Title: Generalized Quantum Stein's Lemma and Second Law of Quantum Resource Theories

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Abstract:

The second law of thermodynamics is the cornerstone of physics, characterizing the convertibility between thermodynamic states through a single function, entropy. Given the universal applicability of thermodynamics, a fundamental question in quantum information theory is whether an analogous second law can be formulated to characterize the convertibility of resources for quantum information processing by a single function. In 2008, a promising formulation was proposed, linking resource convertibility to the optimal performance of a variant of the quantum version of hypothesis testing. Central to this formulation was the generalized quantum Stein's lemma, which aimed to characterize this optimal performance by a measure of quantum resources, the regularized relative entropy of resource. If proven valid, the generalized quantum Stein's lemma would lead to the second law for quantum resources, with the regularized relative entropy of resource taking the role of entropy in thermodynamics. However, in 2023, a logical gap was found in the original proof of this lemma, casting doubt on the possibility of such a formulation of the second law. In this work, we address this problem by developing alternative techniques to successfully prove the generalized quantum Stein's lemma under a smaller set of assumptions than the original analysis. Based on our proof, we reestablish and extend the second law of quantum resource theories, applicable to both static resources of quantum states and a fundamental class of dynamical resources represented by classical-quantum (CQ) channels. These results resolve the fundamental problem of bridging the analogy between thermodynamics and quantum information theory.

The talk is based on the following paper.

https://arxiv.org/abs/2408.02722

Generalized Quantum Stein's Lemma and Second Law of Quantum Resource Theories

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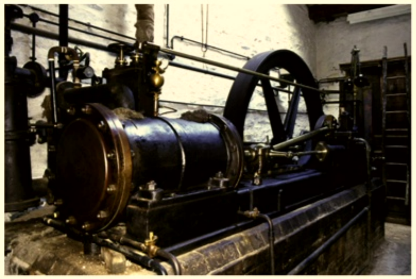
Reference:
Masahito Hayashi, Hayata Yamasaki, arXiv:2408.02722

Second Law of Thermodynamics

19th century
Steam engine
Thermodynamics



Second law: Convertibility between thermodynamic states under adiabatic operations is fully determined by entropy S(X)

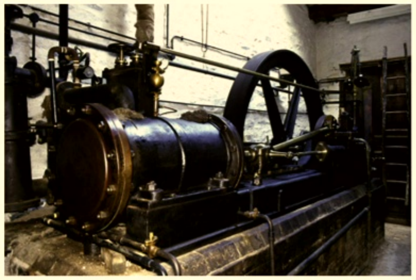
$$X_1 \xrightarrow{\text{adiabatic}} X_2 \Leftrightarrow S(X_1) \leq S(X_2)$$

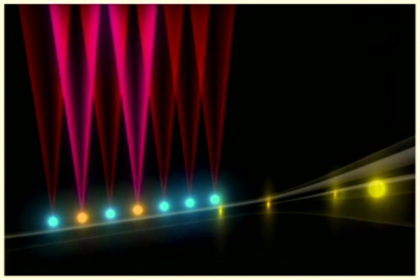
Lieb, Yngvason, arXiv:cond-mat/9708200

Second law of thermodynamics characterizes state convertibility by a single function

Quantum Resource Theories (QRTs)

19th century
Steam engine
Thermodynamics





https://www.nano-qt.com/

21st century
Quantum devices
Quantum information

QRTs: A unified framework for exploring advantages and limitations of quantum mechanics

Free operations: A restricted subset of operations (CPTP maps) $\mathcal{E}_{\mathrm{free}} \in \mathcal{O}$

e.g., local operations and classical communication (LOCC), stabilizer operations,…

→ **Free states**: States $\rho_{\mathrm{free}} = \mathcal{E}_{\mathrm{free}}(\rho)$ obtained from any (non-resourceful) initial state ρ

→ **Resource states**: Non-free states, to assist free operations

QRTs study **manipulation and quantification** of quantum resources in operational approach

Kuroiwa, Yamasaki, arXiv:2002.02458; Chitambar, Gour, arXiv:1806.06107

Entanglement Theory

Entanglement: Resource to assist LOCC ⇔ Separable state

e.g., quantum teleportation

$$\rho_{\mathrm{SEP}} = \sum_x p(x) \left|\psi_x\right\rangle \left\langle\psi_x\right| \otimes \left|\phi_x\right\rangle \left\langle\phi_x\right|$$

$\mathcal{E}_{\mathrm{free}} \in \mathcal{O}$

$\Phi^{\otimes m}$

$$|\Phi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \text{ : Ebit}$$

Distillable entanglement

$$E_{\mathrm{D},\mathcal{O}}(\rho) = \sup\left\{m/n : \Phi^{\otimes m} \approx \mathcal{E}_{\mathrm{free}}\left(\rho^{\otimes n}\right)\right\}$$

Entanglement cost

$$E_{\mathrm{C},\mathcal{O}}(\rho) = \sup\left\{m/n : \rho^{\otimes n} \approx \mathcal{E}_{\mathrm{free}}\left(\Phi^{\otimes m}\right)\right\}$$

$\rho^{\otimes n}$

ρ : Mixed state

Bennet, Bernstein, Popescu, Schumacher, arXiv:quant-ph/9511030

Hayden, Horodecki, Terhal, arXiv:quant-ph/0008134

Yamasaki, Kuroiwa, Hayden, Lami, arXiv:2401.09554

Central goal: Clarify optimal protocols and fundamental limits in manipulating entanglement

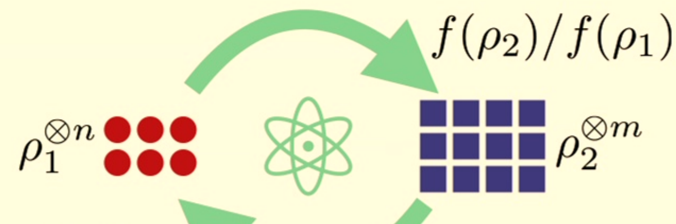
Irreversibility of Entanglement Conversion

Conversion rate: $r_{\mathcal{O}}(\rho_1 \to \rho_2) = \sup\left\{m/n : \rho_2^{\otimes m} \approx \mathcal{E}_{\mathrm{free}}\left(\rho_1^{\otimes n}\right), \mathcal{E}_{\mathrm{free}} \in \mathcal{O}\right\}$

Desired second law: $r_{\mathcal{O}}(\rho_1 \to \rho_2) = f(\rho_2)/f(\rho_1)$

$$\Rightarrow E_{\mathrm{D},\mathcal{O}}(\rho) = E_{\mathrm{C},\mathcal{O}}(\rho)$$

Plenio, Open Problem 20 in arXiv:quant-ph/0504166



Challenge: Irreversibility of entanglement conversion $f(\rho_1)/f(\rho_2)$

$E_{\mathrm{D,LOCC}}(\rho) \neq E_{\mathrm{C,LOCC}}(\rho)$: Bound entangled state

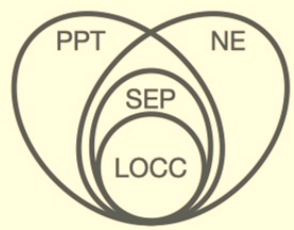
Vidal, Cirac, arXiv:quant-ph/0102036

$E_{\mathrm{D,PPT}}(\rho) \neq E_{\mathrm{C,PPT}}(\rho)$

Wang, Duan, arXiv:1606.09421

$E_{\mathrm{D,NE}}(\rho) \neq E_{\mathrm{C,NE}}(\rho)$

Lami, Regula, arXiv:2111.02438



$(\mathrm{id} \otimes \mathcal{E}_{\mathrm{SEP}})(\rho_{\mathrm{SEP}}) = \rho'_{\mathrm{SEP}}$

$(\mathrm{id} \otimes \mathcal{E}_{\mathrm{PPT}})(\rho_{\mathrm{PPT}}) = \rho'_{\mathrm{PPT}}$

$\mathcal{E}_{\mathrm{NE}}(\rho_{\mathrm{SEP}}) = \rho'_{\mathrm{SEP}}$

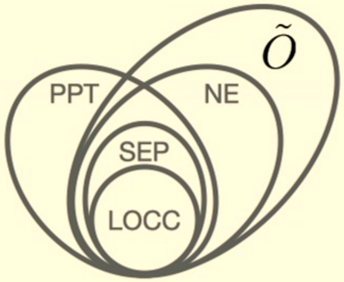
$\boxed{\text{We need \textbf{a more nontrivial relaxation of operations} to obtain a reversible framework}}$

Axiomatic Approach toward Second Law

Idea: Axiomatically define a more relaxed set of operations **for asymptotic scenarios**

- Capture the essence (free state→free state) **only in an asymptotic sense**
- Similar to **axiomatic definition** of adiabatic operations in thermodynamics
  Lieb, Yngvason, arXiv:cond-mat/9708200

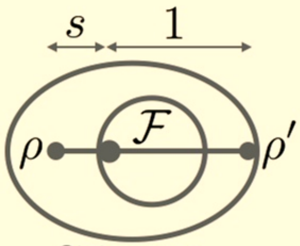
→ **Asymptotically resource-non-generating operations**



$$\tilde{O} = \left\{ \{\mathcal{E}_n\}_{n=1,2,\ldots} : \lim_{n\to\infty} R_{\mathrm{G}}\left(\mathcal{E}_n\left(\rho_{\mathrm{free}}^{(n)}\right)\right) = 0 \right\}$$

Resource measure: **Generalized robustness**

$$R_{\mathrm{G}}(\rho) = \min\left\{ s \geq 0 : \frac{\rho + s\rho'}{1+s} \in \mathcal{F},\ \rho' : \text{any state} \right\}$$



Brandao, Plenio arXiv:0810.2319, arXiv:0710.5827, arXiv:0904.0281; Brandao, Gour, arXiv:1502.03149

Second Law from Generalized Quantum Stein's Lemma

Requirements for QRTs: The set $\mathcal{F}$ of free states (for free operations $\mathcal{O}$) should be

Separable states, Stabilizer states,…

1: Finite-dimensional, closed, and convex

2: Closed under tensor product $\rho_{\mathrm{free}} \otimes \rho'_{\mathrm{free}} \in \mathcal{F}$

3: Including a full-rank state ρ_{full}

4: Closed under partial trace

5: Closed under permutation of subsystems

$\Rightarrow$ We can use **regularized relative entropy of resource** as "entropy"

$$R_{\mathrm{R}}^{\infty}(\rho) := \lim_{n\to\infty} \frac{1}{n} \min_{\rho_{\mathrm{free}}\in\mathcal{F}} D\left(\rho^{\otimes n} \middle\| \rho_{\mathrm{free}}\right)$$

$$D(\rho\|\sigma) := \mathrm{Tr}[\rho(\log\rho - \log\sigma)]$$

Fact: If the "generalized quantum Stein's lemma" (next slides) holds true, we have 2nd law

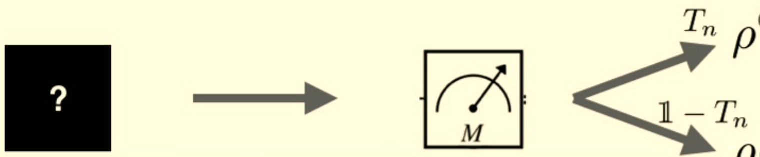
$r_{\tilde{\mathcal{O}}}(\rho_1 \to \rho_2) = R_{\mathrm{R}}^{\infty}(\rho_1)/R_{\mathrm{R}}^{\infty}(\rho_2)$ for any state ρ_j satisfying $R_{\mathrm{R}}^{\infty}(\rho_j) > 0$

Brandao, Gour, arXiv:1502.03149; Regula, Lami, arXiv:2309.07206

Universal framework with R_{R}^{∞} taking **the role of entropy in the 2nd law** of thermodynamics

Setting of Generalized Quantum Stein's Lemma

Task: A variant of hypothesis testing

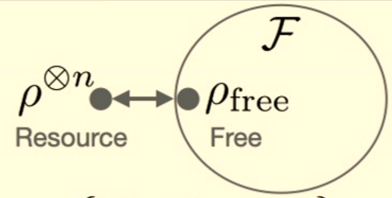


Unknown state

POVM measurement
$\{T_n, \mathbb{1} - T_n\}$

$\rho^{\otimes n}$ IID copies of ρ

$\rho_{\mathrm{free}} \in \mathcal{F}(\mathcal{H}^{\otimes n})$ Non-IID from a (convex) set of free states



$\{T_n, \mathbb{1} - T_n\}$

Minimize type-II error $\beta_\epsilon\left(\rho^{\otimes n} \middle\| \mathcal{F}\right) := \min_{T_n} \max_{\rho_{\mathrm{free}} \in \mathcal{F}} \mathrm{Tr}[T_n \rho_{\mathrm{free}}] \approx \mathrm{e}^{-nR}$

Decaying exponentially

subject to type-I error within ε $\mathrm{Tr}\left[(\mathbb{1} - T_n)\rho^{\otimes n}\right] \leq \epsilon$

Question: Optimal exponent R? ⬌ **Challenge: Non-IID** structure = Hard to analyze

Brandao, Plenio, arXiv:0904.0281

Problem: Generalized Quantum Stein's Lemma

Original statement of generalized quantum Stein's lemma

Brandao, Plenio, arXiv:0904.0281

Assume the set $\mathcal{F}$ satisfies

1: Finite-dimensional, closed, and convex

2: Closed under tensor product

3: Including a full-rank state

4: Closed under partial trace

5: Closed under permutation of subsystems

Then for any $\epsilon \in (0,1)$

$$\Rightarrow \quad \lim_{n\to\infty} -\frac{1}{n}\log \beta_\epsilon\left(\rho^{\otimes n}\middle\|\mathcal{F}\right) = R_{\mathrm{R}}^{\infty}(\rho)$$

- Meaning $\beta_\epsilon\left(\rho^{\otimes n}\middle\|\mathcal{F}\right) \approx \mathrm{e}^{-nR_{\mathrm{R}}^{\infty}(\rho)}$
- Quantum Stein's lemma when $\mathcal{F} = \left\{\sigma^{\otimes n}\right\}$

$$\lim_{n\to\infty} \frac{1}{n}\beta_\epsilon\left(\rho^{\otimes n}\middle\|\sigma^{\otimes n}\right) = D\left(\rho\|\sigma\right)$$

Hiai, Petz, CMP 143, 99 (1991); Ogawa Nagaoka, arXiv:quant-ph/9906090

History of this lemma

- Original announcement by Brandao, Plenio arXiv:0810.2319 (Nat. Phys. 2008)→Full papers arXiv:0710.5827 (CMP2010), **arXiv:0904.0281 (CMP2010)**
- **A logical gap** found in a part of Fang, Gour, Wang arXiv:2110.14842, which was **based on arXiv:0904.0281** → Berta, Brandao, Gour, Lami, Plenio, Regula, Tomamichel arXiv:2205.02813 pointed out **the logical gap of the analysis of generalized quantum Stein's lemma in arXiv:0904.0281**
- **Alternative proof proposed** by Yamasaki, Kuroiwa arXiv:2401.01926 using continuity bounds on quantum relative entropy by Bluhm, Capel, Gondolf, Perez-Hernandez arXiv:2208.00922, arXiv:2305.10140 → **A logical gap** found by the authors of arXiv:2205.02813
- Works to avoid the issue by defining **"operations" beyond the law of quantum mechanics** arXiv:2309.07206, arXiv:2312.04456, arXiv:2405.10599

Results: Proof of Stronger Lemma+More General Second Law

Result 1: Proof of Stronger Version of Generalized Quantum Stein's Lemma

Assume the set $\mathcal{F}$ satisfies

1: Finite-dimensional, closed, and convex

2: Closed under tensor product

3: Including a full-rank state

~~4: Closed under partial trace~~

~~5: Closed under permutation of subsystems~~

Then for any $\epsilon \in (0,1)$

$$\Rightarrow \quad \lim_{n\to\infty} -\frac{1}{n} \log \beta_\epsilon \left(\rho^{\otimes n} \middle\| \mathcal{F}\right) = R_{\mathrm{R}}^{\infty}(\rho)$$

Out proof techniques **eliminate assumptions** required for quantum de Finetti theorem

Soon after posting our work, Lami arXiv:2408.06410 proposed alternative proof of the **previous version imposing all the five assumptions**

Result 2: Under asymptotically free operations, we have **2nd law for dynamical resources**

$r_{\tilde{\mathcal{O}}}(\mathcal{N}_1 \to \mathcal{N}_2) = R_{\mathrm{R}}^{\infty}(\mathcal{N}_1)/R_{\mathrm{R}}^{\infty}(\mathcal{N}_2)$ for any CQ channel $\mathcal{N}_j$ satisfying $R_{\mathrm{R}}^{\infty}(\mathcal{N}_j) > 0$

Opening a way to formalize and utilize **a more generally applicable framework of 2nd law**

Result 1: Generalized Quantum Stein's Lemma

Result 1: Proof of Stronger Version of Generalized Quantum Stein's Lemma

Assume the set $\mathcal{F}$ satisfies

1: Finite-dimensional, closed, and convex

2: Closed under tensor product $\rho_{\mathrm{free}} \otimes \rho'_{\mathrm{free}} \in \mathcal{F}$

3: Including a full-rank state ρ_{full}

Then for any $\epsilon \in (0,1)$

$$\Rightarrow \quad \lim_{n\to\infty} -\frac{1}{n} \log \beta_\epsilon \left(\rho^{\otimes n} \middle\| \mathcal{F}\right) = R_{\mathrm{R}}^{\infty}(\rho)$$

Strong converse part: Optimality $$\limsup_{n\to\infty} -\frac{1}{n} \log \beta_\epsilon \left(\rho^{\otimes n} \middle\| \mathcal{F}\right) \leq \lim_{n\to\infty} \frac{1}{n} \min_{\rho_{\mathrm{free}} \in \mathcal{F}} D\left(\rho^{\otimes n} \middle\| \rho_{\mathrm{free}}\right)$$

Direct part: Achievability $$\liminf_{n\to\infty} -\frac{1}{n} \log \beta_\epsilon \left(\rho^{\otimes n} \middle\| \mathcal{F}\right) \geq \lim_{n\to\infty} \frac{1}{n} \min_{\rho_{\mathrm{free}} \in \mathcal{F}} D\left(\rho^{\otimes n} \middle\| \rho_{\mathrm{free}}\right)$$

Limit on the right-hand side exists due to the subadditivity of quantum relative entropy (Fekete's subadditive lemma)

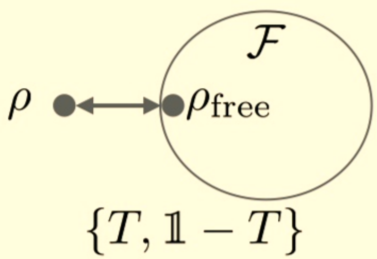
Characterization by Minimax Theorem

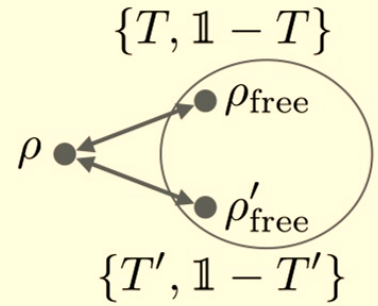
Minimizing worst-case type-II error ⟺ Minimizing type-II error for each state

$$\beta_\epsilon(\rho\|\mathcal{F}) = \min_{T\in\mathcal{T}_{\epsilon,\rho}} \max_{\rho_{\mathrm{free}}\in\mathcal{F}} \mathrm{Tr}[T\rho_{\mathrm{free}}]$$

$$\max_{\rho_{\mathrm{free}}\in\mathcal{F}} \beta_\epsilon(\rho\|\rho_{\mathrm{free}}) = \max_{\rho_{\mathrm{free}}\in\mathcal{F}} \min_{T\in\mathcal{T}_{\epsilon,\rho}} \mathrm{Tr}[T\rho_{\mathrm{free}}]$$

$$\mathcal{T}_{\epsilon,\rho} \coloneqq \{T : 0 \le T \le \mathbb{1},\ \mathrm{Tr}[(\mathbb{1}-T)\rho] \le \epsilon\}$$





Minimax inequality: $\max_{\rho_{\mathrm{free}}\in\mathcal{F}} \beta_\epsilon(\rho\|\rho_{\mathrm{free}}) \le \beta_\epsilon(\rho\|\mathcal{F})$ holds for any set $\mathcal{F}$ in general

Minimax theorem: $\max_{\rho_{\mathrm{free}}\in\mathcal{F}} \beta_\epsilon(\rho\|\rho_{\mathrm{free}}) = \beta_\epsilon(\rho\|\mathcal{F})$ holds for any convex, compact set $\mathcal{F}$

It suffices to analyze $\max_{\rho_{\mathrm{free}}\in\mathcal{F}} \beta_\epsilon(\rho\|\rho_{\mathrm{free}})$ instead of $\beta_\epsilon(\rho\|\mathcal{F})$

Simple Proof of Strong Converse

Goal: $$\limsup_{n\to\infty} -\frac{1}{n}\log \max_{\rho_{\mathrm{free}}\in\mathcal{F}} \beta_\epsilon\left(\rho^{\otimes n}\middle\|\rho_{\mathrm{free}}\right) \leq \lim_{n\to\infty}\frac{1}{n}\min_{\rho_{\mathrm{free}}\in\mathcal{F}} D\left(\rho^{\otimes n}\middle\|\rho_{\mathrm{free}}\right) =: R_{\mathrm{R}}^{\infty}(\rho)$$

- For each m, take the **optimal m-fold non-IID state** $D\left(\rho^{\otimes m}\middle\|\rho_{\mathrm{free}}^{(m)}\right) = \min_{\rho_{\mathrm{free}}\in\mathcal{F}} D\left(\rho^{\otimes m}\middle\|\rho_{\mathrm{free}}\right)$
- To bound β by **additive quantities**, extend it to n-fold states $\tilde{\rho}_{\mathrm{free}}^{(n)} := \underbrace{\rho_{\mathrm{free}}^{(m)\otimes l}}_{\text{Dominant}} \otimes \underbrace{\rho_{\mathrm{full}}^{\otimes n-lm}}_{\text{Remainder}}$
  $lm \leq n < (l+1)m$

→ Core part of the proof

$$\limsup_{n\to\infty} -\frac{1}{n}\max_{\rho_{\mathrm{free}}\in\mathcal{F}} \beta_\epsilon\left(\rho^{\otimes n}\middle\|\rho_{\mathrm{free}}\right) \leq \boxed{\limsup_{n\to\infty} -\frac{1}{n}\beta_\epsilon\left(\rho^{\otimes n}\middle\|\tilde{\rho}_{\mathrm{free}}^{(n)}\right) \leq \frac{1}{m} D\left(\rho^{\otimes m}\middle\|\rho_{\mathrm{free}}^{(m)}\right)} \xrightarrow{m\to\infty} R_{\mathrm{R}}^{\infty}(\rho)$$

We show **simpler derivation** using an additive upper bound of β (Renyi relative entropy)

Ogawa Nagaoka, arXiv:quant-ph/9906090; Cooney, Mosonyi, Wilde, arXiv:1408.3373

Previous proof techniques by Brandao, Plenio arXiv:0904.0281 require additional assumptions that our proof eliminates

Key Idea for Proof of Direct Part

Goal: $$\liminf_{n\to\infty} -\frac{1}{n} \log \max_{\rho_{\mathrm{free}}\in\mathcal{F}} \beta_\epsilon\left(\rho^{\otimes n}\middle\|\rho_{\mathrm{free}}\right) \geq \lim_{n\to\infty} \frac{1}{n} \min_{\rho_{\mathrm{free}}\in\mathcal{F}} D\left(\rho^{\otimes n}\middle\|\rho_{\mathrm{free}}\right)$$

- Fix an **optimal** sequence in **maximizing β** $R_{1,\epsilon} := \liminf_{n\to\infty} -\frac{1}{n}\log\beta_\epsilon\left(\rho^{\otimes n}\middle\|\rho_{\mathrm{free}}^{(n)*}\right)$
- Consider any **sub-optimal** sequence in **reducing D** $R_2 := \liminf_{n\to\infty}\frac{1}{n}D\left(\rho^{\otimes n}\middle\|\rho_{\mathrm{free}}^{(n)}\right) > R_{1,\epsilon}$

Key lemma (next slides): For any fixed $\tilde{\epsilon}\in(0,\epsilon)$, we can **update the sequence**

$$\left\{\rho_{\mathrm{free}}^{(n)}\right\}_n \xrightarrow{\text{more optimal}} \left\{\rho_{\mathrm{free}}^{(n)\prime} := \frac{1}{3}\left(\rho_{\mathrm{free}}^{(n)} + \rho_{\mathrm{free}}^{(n)*} + \rho_{\mathrm{full}}^{\otimes n}\right)\right\}_n$$

$$\liminf_{n\to\infty}\frac{1}{n}D\left(\rho^{\otimes n}\middle\|\rho_{\mathrm{free}}^{(n)\prime}\right) - R_{1,\epsilon} \leq (1-\tilde{\epsilon})(R_2 - R_{1,\epsilon})$$

→ Apply this update k times: $\liminf_{n\to\infty}\frac{1}{n}D\left(\rho^{\otimes n}\middle\|\rho_{\mathrm{free}}^{(n)\prime\prime\cdots}\right) - R_{1,\epsilon} \leq (1-\tilde{\epsilon})^k(R_2 - R_{1,\epsilon}) \xrightarrow{k\to\infty} 0$

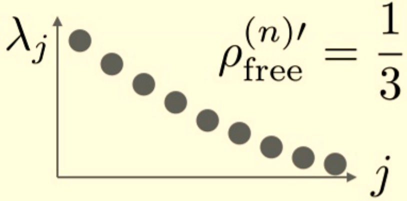
We identify how to **construct the optimal sequence minimizing D** from that maximizing β

Technique: Pinching+Information Spectrum Method

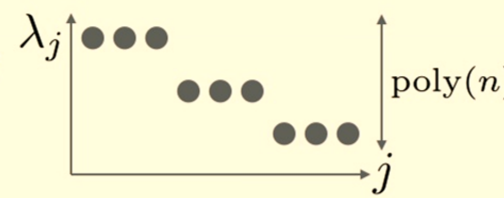
Goal: $$\liminf_{n\to\infty}\frac{1}{n}D\left(\rho^{\otimes n}\middle\|\rho_{\mathrm{free}}^{(n)\prime}\right)-R_{1,\epsilon}\le(1-\tilde{\epsilon})(R_2-R_{1,\epsilon})$$

Technique 1: **Pinching** to make non-commuting operators commute

Hayashi, arXiv:quant-ph/0208020



$$\rho_{\mathrm{free}}^{(n)\prime}=\frac{1}{3}\left(\rho_{\mathrm{free}}^{(n)}+\rho_{\mathrm{free}}^{(n)*}+\rho_{\mathrm{full}}^{\otimes n}\right)\xrightarrow{\text{degenerate}}\tilde{\rho}_{\mathrm{free}}^{(n)\prime}=\sum_{j=0}^{\mathrm{poly}(n)}\lambda_j E_j$$



Pinching map:
$\mathcal{E}_n(\rho^{\otimes n}):=\sum_j E_j\rho^{\otimes n}E_j$

$$\liminf_{n\to\infty}\frac{1}{n}D\left(\rho^{\otimes n}\middle\|\rho_{\mathrm{free}}^{(n)\prime}\right)=\liminf_{n\to\infty}\frac{1}{n}D\left(\mathcal{E}_n(\rho^{\otimes n})\middle\|\tilde{\rho}_{\mathrm{free}}^{(n)\prime}\right)$$

Non-commutative — Commutative

Continuous bounds on 2nd argument disturb D too much ⟺ Pinching may **not disturb D**

Brandao, Plenio, arXiv:0904.0281; Yamasaki, Kuroiwa, arXiv:2401.01926

New goal: $$\liminf_{n\to\infty}\frac{1}{n}D\left(\mathcal{E}_n(\rho^{\otimes n})\middle\|\tilde{\rho}_{\mathrm{free}}^{(n)\prime}\right)-R_{1,\epsilon}\le(1-\tilde{\epsilon})(R_2-R_{1,\epsilon})$$

Technique: Pinching+Information Spectrum Method

Goal: $$\liminf_{n\to\infty} \frac{1}{n} D\left(\mathcal{E}_n\left(\rho^{\otimes n}\right) \middle\| \tilde{\rho}_{\mathrm{free}}^{(n)\prime}\right) - R_{1,\epsilon} \leq (1-\tilde{\epsilon})(R_2 - R_{1,\epsilon})$$

Technique 2: **Information spectrum method** to convert β → probability

Nagaoka, Hayashi, arXiv:quant-ph/0206185

$$\liminf_{n\to\infty} -\frac{1}{n}\log\beta_\epsilon\left(\mathcal{E}_n\left(\rho^{\otimes n}\right)\middle\|\tilde{\rho}_{\mathrm{free}}^{(n)}\right) \leq R_{1,\epsilon}$$

$$P_{n,1} := \left\{\mathcal{E}_n\left(\rho^{\otimes n}\right) \geq \mathrm{e}^{n(R_{1,\epsilon}+\epsilon_2)}\tilde{\rho}_{\mathrm{free}}^{(n)}\right\}$$

$$\liminf_{n\to\infty} \mathrm{Tr}\left[P_{n,1}\mathcal{E}_n\left(\rho^{\otimes n}\right)\right] \leq 1-\epsilon$$

$$\limsup_{n\to\infty} -\frac{1}{n}\log\beta_{1-\epsilon_1}\left(\mathcal{E}_n\left(\rho^{\otimes n}\right)\middle\|\tilde{\rho}_{\mathrm{free}}^{(n)}\right) \leq R_2+\epsilon_0$$

$$P_{n,2} := \left\{\mathcal{E}_n\left(\rho^{\otimes n}\right) \geq \mathrm{e}^{n(R_2+\epsilon_0+\epsilon_2)}\tilde{\rho}_{\mathrm{free}}^{(n)}\right\}$$

$$\limsup_{n\to\infty} \mathrm{Tr}\left[P_{n,2}\mathcal{E}_n\left(\rho^{\otimes n}\right)\right] \leq \epsilon_1$$

→ Core part of the proof

$$\frac{1}{n} D\left(\mathcal{E}_n\left(\rho^{\otimes n}\right) \middle\| \tilde{\rho}_{\mathrm{free}}^{(n)\prime}\right) := \frac{1}{n}\mathrm{Tr}\left[\mathcal{E}_n\left(\rho^{\otimes n}\right)\left(\log\mathcal{E}_n\left(\rho^{\otimes n}\right) - \log\tilde{\rho}_{\mathrm{free}}^{(n)\prime}\right)\right]$$

Commutative

Decomposition

$$\leq \mathrm{Tr}\left[(\mathbb{1}-P_1)\mathcal{E}_n\left(\rho^{\otimes n}\right)\right](R_{1,\epsilon}+\epsilon_2) + \mathrm{Tr}\left[(P_1-P_2)\mathcal{E}_n\left(\rho^{\otimes n}\right)\right](R_2+\epsilon_0+\epsilon_2) + \mathrm{Tr}\left[P_2\mathcal{E}_n\left(\rho^{\otimes n}\right)\right]\times O(1)$$
$$\lessapprox R_{1,\epsilon} + (1-\tilde{\epsilon})(R_2 - R_{1,\epsilon})$$

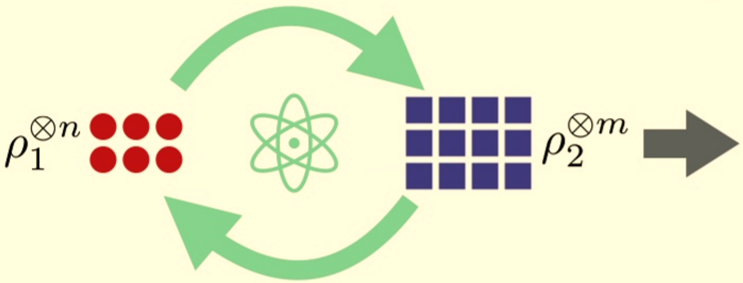
Application to QRTs for Dynamical Resources

Fact: If the generalized quantum Stein's lemma holds, we have 2nd law for QRTs for states

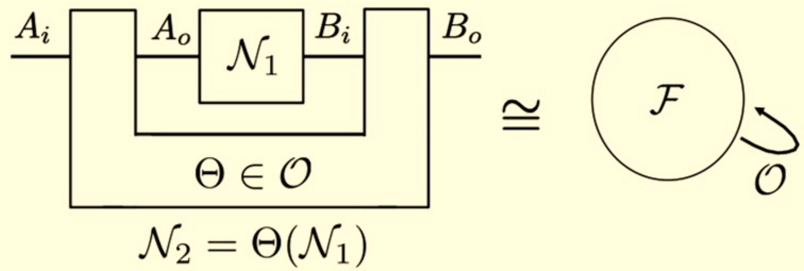
$$r_{\tilde{\mathcal{O}}}(\rho_1 \to \rho_2) = R_{\mathrm{R}}^{\infty}(\rho_2)/R_{\mathrm{R}}^{\infty}(\rho_1) \quad \text{for any state } \rho_j \text{ satisfying } R_{\mathrm{R}}^{\infty}(\rho_j) > 0$$

Brandao, Gour, arXiv:1502.03149; Regula, Lami, arXiv:2309.07206

Static resources: **States**



Dynamical resources: **Channels** in communication scenarios



E.g., Takagi, Hayashi, arXiv:1910.01125; Kuroiwa, Takagi, Adesso, Yamasaki, arXiv:2310.09154, arXiv:2310.09321

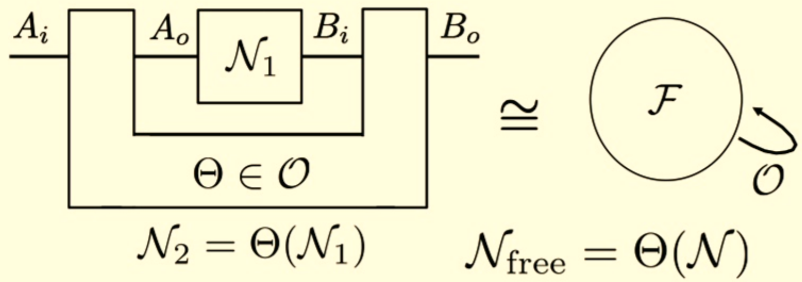
Challenge: Quantum channels are **much harder to analyze** due to unknown inputs

Masahito Hayashi, Hayata Yamasaki, arXiv:2408.02722 17

Formulation of QRTs for Classical-Quantum (CQ) Channels

Setting: Generalize quantum states to CQ channels $\rho \to \mathcal{N}(\sigma) = \sum_j \langle j|\,\sigma\,|j\rangle\,\rho_j$

- **Avoid hardness** of analyzing channels with quantum inputs
- **Including QRTs for states as a special case** when input dimensions are one



Free operations: A restricted subset of operations (CQ→CQ superchannels) $\Theta \in \mathcal{O}$

→ **Free CQ channels**: $\mathcal{N}_{\mathrm{free}} = \Theta(\mathcal{N})$ obtained from any (non-resourceful) CQ channel $\mathcal{N}$

→ **Resource CQ channels**: Non-free

→ **Conversion rate**: $r_{\mathcal{O}}(\mathcal{N}_1 \to \mathcal{N}_2) := \left\{ r : \liminf_{n\to\infty} \frac{1}{2} \left\| J\left(\Theta_n\left(\mathcal{N}_1^{\otimes n}\right)\right) - J\left(\mathcal{N}_2^{\otimes \lceil rn \rceil}\right) \right\|_1 = 0 \right\}$

Trace distance between Choi states

Asymptotically Free Operations

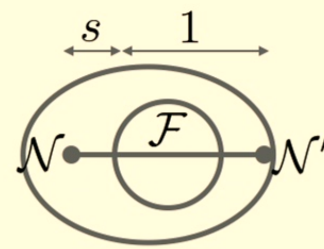
Definition: Asymptotically free operations $\tilde{O} := \left\{ \{\Theta_n\}_{n=1,2,\ldots} \right\}$ with the following properties

- **Asymptotically resource-non-generating property**:

Analogous to asymptotically resource-non-generating operations in QRTs for states

$$\lim_{n\to\infty} R_{\mathrm{G}}\left(\Theta_n\left(\mathcal{N}_{\mathrm{free}}^{(n)}\right)\right) = 0$$

$$R_{\mathrm{G}}(\mathcal{N}) = \min\left\{ s \geq 0 : \frac{\mathcal{N} + s\mathcal{N}'}{1+s} \in \mathcal{F},\ \mathcal{N}' : \text{any CQ channel} \right\}$$



- **Asymptotic continuity**: For any CQ channels satisfying $\lim_{n\to\infty} \frac{1}{2} \|J(\mathcal{N}_n) - J(\mathcal{N}_n')\|_1 = 0$

$$\lim_{n\to\infty} \frac{1}{2} \|J(\Theta_n(\mathcal{N}_n)) - J(\Theta_n(\mathcal{N}_n'))\|_1 = 0$$

In QRTs for states (CQ channels with one-dim inputs), **automatically satisfied thus unnecessary**

Identifying **an appropriate class of superchannels** for which the second law of QRTs holds

Result 2: Second Law for CQ Dynamical Resources

Result 2: Second law of QRTs for states and CQ channels

Assume the set $\mathcal{F}$ satisfies

1: Finite-dimensional, closed, and convex

2: Closed under tensor product $\mathcal{N}_{\mathrm{free}} \otimes \mathcal{N}'_{\mathrm{free}} \in \mathcal{F}$

3: Including full-rank $\mathcal{N}_{\mathrm{full}}(\sigma) := \mathrm{Tr}[\sigma]\rho_{\mathrm{full}}$

$\Rightarrow$

$$r_{\tilde{\mathcal{O}}}(\mathcal{N}_1 \to \mathcal{N}_2) = R_{\mathrm{R}}^{\infty}(\mathcal{N}_1)/R_{\mathrm{R}}^{\infty}(\mathcal{N}_2)$$

for any CQ channel $\mathcal{N}_j$ satisfying

$R_{\mathrm{R}}^{\infty}(\mathcal{N}_j) > 0$

We can use **a generalization of regularized relative entropy of resource** as "entropy"

$$R_{\mathrm{R}}^{\infty}(\mathcal{N}) := \lim_{n\to\infty} \frac{1}{n} \min_{\mathcal{N}_{\mathrm{free}}\in\mathcal{F}} D\left(\underbrace{J(\mathcal{N}) \| J(\mathcal{N}_{\mathrm{free}})}_{\text{Choi states}}\right)$$

Direct part: Achievability $r_{\tilde{\mathcal{O}}}(\mathcal{N}_1 \to \mathcal{N}_2) \geq R_{\mathrm{R}}^{\infty}(\mathcal{N}_1)/R_{\mathrm{R}}^{\infty}(\mathcal{N}_2)$

Brandao, Gour, arXiv:1502.03149 in QRTs for states

Converse part: Optimality $r_{\tilde{\mathcal{O}}}(\mathcal{N}_1 \to \mathcal{N}_2) \leq R_{\mathrm{R}}^{\infty}(\mathcal{N}_1)/R_{\mathrm{R}}^{\infty}(\mathcal{N}_2)$

Regula, Lami, arXiv:2309.07206 in QRTs for states

Direct Part of Resource Conversion

Key lemma: For any CQ channel, regularized relative entropy and robustness are related as

$$R_{\mathrm{R}}^{\infty}(\mathcal{N}) = \min_{\{\tilde{\mathcal{N}}_n\}_n} \left\{ \lim_{n\to\infty} \frac{1}{n} \log\left(1 + R_{\mathrm{G}}\left(\tilde{\mathcal{N}}_n\right)\right) : \lim_{n\to\infty} \frac{1}{2} \left\| J\left(\tilde{\mathcal{N}}_n\right) - J(\mathcal{N}^{\otimes n}) \right\|_1 = 0 \right\}$$

- **Containing the relation for states** as a special case of one-dimensional inputs
- Our proof **guarantees the existence of min and lim**, while previous proof does not

Proposition II.1 and Corollary III.2 of Brandao, Plenio, arXiv:0904.0281; Unknown if this relation extends to channels with quantum inputs

→ We **construct asymptotically free operations** achieving rate $r = (R_{\mathrm{R}}^{\infty}(\mathcal{N}_1) - \delta)/R_{\mathrm{R}}^{\infty}(\mathcal{N}_2)$

$$\Theta_n(\mathcal{N}) \coloneqq \mathrm{Tr}[T_n J(\mathcal{N})]\mathcal{N}_2^{(rn)} + \mathrm{Tr}[(\mathbb{1} - T_n) J(\mathcal{N})]\mathcal{N}_2^{(rn)\prime}$$

A special case of one-dimensional inputs: Operations for states by Brandao, Gour, arXiv:1502.03149

- POVM $\{T_n, \mathbb{1} - T_n\}$ with $\mathrm{Tr}[(\mathbb{1} - T_n) J(\mathcal{N}_1^{\otimes n})] \leq \epsilon_n \to 0,\ \max_{\mathcal{N}_{\mathrm{free}} \in \mathcal{F}} \mathrm{Tr}[T_n J(\mathcal{N}_{\mathrm{free}})] \leq \exp[-n(R_{\mathrm{R}}^{\infty}(\mathcal{N}_1) - \delta/3)]$ — Generalized quantum Stein's lemma
- CQ channel $\mathcal{N}_2^{(rn)}$ with $R_{\mathrm{R}}^{\infty}(\mathcal{N}_2) = \lim_{n\to\infty} \frac{1}{\lceil rn \rceil} \log\left(1 + R_{\mathrm{G}}\left(\mathcal{N}_2^{(rn)}\right)\right),\ \lim_{n\to\infty} \frac{1}{2} \left\| J\left(\mathcal{N}_2^{(rn)}\right) - J\left(\mathcal{N}_2^{\otimes \lceil rn \rceil}\right) \right\|_1 = 0$ — Above lemma on R_{R}^{∞}
- CQ channel $\mathcal{N}_2^{(rn)\prime}$ with $\left(\mathcal{N}_2^{(rn)} + R_{\mathrm{G}}\left(\mathcal{N}_2^{(rn)}\right)\mathcal{N}_2^{(rn)\prime}\right) / \left(1 + R_{\mathrm{G}}\left(\mathcal{N}_2^{(rn)}\right)\right) \in \mathcal{F}$ — By definition of R_{G}

Explicitly constructing superchannels for **optimal dynamical resource conversions**

Converse Part of Resource Conversion

Goal: Any achievable rate r under $\tilde{\mathcal{O}}$ should satisfy $r \leq R_{\mathrm{R}}^{\infty}(\mathcal{N}_1)/R_{\mathrm{R}}^{\infty}(\mathcal{N}_2)$

Conventional QRTs: Under free operations,

$$r_{\mathcal{O}}(\rho_1 \to \rho_2) \leq R_{\mathrm{R}}^{\infty}(\rho_1)/R_{\mathrm{R}}^{\infty}(\rho_2)$$

$\Leftrightarrow$

QRTs with 2nd law: R_{R}^{∞} may increase under asymptotically free operations $\tilde{\mathcal{O}}$

Horodecki, Oppenheim, Horodecki, arXiv:quant-ph/0207177; Kuroiwa, Yamasaki, arXiv:2103.05665

Key lemma: An **asymptotic version** of monotonicity

$$R_{\mathrm{R}}^{\infty}(\mathcal{N}) = \lim_{n\to\infty} \frac{1}{n} \min_{\mathcal{N}_{\mathrm{free}}\in\mathcal{F}} D\left(\underbrace{J(\mathcal{N})\|J(\mathcal{N}_{\mathrm{free}})}_{\text{Choi states}}\right) \geq \liminf_{n\to\infty} \frac{1}{n} R_{\mathrm{R}}\left(\Theta_n\left(\mathcal{N}^{\otimes n}\right)\right)$$

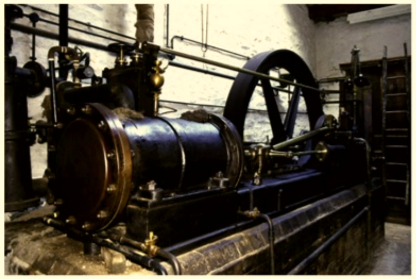
- Satisfying a desired property of “an **asymptotic version of resource measure**”
- Proof **requires asymptotic continuity of Θ**

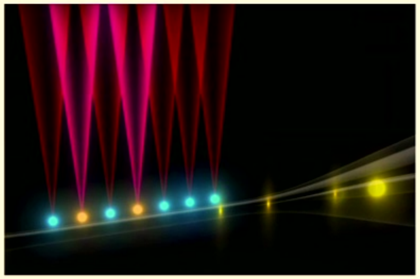
→ Core part of the proof

$$\boxed{R_{\mathrm{R}}^{\infty}(\mathcal{N}_1) \geq \liminf_{n\to\infty} \frac{1}{n} R_{\mathrm{R}}\left(\Theta_n\left(\mathcal{N}_1^{\otimes n}\right)\right)} = \lim_{n\to\infty} \frac{1}{n} R_{\mathrm{R}}\left(\mathcal{N}_2^{\otimes \lceil rn \rceil}\right) = r R_{\mathrm{R}}(\mathcal{N}_2))$$

New Frontiers of General Quantum Resource Theories

19th century
Steam engine
Thermodynamics





21st century
Quantum devices
Quantum information

- Implications for quantum resources: Rate of constant-overhead magic state distillation

$$r_{\tilde{\mathcal{O}}}(\rho \to |T\rangle) = R_{\mathrm{R}}^{\infty}(\rho)/R_{\mathrm{R}}^{\infty}(|T\rangle) \Leftrightarrow$$ Under stabilizer operations (Next week)

Wills, Hsieh, Yamasaki, arXiv:2408.07764

- More applications of generalized quantum Stein's lemma in quantum information theory
- Further extensions of axiomatic framework of QRTs (In preparation)

Proving generalized quantum Stein's lemma is **the first step of its vast applications**

My Research

Social implementation — Advance of IT society by quantum technology

Theoretical foundation = my works

- **Useful quantum algorithm**
  Quantum machine learning with high speed/applicability
- **Implementation of QC**
  Low-overhead/scalable fault-tolerant QC (FTQC)
- **Efficient Q operations**
  Quantitative analysis of use of quantum resources

Experimental foundation — Advance of quantum technology

- Quantum algorithms for exponential speedup in **useful machine-learning subroutines with runtime bounds** arXiv:2004.10756 (NeurIPS2020), arXiv:2106.09028 arXiv:2301.11936 (ICML2023)
- **Provable runtime and energy-consumption advantage** in quantum tasks arXiv:2305.11212, arXiv:2312.03057
- **Constant-space-overhead FTQC with concatenated codes** arXiv:2207.08826 (Nat.Phys.2024), arXiv:2402.09606
- **Polylog-time-overhead** constant-time-overhead FTQC with quantum LDPC codes arXiv:2411.03683
- **Constant-overhead magic state distillation** arXiv:2408.07764 (**next week**)
- **Generalized quantum Stein's lemma** arXiv:2408.02722 (**this talk**), arXiv:2401.01926

さきがけ PRESTO

Summary

- We prove generalized quantum Stein's lemma **with a much smaller set of assumptions**
- Second law of QRTs holds not only for states but also **a fundamental class of channels**
- The **universal axiomatic framework for quantum resources** is now available, much like thermodynamics driving our technological advances ever since Industrial Revolution

References:

Masahito Hayashi, Hayata Yamasaki, arXiv:2408.02722

Reach me out for further discussion, this week & next week

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Thank you for your attention.