Title: Operator algebras and conformal field theory

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Abstract:

The operator algebraic approach to quantum field theory is called algebraic quantum field theory. In this setting, we consider a family of operator algebras generated by observables in spacetime regions. This has been particularly successful in 2-dimensional conformal field theory. We present roles of tensor categories, modular invariance, classification theory, induction machinery and connections to vertex operator algebras.

Operator algebras and conformal field theory

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Operator algebraic approach to chiral conformal field theory

Interactions between two approaches to chiral conformal field theory: vertex operator algebras and conformal nets.

Outline of the talk:

1. Moonshine and vertex operator algebras
2. Quantum fields and chiral conformal field theory
3. Conformal nets
4. Representation theory
5. From vertex operator algebras to conformal nets and back
6. Comparison of representation theories
7. Full conformal field theory and modular invariance

The Monster and the j-function

The Monster group is one of the 26 sporadic finite simple groups and has the largest order, around 8×10^{53}, among them.

The following function, called j-function, has been classically studied.

$$j(\tau) = q^{-1} + 744 + 196884q + 21493760q^2 + 864299970q^3 + \cdots$$

For $q = \exp(2\pi i\tau)$, $\mathrm{Im}\,\tau > 0$, this is characterized by modular invariance property, $j(\tau) = j\left(\frac{a\tau + b}{c\tau + d}\right)$ for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$, and starting with q^{-1}.

Vertex operator algebras (VOA)

The first part of the Moonshine conjecture, proved by Frenkel-Lepowsky-Meurman, asserts that we have some "natural" infinite dimensional graded vector space $V^\natural = \bigoplus_{n=0}^\infty V_n^\natural$ over $\mathbb{C}$ with $\dim V_n^\natural < \infty$ having some algebraic structure whose automorphism group is the Monster group and that the series $\sum_{n=0}^\infty (\dim V_n^\natural) q^{n-1}$ is the j-function minus 744.

A vertex operator algebra gives a precise axiomatization of the above "some algebraic structure". Each $u \in V$ is assumed to produce a vertex operator, $Y(u,z) = \sum_{n\in\mathbb{Z}} u_n z^{-n-1}$, where $u_n \in \mathrm{End}(V)$ and z is a formal variable. We have $u_n v \in V_{k+l-n-1}$ for $u \in V_k$, $v \in V_l$. This is the n-th product of u and v.

Examples of vertex operator algebras

Basic methods of constructions

1. Affine Kac-Moody algebras (Frenkel-Zhu)
2. Virasoro algebra (Frenkel-Zhu)
3. Even lattices (Frenkel-Lepowsky-Meurman)

New constructions from given examples

1. Tensor product construction
2. Orbifold construction (Dijkgraaf-Vafa-Verlinde-Verlinde)
3. Simple current extension (Schellekens-Yankielowicz)
4. Coset construction (Frenkel-Zhu)
5. Extension by a Frobenius algebra (Huang-Kirillov-Lepowsky) (2015)

Quantum Field Theory: (mathematical aspects/axioms)

Mathematical ingredients: Spacetime, its symmetry group, quantum fields on the spacetime.

From a mathematical viewpoint, quantum fields are certain operator-valued distributions on the spacetime. Axiomatization of such operator-valued distributions on a Hilbert space is given by the Wightman axioms.

A pairing $\langle T, f\rangle$ for a quantum field T and a test function f supported in O gives an observable in O. For a fixed O, let $A(O)$ be the von Neumann algebra generated by these observables. We have a family $\{A(O)\}$ of von Neumann algebras of bounded linear operators parameterized with regions O, and it is called a net. We work on its mathematical axiomatization.

Two-dimensional conformal field theory and chiral decomposition

We work on the two-dimensional Minkowski space with the space coordinate x and the time coordinate t. We have a decomposition of a quantum field theory on this spacetime into the product of two light rays $\{x = \pm t\}$ and their one-point compactifications.

In this way, now S^1 is our "spacetime" though the space and the time are mixed into one-dimension. We study a chiral conformal field theory on S^1.

We also have a converse construction from two chiral conformal field theories to one full conformal field theory on the two-dimensional Minkowski space.

Chiral conformal field theory on S^1

This is a quantum field theory on S^1 with the spacetime symmetry group $\mathrm{Diff}(S^1)$. It is described with a family $\{A(I)\}$ of von Neumann algebras parameterized by an interval $I \subset S^1$ subject to certain axioms. Such a family is called a conformal net.

Axioms for a conformal net:

1. $I_1 \subset I_2 \Rightarrow A(I_1) \subset A(I_2)$.
2. $I_1 \cap I_2 = \varnothing \Rightarrow [A(I_1), A(I_2)] = 0$. (locality)
3. $\mathbf{Diff}(S^1)$-covariance (conformal covariance)
4. Positive energy, vacuum vector
5. Irreducibility

The locality axiom comes from the Einstein causality.
Each $A(I)$ is automatically the Araki-Woods III_1 factor.

Examples of conformal nets

Basic methods of constructions

1. Affine Kac-Moody algebras (Gabbiani-Fröhlich, Wassermann)
2. Virasoro algebra (Loke, Xu)
3. Even lattices (K-Longo, Dong-Xu)

New constructions from given examples

1. Tensor product construction
2. Orbifold construction (Xu, Moonshine by K-Longo)
3. Simple current extension (Böckenhauer-Evans)
4. Coset construction (Xu)
5. Extension by a Frobenius algebra (K-Longo 2006, Xu)

Representation theory

Let $\{A(I)\}$ be a conformal net . Each $A(I)$ acts on the same Hilbert space from the beginning. Consider a representation of these von Neumann algebras on another Hilbert space without the vacuum vector.

A classical Doplicher-Haag-Roberts theory adapted to a conformal net shows that each representation gives a subfactor of the Jones theory, and the representation theory produces a braided tensor category.

We are often interested in a situation where we have only finitely many irreducible representations. (Rationality.) K-Longo-Müger 2001 gave an operator algebraic characterization of complete rationality of a conformal net. (We then get a modular tensor category.)

α-induction

We recall a classical notion of induction of a representation of a group and introduce a similar construction for a conformal net.

Let $\{A(I) \subset B(I)\}$ be an inclusion of conformal nets. We can produce an almost representation $\alpha_\lambda^\pm$ of $\{B(I)\}$ from a representation λ of $\{A(I)\}$, using the $\pm$-braiding. ($\alpha^\pm$-induction: Longo-Rehren, Xu, Böckenhauer-Evans-K 1999)

Böckenhauer-Evans-K have shown that the matrix $(Z_{\lambda,\mu})$ defined by $Z_{\lambda,\mu} = \dim \mathrm{Hom}(\alpha_\lambda^+, \alpha_\mu^-)$ is a modular invariant for a $SL(2,\mathbb{Z})$ representation arising from the braiding of the representation category.

Classification theory for small central charge values

Using modular invariants, K-Longo 2004 have obtained the following complete classification of conformal nets with $c < 1$, where c is a numerical invariant called the central charge arising from the Virasoro algebra.

(1) Virasoro nets $\{\mathrm{Vir}_c(I)\}$ with $c < 1$.
(2) Their simple current extensions with index 2.
(3) Four exceptionals at $c = 21/22$, $25/26$, $144/145$, $154/155$.

Three of the four exceptionals in the above (3) are identified with coset constructions, but the remaining one $c = 144/145$ does not seem to be related to any other previously known constructions so far, and is given as an extension by a Frobenius algebra.

Unitarity and energy bounds

Now we construct a conformal net from a VOA V. We first need a positive definite inner product for a Hilbert space. We have to assume to have a nice one, like for many natural examples. This is called unitarity.

Let V be a unitary VOA. The meaning of $Y(u, z)$ should be a Fourier expansion of an operator-valued distribution on S^1. We say that $u \in V$ satisfies energy-bounds if we have positive integers s, k and a constant $M > 0$ such that we have

$$\|u_n v\| \le M(|n| + 1)^s \|(L_0 + 1)^k v\|,$$

for all $v \in V$ and $n \in \mathbb{Z}$. If every $u \in V$ satisfies energy-bounds, we say V is energy-bounded.

Strong locality

For every $u \in V$, we define the operator $Y_0(u, f)$ by $Y_0(u, f)v = \sum_{n\in\mathbb{Z}} \hat{f}_n u_n v$ for $v \in V$, where f is a C^∞ function supported in $I \subset S^1$, $\hat{f}_n$ is its Fourier coefficient and the sum is convergent.

Let $Y(u, f)$ be the closure of $Y_0(u, f)$ on the completion of V. The (possibly unbounded) operators $Y(u, f)$, where $u \in V$ and $\mathbf{supp}\, f \subset I$, generate a von Neumann algebra $A(I)$. The family $\{A(I)\}$ satisfies all the axioms of a local conformal net except for locality. (Conformal covariance is nontrivial.)

If we also have locality, we say the original VOA has strong locality. The name "strong" comes from strong commutativity in functional analysis.

When do we have strong locality?

If a unitary VOA V has a set of nice generators in V_1 and V_2, we have strong locality. (Carpi-K-Longo-Weiner 2018)

This sufficient condition applies to a VOA arising from an affine Kac-Moody algebra or the Virasoro algebra. Strong locality passes to a tensor product and a sub VOA (hence a coset construction and an orbifold construction, in particular). These together show that many known examples of VOAs are strongly local.

Recent works of Carpi, Xu, Gui and Tener show that all known examples of VOA's satisfy strong locality. Some people even conjecture that strong locality would always hold, but I am doubtful.

Going back to a VOA

Suppose we have constructed a conformal net from a strongly local VOA. We now would like to recover the original VOA.

Based on an idea of Fredenhagen-Jörss and with help of the Tomita-Takesaki theory, we can recover a smeared vertex operator $Y(u, f)$ for $u \in V$ and a test function $f \in C^\infty(S^1)$ supported in I, using one of the Virasoro generators, L_{-1}.

Then the vector space V is first recovered as an algebraic direct sum of the eigenspaces of the Virasoro generator L_0. In this way, we can also recover $u_n \in \mathbf{End}(V)$, which gives the entire structure of a VOA. (Carpi-K-Longo-Weiner 2018)

Realization problem of a conformal net

The representation category is an invariant similar to the higher relative commutants of subfactors. (cf. Popa's work.) It also has some formal similarity to K-theory of C^*-algebras and the flow of weights of type III factors.

The assignment map of the representation category cannot be injective, since there are many conformal nets having the trivial representation category (holomorphic theory). But can it be surjective? That is, is a given (unitary) modular tensor category realized as the representation category of a conformal net?

We believe the answer is positive. This would imply there would be a huge variety of new exotic chiral conformal field theories through exotic subfactors.

Representation theories

We have representation categories for both vertex operator algebras and conformal nets. Both give modular tensor categories under some finiteness assumptions. We expect that if a conformal net and a vertex operator algebra mutually correspond in the above sense, their representation categories should be also identified (as braided tensor cateogories).

Such identification has been proved recently by Carpi, Weiner, Xu, Gui and Tener under mild assumptions, and all the Wess-Zumino-Witten models are covered.

Handling of the braiding is a bit subtle even within the operator algebraic approaches, since we have two methods due to Connes and Doplicher-Haag-Roberts.

Full conformal field theory

We have a decomposition $A_L(I) \otimes A_R(J) \subset B(I \times J)$ for a full conformal field theory $B(I \times J)$ on the two-dimensional Minkowski space and chiral conformal field theories $A_L(I), A_R(J)$. For simplicity, we assume $A_L = A_R$ and write simply A for this. We further assume that this A is completely rational.

The vacuum represenation of $B(I \times J)$ decomposes into $\bigoplus_{\lambda,\mu} Z_{\lambda,\mu} \lambda \otimes \mu$, where λ, μ label irreducible representations of A. The multiplicity matrix Z is a modular invariant under a maximality assumption on $B(I \times J)$. That is, the matrix Z commutes with S- and T-matrices arising from A.

Classification of full conformal field theories with $c < 1$

For a given completely rational conformal net $\mathcal{A}$, maximal full conformal field theories $\mathcal{B}$ can be classified as follows.

1. Classify modular invariants Z.
2. Look for a realization of $\mathcal{B}$ for each Z.
3. Study the uniquness problem of $\mathcal{B}$ for each Z.

For example, we can solve the above problems when $\mathcal{A}$ is the Virasoro net with $c < 1$. In this case, we have unique $\mathcal{B}$ for each Z above, labeled by certain pairs of the A-D-E Dynkin diagrams. Now so-called type II modular invariants also appear, unlike the classification theory of chiral conformal field theories. (K-Longo 2004.)

Modular invariance as completeness

For a completely rational conformal net $\mathcal{A}$, we have the Haag duality, $\mathcal{A}(I) = \mathcal{A}(I')'$. If we consider I which is a disjoint union of more than one intervals, then we have $\mathcal{A}(I) \subset \mathcal{A}(I')'$, and the equality here holds if and only if the representation theory of $\mathcal{A}$ is trivial.
(K-Longo-Müger 2001.)

In the case of a full conformal field theory, we further prove that modular invariance of Z is equivalent to certain completeness in the sense of the Haag duality for disconnected regions.
(Benedetti-Casini-Kawahigashi-Longo-Magan 2024.)

The Haag duality violation is related to certain properties of partition functions and the Renyi entropy.