CS}(L) = \exp\left(-\frac{i\pi}{2}w(L)\right) V_B(L; iq^{\frac{1}{2}}).$$

Writhe $w(L) = \#$ crossings (with signs).

Bracket polynomial $V_B(L; x)$ for a non-oriented link L is uniquely defined by

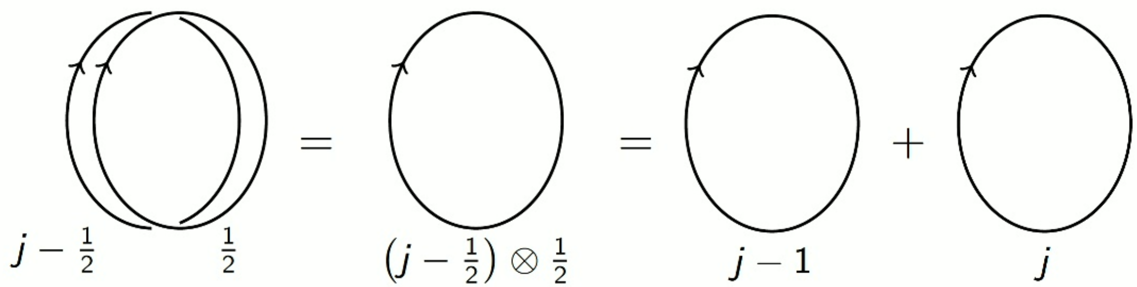
- $V_B(L) = V_B(L')$ if L and L' are regular isotopic;
- $V_B\Big($  $\Big) = -(x^2 + x^{-2})$;
- $V_B\Big($  $\Big) = -x^3 V_B\Big($  $\Big)$, $\quad V_B\Big($  $\Big) = -x^{-3} V_B\Big($  $\Big)$;
- $V_B\Big($  $\Big) = xV_B\Big($  $\Big) + x^{-1}V_B\Big($  $\Big)$.

With some renormalization, $V_B(L; iq^{\frac{1}{2}}) \sim V(L; q^{\frac{1}{2}})$.

What if not all components carry spin 1/2?

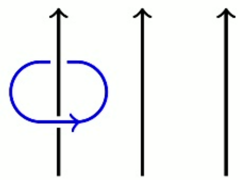
We can always compute $I_{CS}(L)$ using links with spins 1/2, at the cost of adding components.

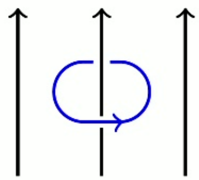
Fusion property of Wilson loops and $\mathfrak{su}(2)$ tensor product decomposition rule:

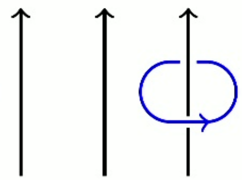


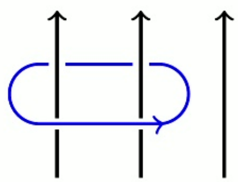
Spin 0 loops can be removed from computation.

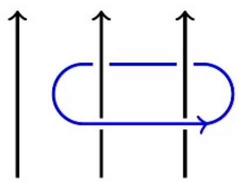
Recall diagrams

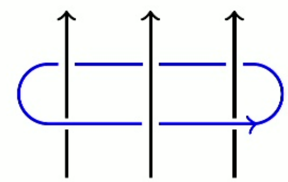
$\mathbb{A}_1 :=$ 

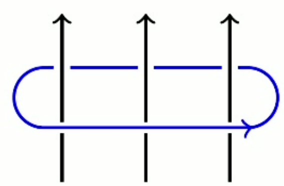
$\mathbb{A}_2 :=$ 

$\mathbb{A}_3 :=$ 

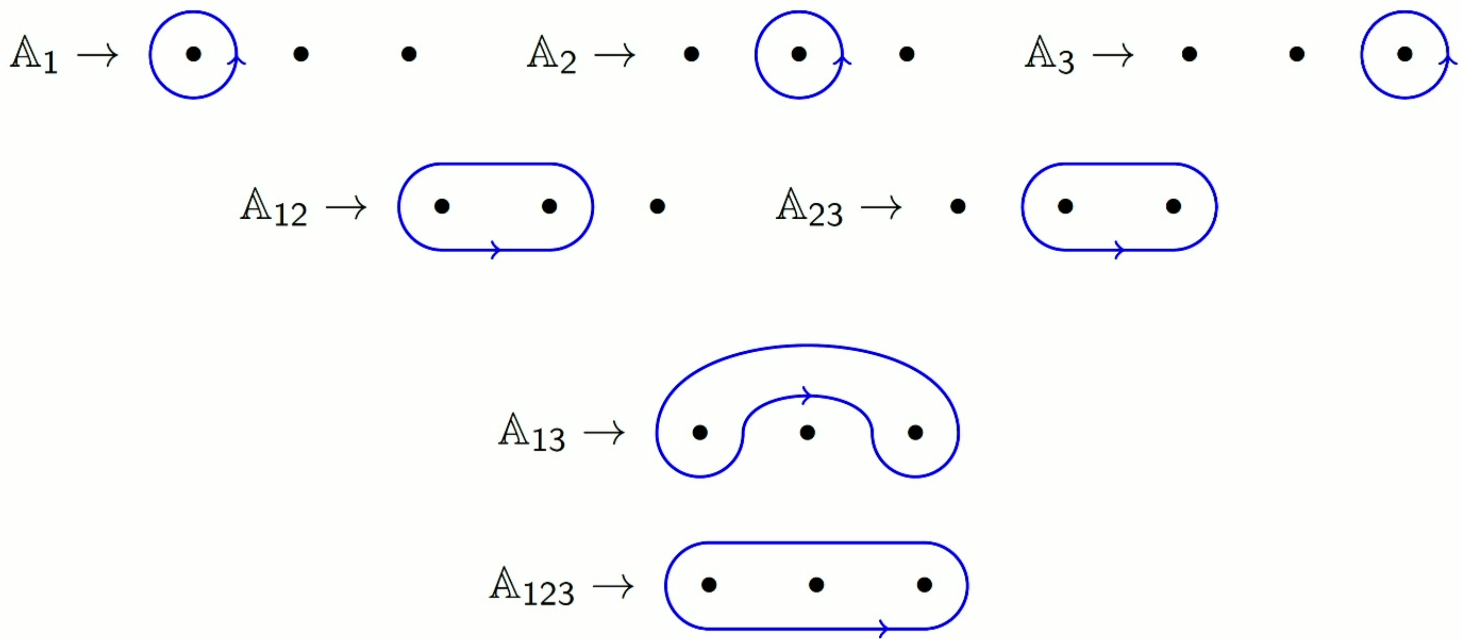
$\mathbb{A}_{12} :=$ 

$\mathbb{A}_{23} :=$ 

$\mathbb{A}_{13} :=$ 

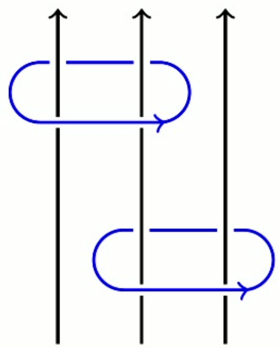
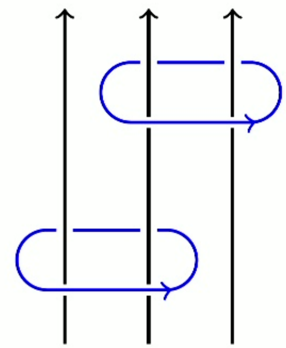
$\mathbb{A}_{123} :=$ 

View from the top:

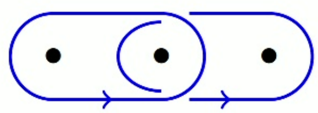
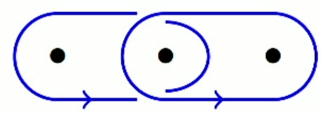


Punctures with spins j_1, j_2, j_3 enclosed by loops with spin $1/2$.

Consider following products

$\mathbb{A}_{12}\mathbb{A}_{23} =$  $\mathbb{A}_{23}\mathbb{A}_{12} =$ 

View from the top:

$\mathbb{A}_{12}\mathbb{A}_{23} \to$  $\mathbb{A}_{23}\mathbb{A}_{12} \to$ 

Result: The expectation values of the AW diagrams in CS theory satisfy the relations of the AW algebra.

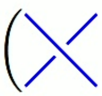
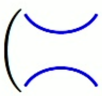
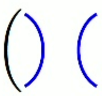
Idea:

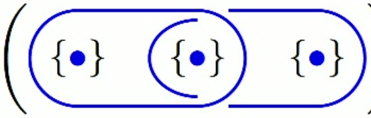
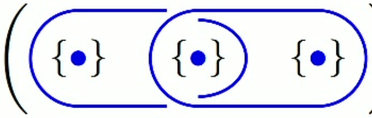
1. $\underset{j_1}{\bullet} \quad \underset{j_2}{\bullet} \quad \underset{j_3}{\bullet} \quad \to \quad \{\bullet\} \quad \{\bullet\} \quad \{\bullet\}$

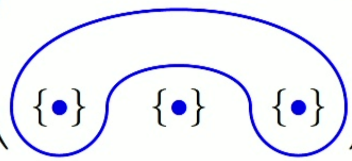
$\{\bullet\}$: set of punctures with spin 1/2.

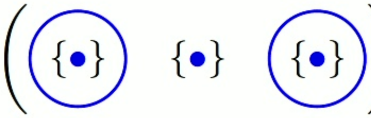
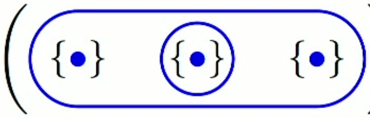
2. Can compute bracket polynomial $V_B(L; iq^{\frac{1}{2}})$.

Simplify crossings using the rule:

$V_B\Big($  $; iq^{\frac{1}{2}}\Big) = iq^{\frac{1}{2}} V_B\Big($  $; iq^{\frac{1}{2}}\Big) - iq^{-\frac{1}{2}} V_B\Big($  $; iq^{\frac{1}{2}}\Big)$

qV_B () $- q^{-1}V_B$ ()

$+ (q^2 - q^{-2})V_B$ ()

$= (q - q^{-1})\Big\{ V_B$ () $+ V_B$ () $\Big\}$

3. Recall

$$I_{\mathrm{CS}}(L) = \exp\left(-\frac{i\pi}{2}w(L)\right) V_B(L; iq^{\frac{1}{2}}).$$

Turns out that all exponential factors simplify.

4. $\{\bullet\} \quad \{\bullet\} \quad \{\bullet\} \quad \rightarrow \quad \underset{j_1}{\bullet} \quad \underset{j_2}{\bullet} \quad \underset{j_3}{\bullet}$

5. Compare with

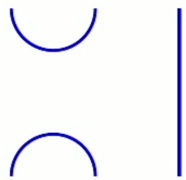
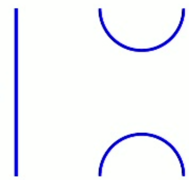
$$[C_{12}, C_{23}]_q + (q^2 - q^{-2})C_{13} = (q - q^{-1})(C_1 C_3 + C_2 C_{123})$$

Connection with Temperley–Lieb algebra

The Temperley–Lieb algebra $TL_3(q)$ is generated by e_1 and e_2 with the following defining relations

$$e_1^2 = (q + q^{-1})e_1, \quad e_2^2 = (q + q^{-1})e_2,$$
$$e_1 e_2 e_1 = e_1, \quad e_2 e_1 e_2 = e_2.$$

Hook diagrams:

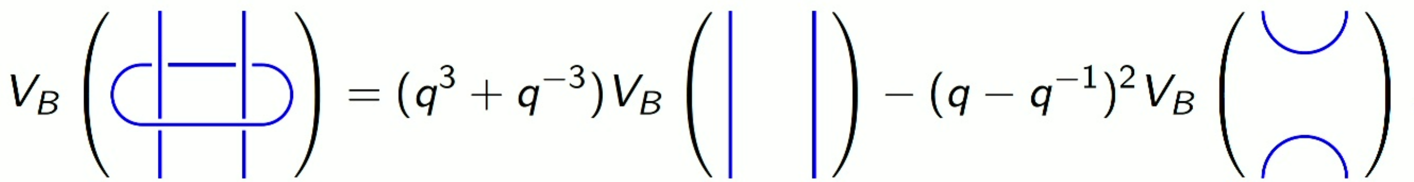
$E_1 =$  , $\quad E_2 =$  .

Bracket polynomial $V_B(L; x = iq^{\frac{1}{2}})$ of the diagrams E_1, E_2 satisfy the defining relations of $TL_3(q)$, with $e_i \mapsto E_i$, for $i = 1, 2$.

In a quotient of the Askey–Wilson algebra, there is an isomorphism

$$C_{12} \mapsto (q^3 + q^{-3}) - (q - q^{-1})^2 e_1,$$
$$C_{23} \mapsto (q^3 + q^{-3}) - (q - q^{-1})^2 e_2.$$

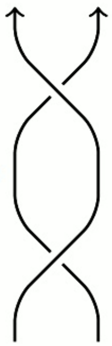
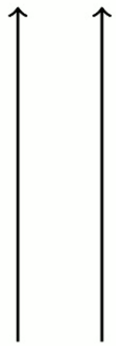
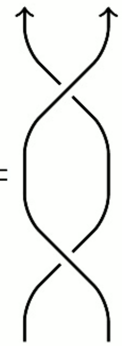
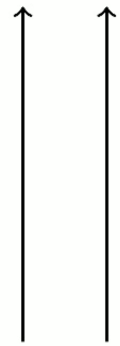
Here, we have a diagrammatic interpretation:

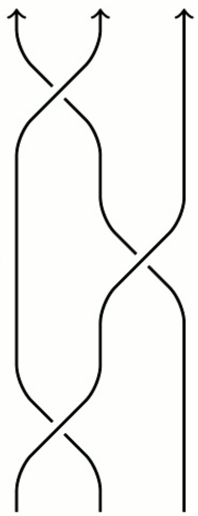
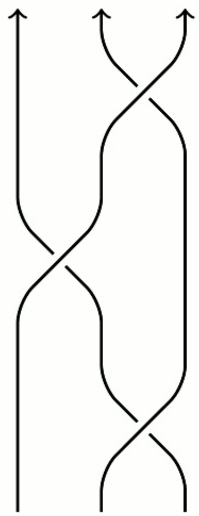


Plan

1. Chern–Simons theory

2. Reshetikhin–Turaev construction

$\sigma_1^{-1}\sigma_1 =$  $=$  $= 1,$ $\quad \sigma_1\sigma_1^{-1} =$  $=$  $= 1.$

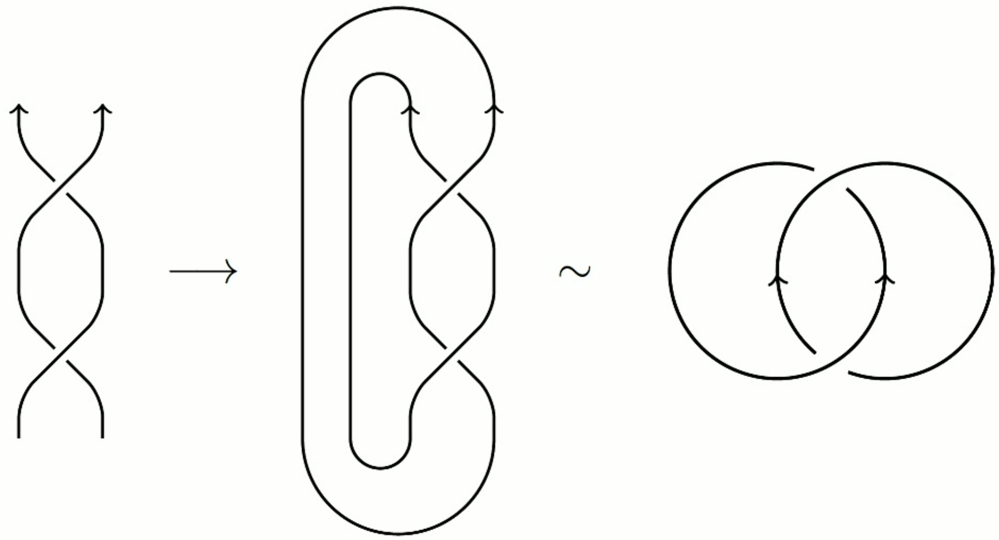
$\sigma_1\sigma_2\sigma_1 =$  $=$  $= \sigma_2\sigma_1\sigma_2$

These correspond to RM of types II and III.

Defining relations of the braid group:

$$\sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{if } |i-j|>1$$
$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}.$$

The closure of a braid is a link.



Alexander's theorem: any link can be represented as the closure of a braid.

Idea: define a mapping $L \mapsto I(L)$ using braid representations of links.

Quantum algebras

$U_q(\mathfrak{sl}_2)$ is the unital associative algebra generated by E, F and q^H, with

$$q^H E = qEq^H, \quad q^H F = q^{-1}Fq^H, \quad [E, F] = [2H]_q,$$

where $[X, Y] = XY - YX$ and $[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}}$.

Casimir element generates the center:

$$Q = (q - q^{-1})^2 FE + q^{2H+1} + q^{-2H-1}.$$

For q not root of unity, finite-dimensional irreps V_j of dimension $2j + 1$.

Quasi-triangular Hopf algebra structure.

Comultiplication $\Delta : U_q(\mathfrak{sl}_2) \to U_q(\mathfrak{sl}_2) \otimes U_q(\mathfrak{sl}_2)$.

Universal R-matrix: invertible element $\mathcal{R} \in U_q(\mathfrak{sl}_2) \otimes U_q(\mathfrak{sl}_2)$ s.t.

$$\Delta^{op}(x) = \mathcal{R}\Delta(x)\mathcal{R}^{-1} \quad \forall x \in U_q(\mathfrak{sl}_2),$$

and

$$(\mathrm{id} \otimes \Delta)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{12}, \qquad (\Delta \otimes \mathrm{id})(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{23}.$$

Quantum Yang–Baxter equation:

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}.$$

Explicit expression:

$$\mathcal{R} = \sum_{k=0}^{\infty} \frac{(q - q^{-1})^k}{[k]_q!} q^{-k(k+1)/2} (F \otimes E)^k (q^{kH} \otimes q^{-kH}) q^{2(H \otimes H)}.$$

Define $\mu := q^{2H}$.

Reshetikhin–Turaev link invariant construction

Consider tensor product

$$V = V_{j_1} \otimes V_{j_2} \otimes \cdots \otimes V_{j_n}.$$

Braided universal R-matrix $\check{\mathcal{R}}_{i,i+1} := \Pi_{i,i+1}\mathcal{R}_{i,i+1}$. ($\Pi_{i,i+1}$ transpositions)
We can show that

$$\check{\mathcal{R}}_{i,i+1}\check{\mathcal{R}}_{j,j+1} = \check{\mathcal{R}}_{j,j+1}\check{\mathcal{R}}_{i,i+1} \quad \text{if } |i-j| > 1,$$
$$\check{\mathcal{R}}_{i,i+1}\check{\mathcal{R}}_{i+1,i+2}\check{\mathcal{R}}_{i,i+1} = \check{\mathcal{R}}_{i+1,i+2}\check{\mathcal{R}}_{i,i+1}\check{\mathcal{R}}_{i+1,i+2}.$$

$$\sigma_i \mapsto \check{\mathcal{R}}_{i,i+1}.$$

$$\check{\mathcal{R}}(\mu \otimes \mu) = (\mu \otimes \mu)\check{\mathcal{R}}.$$

$$\mathrm{Tr}_1^{(j)}(\check{\mathcal{R}}^{\pm 1}(\mu \otimes 1)) = q^{\pm 2j(j+1)}\mathrm{id},$$
$$\mathrm{Tr}_2^{(j)}(\check{\mathcal{R}}^{\pm 1}(1 \otimes \mu^{-1})) = q^{\pm 2j(j+1)}\mathrm{id}.$$

Any link L is the closure of some braid $\sigma(L)$.

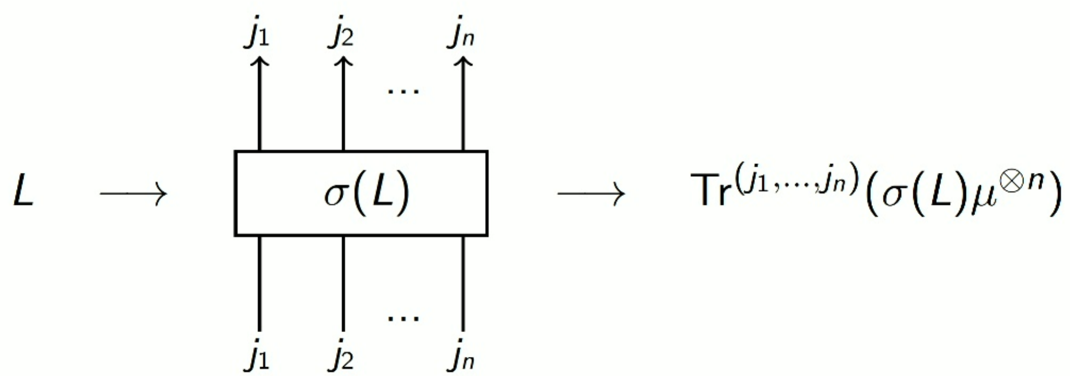
Associate to L the "quantum trace" of the braid $\sigma(L)$:

$$L \mapsto \mathrm{Tr}_q(\sigma(L)) = \mathrm{Tr}^{(j_1,\ldots,j_n)}(\sigma(L)\mu^{\otimes n}).$$

Using Markov's theorem, can show that the map

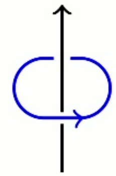
$$L \mapsto I_{\mathrm{RT}}(L) := \mathrm{Tr}_q(\sigma(L))$$

defines an invariant of regular isotopy.

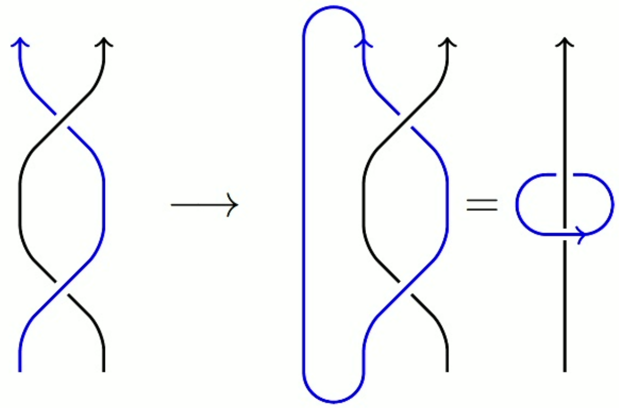


Askey–Wilson algebra in RT construction

Consider the following part of AW diagram:



It corresponds to the partial closure of a braid:

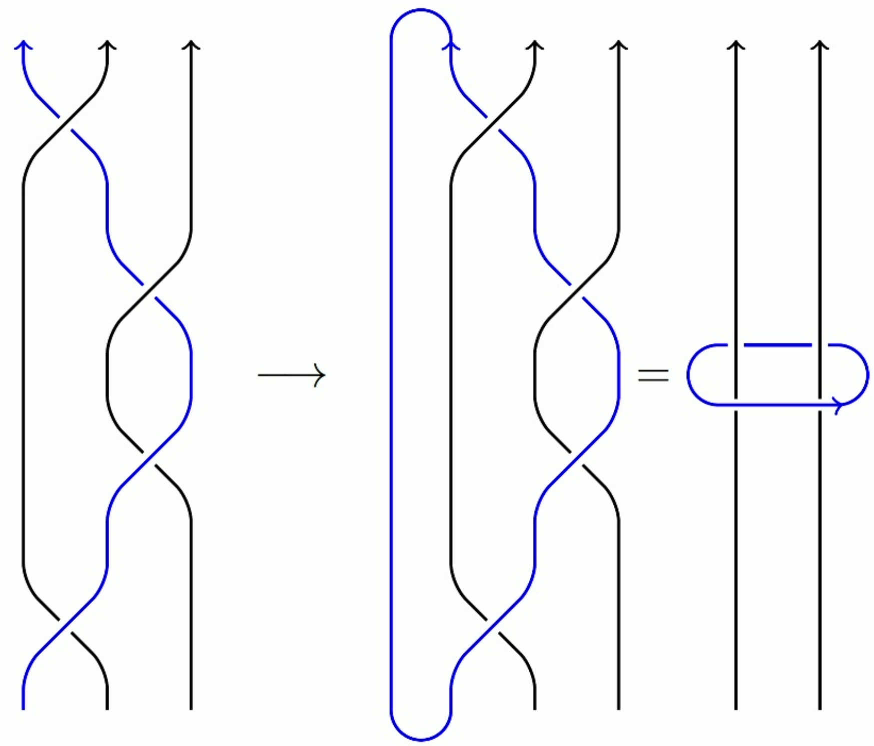


Can compute explicitly the associated partial trace, we find:

$$\mathrm{Tr}_1^{(\frac{1}{2})}(\sigma_1^2(\mu \otimes 1)) = \mathrm{Tr}_1^{(\frac{1}{2})}(\check{\mathcal{R}}_{12}^2(\mu \otimes 1)) = Q \in U_q(\mathfrak{sl}_2)$$

with Q the Casimir element of $U_q(\mathfrak{sl}_2)$.

Similarly



$$\mathrm{Tr}_1^{(\frac{1}{2})}(\check{\mathcal{R}}_{12}\check{\mathcal{R}}_{23}^2\check{\mathcal{R}}_{12}(\mu \otimes 1 \otimes 1)) = \Delta(Q) \in U_q(\mathfrak{sl}_2)^{\otimes 2}$$

In the RT link invariant construction, the AW diagrams are associated to the intermediate Casimir elements of $U_q(\mathfrak{sl}_2)^{\otimes 3}$:

$$Q_1 = Q \otimes 1 \otimes 1, \quad Q_2 = 1 \otimes Q \otimes 1, \quad Q_3 = 1 \otimes 1 \otimes Q,$$
$$Q_{12} = \Delta(Q) \otimes 1, \quad Q_{23} = 1 \otimes \Delta(Q),$$
$$Q_{13} = \check{\mathcal{R}}_{23}^{-1} Q_{12} \check{\mathcal{R}}_{23} = \check{\mathcal{R}}_{12} Q_{23} \check{\mathcal{R}}_{12}^{-1},$$
$$Q_{123} = (\mathrm{id} \otimes \Delta) \circ \Delta(Q).$$

These intermediate Casimir elements belong to the centralizer of $U_q(\mathfrak{sl}_2)$ in $U_q(\mathfrak{sl}_2)^{\otimes 3}$ and are known to satisfy the AW algebra.

Summary

- CS observables (Wilson loops) lead to link invariants.
- $SU(2)$ gauge group on $\mathbb{R}^3$: recovered Askey–Wilson algebra using connection with bracket polynomial and properties of Wilson loops.
- RT construction : take trace over Yang–Baxter representations of the braid group together with an enhancement to obtain link invariants.
- Quantum group $U_q(\mathfrak{sl}_2)$: recovered AW algebra by computing partial traces and recognizing them as intermediate Casimir elements.

Perspectives

- n strands : higher rank Askey–Wilson algebra $AW(n)$ and $U_q(\mathfrak{sl}_2)^{\otimes n}$
- Other gauge groups/quantum algebras? E.g. $U_q(\mathfrak{sl}_N)$.
- Different manifolds? S^3 and q root of unity?